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TRANSITIVE DISTANCE-REGULAR GRAPHS FROM LINEAR GROUPS $L(3, q)$, $q = 2, 3, 4, 5$

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Abstract. In this paper we classify distance-regular graphs, including strongly regular graphs, admitting a transitive action of the linear groups $L(3, 2)$, $L(3, 3)$, $L(3, 4)$ and $L(3, 5)$ for which the rank of the permutation representation is at most 15. We give details about constructed graphs. In addition, we construct self-orthogonal codes from distance-regular graphs obtained in this paper.

1. Introduction

We assume that the reader is familiar with the basic facts of the group theory, the theory of strongly regular graphs and the theory of distance-regular graphs. We refer the reader to [7, 18] for relevant background reading in the group theory, to [3, 19] for the theory of strongly regular graphs, and to [5, 14] for the theory of distance-regular graphs.

A construction of distance-regular graphs (DRGs), and especially strongly regular graphs (SRGs), from finite groups gave an important contribution to the graph theory and the design theory (see [3, 5]). Recently, in [11, 12] the authors found new SRGs admitting a transitive action of some finite simple groups and show how one can use groups as a tool to produce wide range of interesting regular graphs.

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In [9] the authors classify t-designs, $t \geq 2$, admitting a transitive action of the linear groups $L(2,q), q \leq 23$, up to 35 points. Furthermore, they constructed strongly regular graphs admitting a transitive action of the linear groups $L(2,q), q \leq 23$.

The main motivation for this paper is to give further contribution to the classification of transitive DRGs, especially those admitting a transitive action of a simple group. The research presented in the paper can be seen as a continuation of the work done by Cameron, Maimani, Omidi and Tayfeh-Rezaie in [8], and Crnković and Švob in [9], in which transitive structures constructed from linear groups $L(2,q)$ were described. Moreover, we construct self-orthogonal codes from the obtained DRGs which contributes to the study of self-orthogonal and transitive linear codes.

In this paper we construct distance-regular graphs, including strongly regular graphs, admitting a transitive action of the linear groups $L(3,2)$, $L(3,3)$, $L(3,4)$ and $L(3,5)$ for which the rank of the permutation representation is at most 15. We give a details about obtained graphs and give a contribution to the research on transitive structures constructed from finite groups. We refer the reader to [7, 21] for more details about these groups.

Using the method outlined in Section 3 we constructed and classified SRGs and DRGs of diameter $d \geq 3$ from above mentioned simple groups for which the rank of the permutation representation is at most 15 (i.e. the number of orbits of the stabiliser acting on the cosets is at most 15). Moreover, from distance-regular graphs obtained in this paper we constructed self-orthogonal codes using the method introduced in [13].

We used programmes written for Magma [2] and GAP [15]. The constructed SRGs and DRGs can be found at the link:

http://www.math.uniri.hr/~asvob/DRGs_LinGps.zip.

2. Preliminaries

In this section we define coherent configurations and association schemes, which are the tools for the construction of graphs presented in this paper. We also give basic definitions and properties of DRGs and SRGs.

Definition 2.1. *A coherent configuration on a finite non-empty set Ω is an ordered pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ a set of non-empty relations on Ω, such that the following axioms hold.*

(i) $\sum_{i=0}^{t} R_i$ *is the identity relation, where* $\{R_0, R_1, \ldots, R_t\} \subseteq \{R_0, R_1, \ldots, R_d\}$.

(ii) $\mathcal{R}$ *is a partition of* Ω^2.

(iii) *For every relation* $R_i \in \mathcal{R}$, *its converse* $R_i^T = \{(y,x) : (x,y) \in R_i\}$ *is in* $\mathcal{R}$.

(iv) *There are constants* p_{ij}^k *known as the intersection numbers of the coherent configuration* $\mathcal{R}$, *such that for* $(x,y) \in R_k$, *the number of elements* z *in* Ω *for which* $(x,z) \in R_i$ *and* $(z,y) \in R_j$ *equals* p_{ij}^k.

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We say that a coherent configuration is homogeneous if it contains the identity relation, i.e., if $R_0 = I$. If $\mathcal{R}$ is a set of symmetric relations on Ω, then a coherent configuration is called symmetric. A symmetric coherent configuration is homogeneous (see [8]). Symmetric coherent configurations are introduced by Bose and Shimamoto in [1] and called association schemes. An association scheme with relations $\{R_0, R_1, \dots, R_d\}$ is called a d-class association scheme.

Let Γ be a graph with diameter d, and let $\delta(u, v)$ denote the distance between vertices u and v of Γ. The ith-neighborhood of a vertex v is the set $\Gamma_i(v) = \{w : \delta(v, w) = i\}$. Similarly, we define Γ_i to be the ith-distance graph of Γ, that is, the vertex set of Γ_i is the same as for Γ, with adjacency in Γ_i defined by the ith distance relation in Γ. We say that Γ is distance-regular if the distance relations of Γ give the relations of a d-class association scheme, that is, for every choice of $0 \le i, j, k \le d$, all vertices v and w with $\delta(v, w) = k$ satisfy $|\Gamma_i(v) \cap \Gamma_j(w)| = p^k_{ij}$ for some constant p^k_{ij}. In a distance-regular graph, we have that $p^k_{ij} = 0$ whenever $i + j < k$ or $k < |i - j|$. A distance-regular graph Γ is necessarily regular with degree p^0_{11}; more generally, each distance graph Γ_i is regular with degree $k_i = p^0_{ii}$.

An equivalent definition of distance-regular graphs is the existence of the constants $b_i = p^i_{i+1,1}$ and $c_i = p^i_{i-1,1}$ for $0 \le i \le d$ (notice that $b_d = c_0 = 0$). The sequence $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, where d is the diameter of Γ is called the intersection array of Γ. Clearly, $b_0 = k$, $b_d = c_0 = 0$, $c_1 = 0$.

A regular graph is strongly regular with parameters (v, k, λ, μ) if it has v vertices, degree k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices. A strongly regular graph with parameters (v, k, λ, μ) is usually denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$. A strongly regular graph is a distance-regular graph with diameter 2 whenever $\mu \ne 0$. The intersection array of an SRG is given by $\{k, k - 1 - \lambda; 1, \mu\}$.

3. SRGs and DRGs constructed from the groups

Let G be a finite permutation group acting on the finite set Ω. This action induce the action of the group G on the set $\Omega \times \Omega$. For more information see [20]. The orbits of this action are the sets of the form $\{(\alpha g, \beta g) : g \in G\}$. If G is transitive, then $\{(\alpha, \alpha) : \alpha \in \Omega\}$ is one such orbit. If the rank of G is r, then it has r orbits on $\Omega \times \Omega$. Let $|\Omega| = n$ and Δ_i is one of these orbits. We say that the $n \times n$ matrix A_i, with rows and columns indexed by Ω and entries

$$A_i(\alpha, \beta) = \begin{cases} 1, & \text{if } (\alpha, \beta) \in \Delta_i \\ 0, & \text{otherwise.} \end{cases}$$

is called the adjacency matrix for the orbit Δ_i.

Let $A_0, \dots, A_{r-1}$ be the adjacency matrices for the orbits of G on $\Omega \times \Omega$. These satisfy the following conditions.

(i) $A_0 = I$, if G is transitive on Ω. If G has s orbits on Ω, then I is a sum of s adjacency matrices.

(ii) $\sum_i A_i = J$, where J is the all-one matrix.

(iii) If A_i is an adjacency matrix, then so is its transpose A_i^T.

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(iv) If A_i and A_j are adjacency matrices, then their product is an integer-linear-combination of adjacency matrices.

If A_i is symmetric, then the corresponding orbit is called self-paired. Further, if $A_i = A_j^T$, then the corresponding orbits are called mutually paired.

The graphs obtained in this paper are constructed using the method described in [10] which can be rewritten in terms of coherent configurations as shown in [12].

Theorem 3.1. [12, Theorem 1] *Let G be a finite permutation group acting transitively on the set Ω and $A_0, \ldots, A_d$ be the adjacency matrices for orbits of G on $\Omega \times \Omega$. Let $\{B_1, \ldots, B_t\} \subseteq \{A_1, \ldots, A_d\}$ be a set of adjacency matrices for the self-paired or mutually paired orbits. Then $M = \sum_{i=1}^{t} B_i$ is the adjacency matrix of a regular graph Γ. The group G acts transitively on the set of vertices of the graph Γ.*

Using this method one can construct all regular graphs admitting a transitive action of the group G. We will be interested only in those regular graphs that are distance-regular, and especially strongly regular.

Remark 3.2. *Because of the large number of possibilities for building the first row of the adjacency matrix of a DRG, the only way to obtain the classification of DRGs given in this paper was with the use of computers. The running time complexity of the algorithm used for the construction of graphs depends on a number of parameters, such as the size of the used subgroup, the number of orbits of a vertex stabilizer, the number of vertices of the graphs and the number of self-paired and mutually paired orbits in a particular case. We have been able to complete the construction up to the rank 15 i.e. up to 15 orbits of a vertex stabiliser. For the ranks greater than 15, the number of possibilities for the first row of the adjacency matrix was too big to complete the classification in a reasonable amount of time.*

3.1. **SRGs and DRGs from the group $L(3,2)$.** The group $L(3,2)$ is the simple group of order 168, isomorphic to the group $L(2,7)$. Up to conjugation it has 15 subgroups. In Table 1 we give the list of all the subgroups $H_i^1 \leq L(3,2)$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

Table 1. Subgroups of the group $L(3,2)$

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^1	D_8	8	21	6	no
H_2^1	Z_7	7	24	6	no
H_3^1	S_3	6	28	7	no
H_4^1	Z_4	4	42	7	no

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Using the method described in Theorem 3.1 we obtained all DRGs on which the group $L(3,2)$ acts transitively and for which the rank of the permutation representation of the group is at most 15, i.e. we gave the classification of such DRGs.

Theorem 3.3. *Up to isomorphism there are exactly three strongly regular graphs and exactly four distance-regular graphs of diameter $d \geq 3$ admitting an transitive action of the group $L(3,2)$, having the rank at most 15. The SRGs have parameters $(21,10,3,6)$, $(21,10,5,4)$ and $(28,12,6,4)$, and the DRGs have 21, 24, 28 and 42 vertices, respectively. Details about the obtained strongly regular graphs are given in Table 2 and details about the obtained DRGs with $d \geq 3$ are given in Table 3.*

Table 2. SRGs constructed from the group $L(3,2)$

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^1 = \Gamma(L(3,2), H_1^1)$	(21,10,3,6)	S_7
$\Gamma_2^1 = \Gamma(L(3,2), H_1^1)$	(21,10,5,4)	S_7
$\Gamma_3^1 = \Gamma(L(3,2), H_3^1)$	(28,12,6,4)	S_8

Table 3. DRGs constructed from the group $L(3,2)$, $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_4^1 = \Gamma(L(3,2), H_1^1)$	21	3	$\{4,2,2;1,1,2\}$	$L(3,2):Z_2$
$\Gamma_5^1 = \Gamma(L(3,2), H_2^1)$	24	3	$\{7,4,1;1,2,7\}$	$L(3,2):Z_2$
$\Gamma_6^1 = \Gamma(L(3,2), H_3^1)$	28	4	$\{3,2,2,1;1,1,1,2\}$	$L(3,2):Z_2$
$\Gamma_7^1 = \Gamma(L(3,2), H_4^1)$	42	3	$\{13,8,1;1,4,13\}$	$L(3,2)$

Proof. Each transitive action of the group $L(3,2)$ is permutation isomorphic to an action of the group $L(3,2)$ on cosets of one of its subgroup. There are 15 conjugacy classes of subgroups of $L(3,2)$, but only 12 of them lead to a permutation representation of rank at most 15. Applying the method described in Theorem 3.1 to the permutation representations on cosets of these 12 subgroups we obtain the results. □

Remark 3.4. *The graph Γ_1^1 is a rank 3 graph and the unique strongly regular graph with these parameters. The strongly regular graphs Γ_2^1 and Γ_3^1 are the triangular graphs $T(7)$ and $T(8)$, respectively.*

Remark 3.5. *The graph Γ_4^1 is the hexagon of order $(2,1)$, the unique DRG with the given intersection array. The DRGs Γ_5^1 and Γ_7^1 belong to the family od distance-regular graphs of diameter 3 that are antipodal but not bipartite graphs. The graph Γ_6^1 is a Coxeter graph, the unique DRG with the given intersection array.*

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3.2. **SRGs and DRGs from the group $L(3,3)$.** The group $L(3,3)$ is the simple group of order 5616. Up to conjugation it has 51 subgroups. In Table 4 we give the list of all the subgroups $H_i^2 \leq L(3,3)$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

TABLE 4. Subgroups of the group $L(3,3)$

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^2	$((E_9 : Z_3) : Z_2) : Z_2$	108	52	6	no
H_2^2	$E_9 : Q_8$	72	78	7	no
H_3^2	$GL(2,3)$	48	117	8	no
H_4^2	$Z_{13} : Z_3$	39	144	8	yes

Using the method described in Theorem 3.1 we obtained all DRGs for which the rank of the permutation representation of the group is at most 15, i.e. we gave the classification of such DRGs.

Theorem 3.6. *Up to isomorphism there are exactly three strongly regular graphs and exactly one distance-regular graph of diameter $d \geq 3$ admitting an transitive action of the group $L(3,3)$, having the rank at most 15. The SRGs have parameters $(78,22,11,4)$, $(117,36,15,9)$ and $(144,39,6,12)$, and the DRG has 52 vertices. Details about the obtained strongly regular graphs are given in Table 5 and details about the obtained DRG with $d \geq 3$ are given in Table 6.*

TABLE 5. SRGs constructed from the group $L(3,3)$

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^2 = \Gamma(L(3,3), H_2^2)$	(78,22,11,4)	S_{13}
$\Gamma_2^2 = \Gamma(L(3,3), H_3^2)$	(117,36,15,9)	$L(4,3) : Z_2$
$\Gamma_3^2 = \Gamma(L(3,3), H_4^2)$	(144,39,6,12)	$L(3,3) : Z_2$

TABLE 6. DRGs constructed from the group $L(3,3)$, $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_4^2 = \Gamma(L(3,3), H_1^2)$	52	3	$\{6,3,3;1,1,2\}$	$L(3,3) : Z_2$

Proof. There are 51 conjugacy classes of subgroups of $L(3,3)$, but only 21 of them lead to a permutation representation of rank at most 15. Applying the method described in Theorem 3.1 to the permutation representations on cosets of these 21 subgroups we obtain the results. □

Remark 3.7. *The graph Γ_1^2 is the triangular graph $T(13)$ and graph Γ_3^2 is known as $L(3,3)$ rank 8 graph. We refer the reader to [4] for more information. The strongly regular graph Γ_2^2 is a rank 3 graph.*

Remark 3.8. *The graph Γ_4^2 is the generalised hexagon of order $(3,1)$, the unique DRG with the given intersection array.*

3.3. **SRGs and DRGs from the group $L(3,4)$.** The group $L(3,4)$ is a simple group of order 20160. Up to conjugation it has 95 subgroups. In Table 7 we give the list of all the subgroups $H_i^3 \leq L(3,4)$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

TABLE 7. Subgroups of the group $L(3,4)$

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^3	A_6	360	56	3	yes
H_2^3	$(((Z_4 \times Z_4):Z_2):Z_2):Z_3$	192	105	6	no
H_3^3	$(Z_3 \times (Z_9:Z_3)):Z_2$	164	120	4	yes
H_4^3	$(E_{16}:Z_5):Z_2$	160	126	5	no
H_5^3	$(E_{16}:Z_3):Z_2$	96	210	9	no
H_6^3	$E_9:Q_8$	72	280	8	yes

Using the method described in Theorem 3.1 we obtained all DRGs for which the rank of the permutation representation of the group is at most 15, i.e. we gave the classification of such DRGs.

Theorem 3.9. *Up to isomorphism there are exactly six strongly regular graphs and exactly two distance-regular graphs of diameter* $d \geq 3$ *admitting an transitive action of the group* $L(3,4)$*, having the rank at most 15. The SRGs have parameters* $(56,10,0,2)$*,* $(105,32,4,12)$*,* $(120,42,8,18)$*,* $(126,45,12,18)$*,* $(210,38,19,4)$ *and* $(280,36,8,4)$*, and the DRGs have* 105 *and* 280 *vertices. Details about the obtained strongly regular graphs are given in Table 8 and details about the obtained DRGs with* $d \geq 3$ *are given in Table 9.*

TABLE 8. SRGs constructed from the group $L(3,4)$

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^3 = \Gamma(L(3,4), H_1^3)$	(56,10,0,2)	$(L(3,4):Z_2):Z_2$
$\Gamma_2^3 = \Gamma(L(3,4), H_2^3)$	(105,32,4,12)	$((L(3,4):Z_3):Z_2):Z_2$
$\Gamma_3^3 = \Gamma(L(3,4), H_3^3)$	(120,42,8,18)	$(L(3,4):Z_)):Z_2$
$\Gamma_4^3 = \Gamma(L(3,4), H_4^3)$	(126,45,12,18)	$(U(4,3):Z_2):Z_2$
$\Gamma_5^3 = \Gamma(L(3,4), H_5^3)$	(210,38,19,4)	S_{21}
$\Gamma_6^3 = \Gamma(L(3,4), H_6^3)$	(280,36,8,4)	$(U(4,3):Z_4):Z_2$

TABLE 9. DRGs constructed from the group $L(3,4)$, $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_7^3 = \Gamma(L(3,4), H_2^3)$	105	3	$\{8,4,4;1,1,2\}$	$((L(3,4):Z_3):Z_2):Z_2$
$\Gamma_8^3 = \Gamma(L(3,4), H_6^3)$	280	4	$\{9,8,6,3;1,1,3,8\}$	$((L(3,4):Z_3):Z_2):Z_2$

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Proof. There are 95 conjugacy classes of subgroups of $L(3,4)$, but only 27 of them lead to a permutation representation of rank at most 15. Applying the method described in Theorem 3.1 to the permutation representations on cosets of these 27 subgroups we obtain the results. □

Remark 3.10. *The strongly regular graphs* Γ_1^3, Γ_2^3 *and* Γ_3^3 *are the unique graphs with this parameters. The graph* Γ_1^3 *is known as the Sims-Gewirtz graph. The SRG* Γ_5^3 *is the triangular graph* $T(21)$ *and the graph* Γ_6^3 *is a* $U(4,3)$ *polar graph. The graphs* Γ_1^3, Γ_4^3 *and* Γ_6^3 *are rank 3 graphs.*

Remark 3.11. *The DRGs* Γ_7^3 *and* Γ_8^3 *are the unique graphs with these parameters. The graph* Γ_7^3 *is the generalised hexagon of order* $(4,1)$ *whose distance 2 graph is the strongly regular with parameters* $(105,32,4,12)$ *which is isomorphic to the graph* Γ_2^3*. Further,* Γ_8^3 *is called a unital graph.*

3.4. **SRGs and DRGs from the group** $L(3,5)$**.** The group $L(3,5)$ is a simple group of order 372000. Up to conjugation it has 140 subgroups. In Table 10 we give the list of all the subgroups $H_i^4 \leq L(3,5)$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

Table 10. Subgroups of the group $L(3,5)$

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^4	$((E_{25}:Z_5):Z_4):Z_4$	2000	186	6	no
H_2^4	$((Z_5:Z_4)\times(Z_5:Z_4)):Z_2$	800	465	8	no
H_3^4	$GL(2,5)$	480	775	10	no

Using the method described in Theorem 3.1 we obtained all DRGs for which the rank of the permutation representation of the group is at most 15, i.e. we gave the classification of such DRGs.

Theorem 3.12. *Up to isomorphism there are exactly two strongly regular graphs and exactly one distance-regular graph of diameter* $d \geq 3$ *admitting an transitive action of the group* $L(3,5)$*, having the rank at most 15. The SRGs have parameters* $(465,58,29,4)$ *and* $(775,150,45,25)$ *and the DRG has* 186 *vertices. Details about the obtained strongly regular graphs are given in Table 11 and details about the obtained DRG with* $d \geq 3$ *are given in Table 12.*

Table 11. SRGs constructed from the group $L(3,5)$

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^4=\Gamma(L(3,5),H_2^4)$	(465,58,29,4)	S_{31}
$\Gamma_2^4=\Gamma(L(3,5),H_3^4)$	(775,150,45,25)	$L(3,5):Z_2$

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TABLE 12. DRGs constructed from the group $L(3,5)$, $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_3^4 = \Gamma(L(3,5), H_1^4)$	186	3	$\{10,5,5;1,1,2\}$	$L(3,5) : Z_2$

Proof. There are 140 conjugacy classes of subgroups of $L(3,5)$, but only 36 of them lead to a permutation representation of rank at most 140. Applying the method described in Theorem 3.1 to the permutation representations on cosets of these 36 subgroups we obtain the results. □

Remark 3.13. *The strongly regular graph Γ_2^4 is the triangular graph $T(31)$, while Γ_3^4 can be described as a graph obtained from lines in $AG(3,5)$.*

Remark 3.14. *The distance-regular graph Γ_3^4 is the generalised hexagon of order $(5,1)$, the unique DRG with the given intersection array.*

Remark 3.15. *All SRGs and DRGs constructed in this paper are vertex-transitive, but some of them are also edge-transitive. The graphs that are not edge-transitive are Γ_7^1 and Γ_2^4.*

4. Self-orthogonal codes obtained from DRGs

Suppose A is a symmetric real matrix whose rows and columns are indexed by the elements of $X = \{1, \ldots, n\}$. Let $\{C_0, \ldots, C_{t-1}\}$ be a partition of X.

A partition $\Pi = \{C_0, C_1, \ldots, C_{t-1}\}$ of the n vertices of a graph G is *equitable* (or regular) if for every pair of (not necessarily distinct) indices $i, j \in \{0, 1, \ldots, t-1\}$ there is a nonnegative integer $b_{i,j}$ such that each vertex $v \in C_i$ has exactly $b_{i,j}$ neighbors in C_j, regardless of the choice of v. The $t \times t$ quotient matrix $B = (b_{i,j})$ is well-defined if and only if the partition Π is equitable. An equitable (or regular) partition of an association scheme $(X, \mathcal{R})$ is a partition of X which is equitable with respect to each of the graphs Γ_i, $i \in \{1, 2, ..., d\}$ corresponding to the association scheme $(X, \mathcal{R})$ with d classes.

A q-ary *linear code* C of dimension k for a prime power q, is a k-dimensional subspace of a vector space $\mathbb{F}_q^n$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$. The *Hamming distance* between words x and y is the number $d(x,y) = |\{i : x_i \neq y_i\}|$. The *minimum distance* of the code C is defined by $d = \min\{d(x,y) : x, y \in C,\ x \neq y\}$. The *weight* of a codeword x is $w(x) = d(x,0) = |\{i : x_i \neq 0\}|$. For a linear code, $d = \min\{w(x) : x \in C, x \neq 0\}$. The *dual* code $C^\perp$ of a code C is the orthogonal complement of C under the standard inner product $\langle \cdot \rangle$, i.e. $C^\perp = \{v \in \mathbb{F}_q^n | \langle v, c \rangle = 0 \text{ for all } c \in C\}$. A code C is *self-orthogonal* if $C \subseteq C^\perp$ and *self-dual* if $C = C^\perp$. For background reading in coding theory we refer the reader to [17].

In [13] the authors gave a method of constructing self-orthogonal codes from equitable partitions of association schemes. In this Section we use the method given in [13, Theorem 2] and apply the construction on distance-regular graphs constructed in previous sections. The obtained results are presented in Tables 13 and 14, together with the following information: the DRG from which the code

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is constructed, an automorphism group used for obtaining quotient matrices and index i that gives the quotient matrix that is used as the generator matrix for the code. For details on applied method we refer the reader to [13].

A q-ary linear code of length n, dimension k, and distance d is called a $[n,k,d]_q$ code. An $[n,k]$ linear code C is *optimal* if the minimum weight of C achieves the theoretical upper bound on the minimum weight of $[n,k]$ linear codes, and is said to be a *best known* linear $[n,k]$ code if C has the highest minimum weight among all known $[n,k]$ linear codes. A catalogue of best known and optimal codes is maintained at [16], to which we compare the minimum weight of all codes constructed in this paper. The self-orthogonal codes in Tables 13 and 14 marked with $*$ are optimal. Some of the obtained codes are self-dual.

Table 13. Self-orthogonal codes from the graphs obtained from $L(3,2)$, $L(3,3)$, L(3,5)

The graph	$H \leq \mathrm{Aut}(\Gamma)$, i	The code	The graph	$H \leq \mathrm{Aut}(\Gamma)$, i	The code
Γ_3^1	I, 1	$[28,6,12]_2*$	Γ_3^2	E_4, 2	$[36,7,8]_2$
Γ_2^2	I, 1	$[117,21,36]_3$	Γ_3^2	S_3, 1	$[24,5,12]_3$
Γ_2^2	I, 2	$[117,26,24]_2$	Γ_3^2	S_3, 2	$[24,6,8]_2$
Γ_2^2	Z_3, 1	$[39,7,12]_3$	Γ_3^2	Q_8, 1	$[18,4,9]_3$
Γ_2^2	Z_3, 2	$[39,8,8]_2$	Γ_3^2	Q_8, 2	$[18,4,4]_2$
Γ_2^2	E_9, 2	$[13,2,8]_2*$	Γ_3^2	D_8, 1	$[18,4,6]_3$
Γ_2^2	Z_{13}, 1	$[9,3,6]_3*$	Γ_3^2	D_8, 2	$[18,2,8]_2$
Γ_2^2	Z_{13}, 2	$[9,2,4]_2$	Γ_3^2	Z_8, 1	$[18,5,6]_3$
Γ_3^2	I, 1	$[144,32,36]_3$	Γ_3^2	Z_8, 2	$[18,4,4]_2$
Γ_3^2	I, 2	$[144,40,32]_3$	Γ_3^2	D_{12}, 1	$[12,3,6]_3$
Γ_3^2	Z_2, 1	$[72,16,18]_3$	Γ_3^2	D_{12}, 2	$[12,2,4]_2$
Γ_3^2	Z_2, 2	$[72,18,16]_2$	Γ_3^2	Z_{12}, 2	$[12,4,4]_2$
Γ_3^2	Z_3, 1	$[48,9,12]_3$	Γ_3^2	QD_{16}, 1	$[9,2,6]_3*$
Γ_3^2	Z_3, 2	$[48,16,12]_2$	Γ_2^4	Z_{31}, 1	$[25,5,13]_5$
Γ_3^2	Z_4, 1	$[36,9,12]_3$	Γ_2^4	Z_{31}, 2	$[25,4,12]_2*$
Γ_3^2	Z_4, 2	$[36,8,8]_2$	Γ_2^4	Z_5, 1	$[155,19,30]_5$
Γ_3^2	E_4, 1	$[36,8,12]_3$	Γ_2^4	Z_5, 2	$[155,24,44]_2$

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TABLE 14. Self-orthogonal codes from the graphs obtained from $L(3,4)$

The graph	$H \leq \mathrm{Aut}(\Gamma)$, i	The code	The graph	$H \leq \mathrm{Aut}(\Gamma)$, i	The code
Γ_1^3	Z_2,1	$[28,9,8]_2$	Γ_4^3	S_3, 2	$[21,2,8]_2$
Γ_1^3	Z_2, 2	$[28,9,12]_3$	Γ_4^3	Z_6, 1	$[21,3,12]_3$
Γ_1^3	I, 1	$[56,20,10]_2$	Γ_4^3	Z_6, 2	$[21,4,8]_2$
Γ_1^3	I, 2	$[56,19,18]_3$	Γ_4^3	Z_7, 1	$[18,3,12]_3*$
Γ_2^3	Z_3, 1	$[35,6,16]_2*$	Γ_4^3	Z_7, 2	$[18,4,8]_2*$
Γ_2^3	Z_3, 2	$[35,6,18]_3$	Γ_4^3	E_9, 2	$[14,2,8]_2$
Γ_2^3	Z_5, 1	$[21,2,8]_2$	Γ_4^3	Z_{14}, 1	$[9,3,6]_3*$
Γ_2^3	Z_5, 2	$[21,4,9]_3$	Γ_4^3	Z_{14}, 2	$[9,3,4]_2*$
Γ_2^3	Z_7, 2	$[15,2,6]_3$	Γ_6^3	I, 1	$[280,70,36]_2$
Γ_2^3	I, 1	$[105,18,32]_2$	Γ_6^3	Z_5, 1	$[56,14,8]_2$
Γ_2^3	I, 2	$[105,20,33]_3$	Γ_6^3	Z_5, 2	$[56,13,12]_3$
Γ_3^3	I, 1	$[120,20,30]_2$	Γ_6^3	Z_7, 1	$[40,10,12]_2$
Γ_3^3	Z_5, 1	$[24,4,6]_2$	Γ_6^3	Z_7, 2	$[40,9,12]_3$
Γ_4^3	I, 2	$[126,34,24]_2$	Γ_7^3	Z_3, 2	$[35,6,16]_2*$
Γ_4^3	I, 1	$[126,21,36]_3$	Γ_7^3	Z_5, 2	$[21,2,8]_2$
Γ_4^3	Z_2, 2	$[63,14,16]_2$	Γ_7^3	I, 2	$[105,18,32]_2$
Γ_4^3	Z_2, 1	$[63,9,30]_3$	Γ_8^3	Z_5, 3	$[56,2,24]_2$
Γ_4^3	Z_3, 2	$[42,10,8]_2$	Γ_8^3	Z_5, 4	$[56,16,18]_3$
Γ_4^3	Z_3, 1	$[42,7,15]_3$	Γ_8^3	Z_7, 4	$[40,12,12]_3$
Γ_4^3	S_3, 1	$[21,2,12]_3$	Γ_8^3	I, 3	$[280,18,112]_2$

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