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NILPOTENT GRAPHS OF SKEW POLYNOMIAL RINGS OVER NON-COMMUTATIVE RINGS

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ABSTRACT. Let R be a ring and α be a ring endomorphism of R. The undirected nilpotent graph of R, denoted by $\Gamma_N(R)$, is a graph with vertex set $Z_N(R)^*$, and two distinct vertices x and y are connected by an edge if and only if xy is nilpotent, where $Z_N(R) = \{x \in R \mid xy \text{ is nilpotent, for some y} \in \text{R}^*\}$. In this article, we investigate the interplay between the ring theoretical properties of a skew polynomial ring $R[x;\alpha]$ and the graph-theoretical properties of its nilpotent graph $\Gamma_N(R[x;\alpha])$. It is shown that if R is a symmetric and α-compatible with exactly two minimal primes, then $diam(\Gamma_N(R[x,\alpha])) = 2$. Also we prove that $\Gamma_N(R)$ is a complete graph if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

1. Introduction

Throughout this paper R denotes an associative ring with unity and α is a ring endomorphism of R. We denote by $R[x;\alpha]$ the skew polynomial ring whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xr = \alpha(r)x$ for any $r \in R$. We write $Z(R)$ and $Z(R)^* = Z(R) \setminus \{0\}$ for the set of zero-divisors of R including zero and the set of nonzero zero-divisors of R, respectively. An element x is called nilpotent if $x^m = 0$ for some positive integer m and the set of all nilpotent elements in R is denoted by $\text{Nil}(R)$. A ring R is called an *NI*

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ring if Nil(R) forms an ideal. For $X \subset R$, the ideal generated by X is denoted by $< X >$. Note that $R^* = R \setminus \{0\}$.

Recall that a graph is said to be *connected* if for each pair of distinct vertices x and y, there is a finite sequence of distinct vertices $x = x_1, x_2, \ldots, x_n = y$ such that each pair x_i, x_j is an edge. Such a sequence is said to be a *path* and for two distinct vertices a and b in the simple graph Γ, the *distance* between a and b denoted by $d(a, b)$, is the length of a shortest path connecting a and b, if such a path exists, otherwise we put $d(a, b) = \infty$. Recall that the *diameter* of a graph Γ is defined as follows:

$$diam(\Gamma) = sup\{d(a, b) | \; a \text{ and b are distinct vertices of } \Gamma\}.$$

The *girth* of a simple connected graph Γ, denoted by $gr(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle, otherwise $gr(\Gamma) = \infty$.

In 1988, Beck began to investigate the possibility of coloring a commutative ring R by associating to the ring a zero-divisor graph, defined as a simple graph, the vertices of which are the elements of the ring R, with two distinct elements x and y being adjacent if and only if $xy = 0$ [4]. However, in 1999, Anderson and Livingston [2] modified and studied the zero-divisor graph $\Gamma(R)$, whose vertices are the nonzero zero-divisors $Z(R)^*$ of commutative ring R.

If R is a noncommutative ring, Redmond [16], defined an undirected zero-divisor graph of a non-commutative ring, the graph $\Gamma(R)$, with vertices in the set $Z(R)^* = Z(R) \setminus \{0\}$ such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$ or $ba = 0$. Several papers are devoted to studying the relationship between the zero-divisor graphs and algebraic properties of rings [2, 1, 3, 4].

In [8], Chen defined a kind of graph structure of rings. He let all the elements of ring R be the vertices of the graph and two vertices x and y are adjacent if and only if xy is nilpotent. However, in 2010, Li and Li [13] modified and studied the nilpotent graph $\Gamma_N(R)$ of R as a graph with vertex set $Z_N(R)^*$, and two distinct vertices x and y were adjacent if and only if xy is nilpotent, where $Z_N(R) = \{x \in R \mid xy \text{ is nilpotent, for some y} \in \text{R}^*\}$. Note that the usual zero-divisor graph $\Gamma(R)$ is a subgraph of the graph $\Gamma_N(R)$. There is a considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. The first such extensions that come to mind are those of matrix, polynomilal and power series extensions. Nikmehr [15] studied the diameter and the girth of nilpotent graph of matrix rings over noncommutative rings. Also, Axtell et al. [3] examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings.

According to Cohn [8], a ring R is called *reversible* if $ab = 0$ implies that $ba = 0$ for each $a, b \in R$. Clearly, *reduced* rings (i.e. rings with no nonzero nilpotent elements) and commutative rings are reversible. Moreover, a ring R is called *symmetric* if $abc = 0$ implies that $acb = 0$ for each $a, b, c \in R$. Simple computations show that reversible rings are symmetric. Following [10], a ring R is called α*-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Recall that an ideal P of R is completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. In this paper, our first main result states that,

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$\Gamma_N(R)$ is a complete graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We also prove that if R is a symmetric and α-compatible with exactly two minimal primes, then $diam(\Gamma_N(R[x,\alpha])) = 2$. Moreover, we examine the preservation and lack thereof of the girth of the nilpotent graph of a noncommutative ring under extension to skew polynomial ring.

2. On Nilpotent Properties of Skew Polynomial Rings

In this section, we show that if R is symmetric, then $\Gamma_N(R)$ is always connected and we determine which complete graphs may be realized as $\Gamma_N(R)$. We also examine the preservation and lack thereof of the diameter and girth of the nilpotent graph of a noncommutative ring under extension to skew polynomial ring. The following lemma, which has been proved in [10, Lemma 2.1], will be helpful in our main results.

Lemma 2.1. *Let R be an α-compatible ring. Then we have the following:*

(1) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n.*
(2) *If $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$.*

Note that the set of nilpotent elements of a ring is not ideal in general, (see [17]). According to [5], a ring R is called *semi-commutative* if $ab = 0$ implies $aRb = 0$. If R is a semi-commutative ring, then $\mathrm{Nil}(R)$ is an ideal of R, by [18, Lemma 3.1]. It is well known that every symmetric ring is semicommutative. Then we have the following result:

Lemma 2.2. *Let R be a symmetric ring. Then* $\mathrm{Nil}(R)$ *is an ideal of R.*

Now we bring the following lemma, which has a key role in our main results.

Lemma 2.3. *Let R be a symmetric α-compatible ring. Then* $\mathrm{Nil}(R[x,\alpha])$ *is an ideal.*

Proof. The proof is a consequence of the following claims:
Claim 1. If $ab \in \mathrm{Nil}(R)$, then $a\alpha^j(b) \in \mathrm{Nil}(R)$, for all positive integers j.
Proof of Claim 1. If $ab \in \mathrm{Nil}(R)$, then $(ab)^k = 0$, for some $k \in \mathbb{N}$. So
$a\alpha^j(b)\alpha^j(abab\cdots ab) = 0$, and thus $a\alpha^j(b)abab\cdots ab = 0$. Continuing this procedure yields that $(a\alpha^j(b))^k = 0$, this means $a\alpha^j(b) \in \mathrm{Nil}(R)$.
Claim 2. If $a\alpha^m(b) \in \mathrm{Nil}(R)$, for some positive integer m, then $ab \in \mathrm{Nil}(R)$.
Proof of Claim 2. The proof is similar to that of Claim 1.
Claim 3. $\mathrm{Nil}(R)[x;\alpha]$ is an ideal of $R[x;\alpha]$.
Proof of Claim 3. Recall that an ideal I of a ring R is called *α-ideal* if $\alpha(I) \subseteq I$. If an ideal I of R is α-ideal, then $I[x;\alpha]$ is an ideal of $R[x;\alpha]$. On the other hand, $\mathrm{Nil}(R)$ is an α-ideal, by Claim 2. Thus $\mathrm{Nil}(R)[x;\alpha]$ is an ideal of $R[x;\alpha]$, and the result follows.

Claim 4. $\mathrm{Nil}(R[x;\alpha]) = \mathrm{Nil}(R)[x;\alpha]$.
Proof of Claim 4. First we show that $\mathrm{Nil}(R[x,\alpha]) \subseteq \mathrm{Nil}(R)[x;\alpha]$. Suppose $f(x) \in \mathrm{Nil}(R[x;\alpha])$.

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There exists some positive integer k such that $f(x)^k = (a_0 + a_1x + \cdots + a_mx^m)^k = 0$. Then we have $h(x) + a_m\alpha^m(a_m)\alpha^{2m}(a_m)\cdots$
$\alpha^{(k-1)m}(a_m)x^{km} = 0$, where $h(x)$ is a polynomial in $R[x;\alpha]$ with degree at most $(km-1)$. Hence $a_m\alpha^m(a_m)\alpha^{2m}(a_m)\cdots\alpha^{(k-1)m}(a_m) = 0$. Since R is α-compatible, $a_m \in \mathrm{Nil}(R)$. Moreover, since $\mathrm{Nil}(R)[x;\alpha]$ is an ideal of $R[x;\alpha]$, by Claim 3, and for each $g_1(x)$ and $g_2(x) \in R[x;\alpha]$, $a_mg_1(x), g_1(x)a_m$, $f(x)^k \in \mathrm{Nil}(R)[x;\alpha]$, we have

$$(a_0 + a_1x + \cdots + a_{m-1}x^{m-1})^k = h'(x) + a_{m-1}\alpha^{m-1}(a_{m-1}) \cdots \alpha^{(k-1)(m-1)}(a_{m-1})x^{k(m-1)} \in \mathrm{Nil}(R)[x;\alpha],$$

where $h'(x)$ is a polynomial in $R[x;\alpha]$ with degree at most $k(m-1)-1$. Hence

$$a_{m-1}\alpha^{m-1}(a_{m-1})\cdots\alpha^{(k-1)(m-1)}(a_{m-1}) \in \mathrm{Nil}(R)$$

, and similar discussion yields that $a_{m-1} \in \mathrm{Nil}(R)$. By continuing this procedure, we have $a_i \in \mathrm{Nil}(R)$, for each $1 \leq i \leq m$. This means that $f(x) \in \mathrm{Nil}(R)[x;\alpha]$.

Next we show that $\mathrm{Nil}(R)[x,\alpha] \subseteq \mathrm{Nil}(R[x,\alpha])$. Suppose that ${a_i}^{m_i} = 0$, for $i = 0,\ldots,m$. Let $k = m_0 + \cdots + m_n + 1$. Then we have $(f(x))^k = \sum[({a_0}^{i_{0_1}}(a_1x)^{i_{1_1}}\cdots$
$(a_mx^m)^{i_{m_1}})({a_0}^{i_{0_2}}(a_1x)^{i_{1_2}}\cdots(a_mx^m)^{i_{m_2}})\cdots({a_0}^{i_{0_k}}(a_1x)^{i_{1_k}}\cdots(a_mx^m)^{i_{m_k}})]$, where
$i_{0_r} + i_{1_r} + \cdots + i_{m_r} = 1$, for $r = 1,\ldots,k$ and $i_{r_s} \in \{0,1\}$. It is easy to see that each coefficent of $[f(x)]^k$ is a sum of such elements $((f^{u_{0_1}}_{u_{0_1}}(a_0))^{i_{0_1}}(f^{u_{1_1}}_{u_{1_1}}(a_1))^{i_{1_1}}\cdots$
$(f^{u_{m_1}}_{u_{m_1}}(a_m))^{i_{m_1}})\cdots((f^{u_{0_k}}_{u_{0_k}}(a_0))^{i_{0_k}}(f^{u_{1_k}}_{u_{1_k}}(a_1))^{i_{1_k}}\cdots(f^{u_{m_k}}_{u_{m_k}}(a_m))^{i_{m_k}})$, where $i_{0_r} + i_{1_r} + \cdots + i_{m_r} = 1$, and $f^j_j = \alpha^j$. It can be easily checked that there exists $a_h \in \{a_0, a_1, \ldots, a_m\}$ such that $i_{h_1}+i_{h_2}+\cdots+i_{h_k} \geq m_h$. Since ${a_h}^{m_h} = 0$, we deduce that ${a_h}^{i_{h_1}+i_{h_2}+\cdots+i_{h_k}} = 0$, and hence, by Lemma 2.1, we have $((f^{u_{h_1}}_{u_{h_1}}(a_h))^{i_{h_1}}(f^{u_{h_2}}_{u_{h_2}}(a_h))^{i_{h_2}}$
$\cdots(f^{u_{h_k}}_{u_{h_k}}(a_h))^{i_{h_k}}) = 0$. Thus each cofficient of $(f(x))^k$ is zero, since R is a symmetric ring. Hence $[f(x)]^k = 0$. Therefore, $f(x) \in \mathrm{Nil}(R[x;\alpha])$ as desired. □

Note that reduced rings are symmetric, so we can deduce the following result. Notice that our proof is simple and short.

Corollary 2.4. *Let R be a reduced ring with α-compatible property. Then the skew polynomial ring $R[x,\alpha]$ is reduced.*

Remark 2.5. *Let R be a non-reduced ring. If R is symmetric, then $Z_N(R) = R$. In fact, if R is a non-reduced symmetric ring, then $\mathrm{Nil}(R)$ is an ideal of R and so r and x are adjacent for any $r \in R^*$ and $x \in \mathrm{Nil}^*(R)$. Therefore, $Z_N(R) = R$.*

Remark 2.6. *Let R be a NI ring. If there exists a nonzero nilpotent element of R, then all nonzero elements of R are adjacent to it. Thus, for any NI ring R whose $Z_N(R)$ contains at least one nonzero nilpotent element, then $\Gamma_N(R)$ is connected, $diam(\Gamma_N(R)) \leq 2$, and $gr(\Gamma_N(R)) = 3$ or ∞.*

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Theorem 2.7. *Let R be a ring. Then $\Gamma_N(R)$ is complete if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. ($\Leftarrow$) By definition.

($\Rightarrow$) Suppose that $\Gamma_N(R)$ is complete. We claim that R is a reduced ring. To prove this, suppose that $x \in \mathrm{Nil}(R)^*$, for some $x \in Z(R)^*$. Since $x \in \mathrm{Nil}(R)^*$, we deduced that there exists a positive integer $n \in \mathbb{N}$ such that $x^n = 0$ and $x^i \neq 0$, for all $1 \leq i \leq n-1$. It is clear that $(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1$. Thus $1 \neq 1-x \in \mathrm{U}(R)$ and so $|\mathrm{U}(R)| \geq 2$. Let $a, b \in \mathrm{U}(R)$, and let $x \in \mathrm{Nil}^*(R)$. Since $a.a^{-1}x$ and $b.b^{-1}x$ are contained in $\mathrm{Nil}(R)$, $a^{-1}x$ and $b^{-1}x$ are also nonzero, we conclude that $a, b \in Z_N^*(R)$, with $a^{-1}, b^{-1} \in \mathrm{U}(R)$. Also we have $ab \notin \mathrm{Nil}(R)$. But since $\Gamma_N(R)$ is complete, we reach a contradiction. This proves the claim. Thus $\Gamma_N(R) = \Gamma(R)$ and so by [1, Theorem 5], $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

We have also the following theorem, its proof is essentially same as that of [14, Theorem 2.2].

Theorem 2.8. *Let R be a reduced ring. If $Z(R)$ is not an ideal, then R has exactly two minimal primes if and only if diam$(\Gamma(R))$ is less than or equal to 2.*

Following [6], an ideal I of R is called *completely semiprime ideal* if $\frac{R}{I}$ is a reduced ring. Lam et al. [12, Lemma 12.6], proved that, if R is a reduced ring, then each minimal prime ideal of R is completely prime. Also each minimal prime ideal is a union of annihilators. Thus, if P is a minimal prime ideal of a reduced α-compatible ring R, then $\alpha(P) \subseteq P$, and so $P[x;\alpha]$ is an ideal of $R[x;\alpha]$, by [9, Theorem 3.1]. One can easily prove that $P[x;\alpha]$ is a minimal prime ideal of $R[x;\alpha]$.

Theorem 2.9. *Let R be a symmetric and α-compatible ring . If R has exactly two minimal primes, then diam$(\Gamma_N(R[x;\alpha])) = 2$.*

Proof. First suppose that R is a non-reduced ring. Lemma 2.3 and Remark 2.6 yield that *diam*$(\Gamma_N(R[x;\alpha])) \leq 2$. Also by Theorem 2.7, we conclude that *diam*$(\Gamma_N(R[x;\alpha])) \geq 2$ and so *diam*$(\Gamma_N(R[x;\alpha])) = 2$.
Next suppose that R is reduced and $\mathrm{Min}(R) = \{P_1, P_2\}$. Clearly, $\Gamma(R) = \Gamma_N(R), Z_N(R) = Z(R)$ and these primes are completely primes. Since R is reduced, $< 0 >$ is a completely semiprime ideal. Thus $Z(R)$ is the union of the completely prime ideals, by [6, Theorem 2.10]. This together with [6, Corollary 2.3] imply that $P \cup Q = Z(R)$. By using Corollary 2.4 and a similar argument as used in the proof of part (3) of [11, Theorem 2.7], we have $R[x,\alpha]$ is a reduced ring, and $P[x,\alpha]$ and $Q[x,\alpha]$ are two minimal primes of $R[x,\alpha]$. Then by [11, Theorem 2.7] $Z_N(R[x,\alpha])$ is not ideal of $R[x,\alpha]$, and $|P[x,\alpha]| \geq 3$ and $|Q[x,\alpha]| \geq 3$. Let $f(x), g(x)$ be distinct nonzero elements of $Q[x,\alpha]$. If $f(x)g(x) = 0$, then by reduced property of $R[x,\alpha]$ we obtain $f(x)R[x,\alpha]g(x) = 0$, and then $f(X) \in P[x,\alpha]$ or $g(X) \in P[x,\alpha]$, which is a contradiction. Thus, $d(f(x), g(x)) \geq 2$ and so *diam*$(\Gamma_N(R[x,\alpha])) =$ *diam*$(\Gamma(R[x,\alpha])) = 2$, by Theorem 2.8. $\square$

Theorem 2.10. *Let R be a symmetric α-compatible ring. Then $2 \leq$ diam$(\Gamma_N(R[x,\alpha])) \leq 3$.*

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Proof. First we show that $2 \leq diam(\Gamma_N(R[x,\alpha]))$. It is easy to check that

$$diam(\Gamma_N(R)) \leq diam(\Gamma_N(R[x,\alpha])).$$

So if $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then, by Theorem 2.7, $2 \leq diam(\Gamma_N(R))$ and so $2 \leq diam(\Gamma_N(R[x,\alpha]))$. Now, let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since R is reduced with exactly two minimal primes, by Theorem 2.9, $2 = diam(\Gamma_N(R[x,\alpha]))$.

Next we show that $diam(\Gamma_N(R[x,\alpha])) \leq 3$. To prove this, we consider two following cases:

Case 1: R is a non-reduced ring. By a similar argument as used in the proof of Theorem 2.9, we conclude that $diam(\Gamma_N(R[x,\alpha])) = 2 \leq 3$.

Case 2: R is a reduced ring. It follows from [16, Theorem 3.2]. □

Theorem 2.11. *Let R be a symmetric ring. Then the following hold:*

(1) *$\Gamma_N(R)$ is connected,*
(2) *$diam(\Gamma_N(R)) \leq 3$,*
(3) *If $\Gamma_N(R)$ contains a cycle, then $gr(\Gamma_N(R)) \leq 4$. Moreover, if R is non-reduced, then*

$$gr(\Gamma_N(R)) = 3.$$

Proof. If R is reduced, then $\Gamma_N(R)$ is actually the usual zero-divisor graph $\Gamma'(R)$. We have $\Gamma_N(R)$ is connected, $diam(\Gamma_N(R)) \leq 3$ and $gr(\Gamma_N(R)) \leq 4$ by [16, Theorem 3.2]. If R is non-reduced, then since R is symmetric, by Remark 2.6 and Lemma 2.3, we conclude that $\Gamma_N(R)$ is connected, $diam(\Gamma_N(R)) \leq 2$ and $gr(\Gamma_N(R)) = 3$ or ∞. □

Lemma 2.12. *Let R be a reduced and α-compatible ring. If $Z_N^*(R) \neq \varnothing$, then $gr(\Gamma_N(R[x,\alpha])) \leq 4$.*

Proof. Since R is reduced and $Z_N^*(R) \neq \varnothing$, there exist nonzero elements $a, b \in Z_N(R)$ such that $ab = 0$ and so by α-compatibility of R, $b\alpha(a) = a\alpha(b) = 0$. Now consider the cycle $a - bx - ax - b - a$ of length four in $\Gamma_N(R[x,\alpha])$. Therefore $gr(\Gamma_N(R[x,\alpha])) \leq 4$. □

Theorem 2.13. *Let R be a symmetric and α-compatible ring. Then $gr(\Gamma_N(R)) \geq gr(\Gamma_N(R[x.\alpha]))$. In addition, if $\Gamma_N(R)$ contains a cycle, then*

$$gr(\Gamma_N(R)) = gr(\Gamma(R[x,\alpha])).$$

Proof. If $Z_N^*(R) = \varnothing$, then $gr(\Gamma_N(R)) = \infty = gr(\Gamma_N(R[x,\alpha]))$. So we may assume that $Z_N^*(R) \neq \varnothing$. Since the graph $\Gamma_N(R)$ is a subgraph of $\Gamma_N(R[x,\alpha])$, we have $gr(\Gamma_N(R)) \geq gr(\Gamma_N(R[x,\alpha]))$. We consider the following two cases:

Case 1: R is a non-reduced ring. By Theorem 2.11 (3), $gr(\Gamma_N(R)) = 3$. Since $gr(\Gamma_N(R)) \geq gr(\Gamma_N(R[x,\alpha]))$, we conclude that $gr(\Gamma_N(R[x,\alpha])) = gr(\Gamma_N(R)) = 3$.

Case 2: R is a reduced ring. By Lemma 2.12, we have $gr(\Gamma_N(R[x,\alpha])) \leq 4$. Now suppose that $gr(\Gamma_N(R[x,\alpha])) = 4$. Since $\Gamma_N(R)$ contains a cycle, by Theorem 2.11 (3), $gr(\Gamma_N(R)) \leq 4$. This together with $gr(\Gamma_N(R)) \geq gr(\Gamma_N(R[x,\alpha])) = 4$ imply that $gr(\Gamma_N(R)) = 4$.

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So it suffices to show that if $gr(\Gamma_N(R[x,\alpha])) = 3$, then we have $gr(\Gamma_N(R)) = 3$. Suppose that

$$f(x) - g(x) - h(x) - f(x).$$

is a cycle of length 3 in $R[x,\alpha]$, where $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=o}^{m} b_j x^j$ and $h(x) = \sum_{k=o}^{l} c_k x^k$. Also, we may assume a_n, b_m and c_l are nonzero elements in R. From $f(x)g(x) = 0$ we have $a_n\alpha^n(b_m) = 0$, since it is the leading coefficient of $f(x)g(x)$. So by α-compatibility of R we have $a_n b_m = 0$. By a similar way, we have $b_m c_l = 0$ and $c_l a_n = 0$. Clearly, the coefficients a_n, b_m and c_l are distinct, since R is reduced. Now form the cycle $a_n - b_m - c_l - a_n$. This is a cycle of length 3 in $\Gamma_N(R)$.

□

Corollary 2.14. *Let R be a symmetric and α-compatible ring. If $gr(\Gamma_N(R)) = \infty$ and $Z_N^*(R) \neq \varnothing$, then either $|Nil(R)| = 2$ and $gr(\Gamma_N(R[x,\alpha])) = 3$ or R is reduced and $gr(\Gamma_N(R[x,\alpha])) = 4$.*

Proof. We consider two following cases:
Case (1): R is a non-reduced ring. By Lemma 2.2, $\mathrm{Nil}(R)$ is an ideal of R. We claim that $|\mathrm{Nil}(R)| = 2$. To see this, suppose to the contrary, $|\mathrm{Nil}(R)| \geq 3$. This imply that $|\mathrm{U}(R)| \geq 3$. Assume that $a, b \in \mathrm{Nil}^*(R)$ and $1 \neq c \in \mathrm{U}(R)$. By Remark 2.6, a and b are adjacent to all vertices contained $Z_N^*(R) = R^*$. Thus
$a - b - c - a$ forms a triangle and so $gr(\Gamma_N(R)) = 3$, a contradiction. Therefore $|\mathrm{Nil}(R)| = 2$. Now we show that $gr(\Gamma_N(R[x,\alpha])) = 3$. Suppose $a \in \mathrm{Nil}^*(R)$. Since $a^2 = 0$, $a - ax - ax^2 - a$ forms a triangle for nonzero element $a \in \mathrm{Nil}(R)$, and so the claim is proved.
Case (2): R is a reduced ring. By Lemma 2.12, we have $gr(\Gamma_N(R[x,\alpha])) \leq 4$. Now if $gr(\Gamma_N(R[x,\alpha])) = 3$, then by a similar way as used in Case (2) of Theorem 2.13, we can show that $gr(\Gamma_N(R)) = 3$, a contradiction, and so $gr(\Gamma_N(R[x,\alpha])) = 4$. □

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