Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 9 No. 1 (2020), pp. 31-39.
© 2020 University of Isfahan

www.combinatorics.ir

www.ui.ac.ir

BOUNDS FOR METRIC DIMENSION AND DEFENSIVE k-ALLIANCE OF GRAPHS UNDER DELETED LEXICOGRAPHIC PRODUCT

KINKAR CHANDRA DAS AND MOSTAFA TAVAKOLI*

Communicated by Bijan Taeri

Abstract. Metric dimension and defensive k-alliance number are two distance-based graph invariants which have applications in robot navigation, quantitative analysis of secondary RNA structures, national defense and fault-tolerant computing. In this paper, some bounds for metric dimension and defensive k-alliance of deleted lexicographic product of graphs are presented. We also show that the bounds are sharp.

1. Introduction

Throughout this paper all graphs considered are assumed to be finite, simple, undirected and connected. Suppose that G is such a graph. The **distance** $d_G(u, v)$ between two vertices u and v in G is defined as the length of a shortest path that connects vertices u and v, and the **diameter** of G is the largest shortest-path distance between any two vertices in G. For a nonempty set $A \subseteq V(G)$, and a vertex $v \in V(G)$, $N_A(v)$ denotes the set of neighbors of v in A, i.e., $N_A(v) = \{u \in A \,|\, uv \in E(G)\}$, and $\bar{A}$ denotes as usual the complement of A in $V(G)$, i.e., $\bar{A} = V(G) \backslash A$. The degree of a vertex v in G is denoted by $deg_G(v)$ and δ_G denotes the minimum degree of all vertices in G.

The **lexicographic product** of graphs is one of the oldest and most investigated graph operations that introduced by Felix Hausdorff in 1914 [6]. Feigenbaum and Schäffer [3] proved that the

MSC(2010): Primary: 05C12; Secondary: 05C76.
Keywords: Deleted lexicographic product, Metric dimension, Defensive k-alliance.
Received: 23 February 2019, Accepted: 29 October 2019.
∗Corresponding author

DOI: http://dx.doi.org/10.22108/toc.2019.115674.1622

.

complexity of testing whether an arbitrary graph can be written nontrivially as the composition of two smaller graphs is the same, to within polynomial factors, as the complexity of testing whether two graphs are isomorphic.

Frelih and Miklavič [4] proposed another lexicographic-like product that is called the **deleted lexicographic product**. To define, we assume that G and H are graphs and $|V(H)| = n$. The deleted lexicographic product $G[H] - nG$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$ and $u = (g, h)$ is adjacent with $v = (g', h')$ whenever ($g = g'$ and h is adjacent with h') or ($h \neq h'$ and g is adjacent with g'), see Figure 1. One can use this product to form some known graphs like ladder and octahedron graphs.

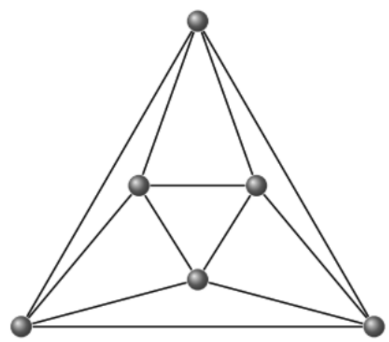

FIGURE 1. The deleted lexicographic product of C_3 and P_2.

A **graph invariant** is a number which is invariant under graph isomorphisms. However, they are not usually preserved under graph homomorphisms.

Let $G = (V(G), E(G))$ be a simple connected graph of order n. Given a set of vertices $S = \{v_1, v_2, \ldots, v_m\}$ of G, the **k-metric representation** of a vertex $v \in V(G)$ with respect to S is the vector $r_k(v|S) = (d_1, d_2, \ldots, d_m)$, where $d_i = \min\{d_G(v, v_i), k\}$, $i \in \{1, 2, \ldots, m\}$. We say that S is a **k-resolving set** for G if for every pair of distinct vertices $u, v \in V$, $r_k(u|S) \neq r_k(v|S)$. A k-resolving set S for G with minimum cardinality is called a **k-basis** of G, and its cardinality is the **k-metric dimension** of G, denoted by $\dim_k(G)$. If k is greater than or equal to diameter of G, then S is a called a **resolving set**, $r_k(v|S)$ is a called a **metric representation** and denoted by $r(v|S)$, and $\dim_k(G)$ is called the **metric dimension** and denoted by $\dim(G)$. The metric dimension was first introduced by Slater in 1975 [14]. He introduced the metric representation of a vertex $v \in V(G)$ with respect to S as the vector $r(v|S) = \Big(d_G(v, v_1), d_G(v, v_2), \ldots, d_G(v, v_k)\Big)$ and defined concepts of resolving set and metric dimension based on this vector, see [2, 11] for more information.

About twenty years later, Khuller et al. [8] have considered the application of the metric dimension of a connected graph in robot navigation. After that, many studies were done on this invariant. Since, the problem of finding a resolving set is an NP-hard problem, then mathematicians tend to use graph products to obtain a resolving set of big graphs in terms of invariants of their factors. For example, this matter was studied over the Cartesian product of graphs in [1], over the lexicographic product

DOI: http://dx.doi.org/10.22108/toc.2019.115674.1622

of graphs in [12] and over the hierarchical product of graphs in [15]. In this paper, we investigate these sets over deleted lexicographic product of graphs.

A set D_G of vertices in a graph G is a **dominating set** if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The **domination number** $\gamma(G)$ is the number of vertices in smallest dominating set of G. The dominating sets in graphs are natural models for facility location problems in operational research [5, 16, 10].

A nonempty set $A \subseteq V(G)$ is a **defensive k-alliance** in G, $k \in \{-\delta_G, \ldots, \delta_G\}$, if for every $v \in A$, $|N_A(v)| \geq |N_{\bar{A}}(v)| + k$. Following Kristiansen et al. [9], defensive (-1)-alliance and defensive 0-alliance are called **defensive alliance** and **strong defensive alliance**, respectively. The **defensive k-alliance number** of G, denoted by a $a_k(G)$, is defined as the minimum cardinality of a defensive k-alliance in G. If G does not contain defensive k-alliance set, then we set $a_k(G) = \infty$. One can use references [7, 9, 13] to study applications of alliances in quantitative analysis of secondary RNA structures, national defense and fault-tolerant computing. The mathematical properties of alliances in graphs were first studied by Kristiansen, Hedetniemi and Hedetniemi [9]. Also, defensive alliances of Cartesian product was investigated in [17]. In this paper, we study alliances in deleted lexicographic product of graphs. One of the advantages of our result is that defensive alliances of big graphs can be investigated using the small graphs forming them.

As usual, P_n, C_n, and K_n denote, respectively, the path, the cycle, and the complete graph on n vertices, and the ladder graph L_n on $2n$ vertices. Our other notations are standard and taken mainly from the standard books of graph theory.

2. Main Results

In this section, our main results are presented. We start by a simple lemma that will be used later.

Lemma 2.1. *Let G and H be two graphs with $|V(G)| \geq 2$, $|V(H)| = n \geq 2$ and $W = G[H] - nG$. Then*

(1) $deg_W\Big((g,h)\Big) = deg_H(h) + \Big(|V(H)| - 1\Big)deg_G(g)$,
(2) *If $H = \bar{K}_2$, then W is connected if and only if G is connected and nonbipartite.*
(3) *If H has at least three vertices, then*

$$d_W\Big((g,h),(g',h')\Big) = \begin{cases} 1 & \text{if } \Big(g = g', hh' \in E(H)\Big) \text{ or } \Big(gg' \in E(G), h \neq h'\Big), \\ 2 & \text{if } \Big(g = g', hh' \notin E(H)\Big) \text{ or } \Big(gg' \in E(G), h = h'\Big), \\ d_G(g,g') & \text{if } \Big(gg' \notin E(G) \text{ and } g \neq g'\Big). \end{cases}$$

DOI: http://dx.doi.org/10.22108/toc.2019.115674.1622

(4) *If H has exactly two vertices, then*

$$d_W((g,h),(g',h')) = \begin{cases} 1 & \text{if } \Big(g=g', hh' \in E(H)\Big) \text{ or } \Big(gg' \in E(G), h \neq h'\Big), \\ d_G(g,g') & \text{if } \Big(g \neq g', h = h', 2 \mid d_G(g,g')\Big) \\ & \text{or } \Big(g \neq g', h \neq h', 2 \nmid d_G(g,g')\Big), \\ d_G(g,g')+1 & \text{if } \Big(g \neq g', h \neq h', 2 \mid d_G(g,g')\Big) \\ & \text{or } \Big(g \neq g', h = h', 2 \nmid d_G(g,g')\Big). \end{cases}$$

Proof. The Part (1) is a direct consequence of definition and Part (2) is a result in [4]. To prove (3), we assume that (g,h) and (g',h') are two vertices in W. If $\Big(g = g', hh' \in E(H)\Big)$ or $\Big(gg' \in E(G), h \neq h'\Big)$, then by definition of the deleted lexicographic product, (g,h) and (g',h') are adjacent. If $g = g'$, $hh' \notin E(H)$, then $(g,h)(g'',h'')(g',h')$ is a path of length 2, where $gg'' \in E(G)$ and $h \neq h'' \neq h'$. Again by definition of the deleted lexicographic product, there is not a path of length 1 between (g,h) and (g',h'); therefore $d_W\Big((g,h),(g',h')\Big) = 2$. Similarly, if $gg' \in E(G), h = h'$, then $d_W\Big((g,h),(g',h')\Big) = 2$. Now suppose $gg' \notin E(G)$, $g \neq g'$ and f is a mapping from W to G such that $f\Big((g,h)\Big) = g$. The fact that if $(g,h)(g',h') \in E(W)$ then $\Big(gg' \in E(G)$ and $h \neq h'\Big)$ or $\Big(g = g'$ and $hh' \in E(H)\Big)$, implies that f is a weak homomorphism and so $d_W\Big((g,h),(g',h')\Big) \geq d_G(g,g')$. Thus it is enough to prove $d_W\Big((g,h),(g',h')\Big) \leq d_G(g,g')$. To do this, let $gv_1v_2\cdots v_tg'$ be a shortest (g,g')-path in G. Then $(g,h)(v_1,h'')(v_2,h)\cdots(v_t,h'')(g',h')$ is a path of length $d_G(g,g')$ in W, where $h \neq h'' \neq h'$; therefore $d_W\Big((g,h),(g',h')\Big) \leq d_G(g,g')$.
By a similar argument a Part (3), one can prove the Part (4). □

Recall that a subset X of vertices of a graph G is called an independent set for G if no two distinct vertices of X are adjacent. The size of a largest independent set is called the **independence number** of G and denoted by $\alpha(G)$.

Theorem 2.2. *Let G be a graph with $m > 2$ vertices, H be a graph with n vertices, S be a $2-$basis of H and $W = G[H] - nG$. Then*

$$\dim(W) \leq \begin{cases} \Big(2\gamma(G) - m\Big)\dim_2(H) + \Big(n + |Y| + |Y'|\Big)\Big(m - \gamma(G)\Big) + (\sigma_1 + \sigma_2)\gamma(G) \\ \text{if } \dim_2(H) \geq \frac{n}{2}, \\ m\dim_2(H) + \sigma_1\Big(m - \alpha(G)\Big) + \sigma_2 m \text{ otherwise,} \end{cases}$$

where $\sigma_1 = 1$ if there exists $h \in V(H)$ such that $r_2(h|S) = (1, 1, \ldots, 1)$, and $\sigma_1 = 0$ otherwise; $\sigma_2 = 1$ if there exists $h \in V(H)$ such that $r_2(h|S) = (2, 2, \ldots, 2)$, and $\sigma_2 = 0$ otherwise; and

$$Y = \Big\{h \in V(H) \mid r_2(h|S) = (2,2,\ldots,2) \text{ or } \Big(h \in S \text{ and } r_2\Big(h|(V(H)\setminus S)\Big) = (2,2,\ldots,2)\Big)\Big\},$$

$$Y' = \Big\{h \in V(H) \mid r_2(h|S) = (1,1,\ldots,1) \text{ or } \Big(h \in S \text{ and } r_2\Big(h|(V(H)\setminus S)\Big) = (1,1,\ldots,1)\Big)\Big\}.$$

Proof. We first assume that $\dim_2(H) \geq \frac{n}{2}$. Then we should provide a resolving set for W of size $\Big(2\gamma(G) - m\Big)\dim_2(H) + \Big(n + |Y| + |Y'|\Big)\Big(m - \gamma(G)\Big) + \Big(\sigma_1 + \sigma_2\Big)\gamma(G)$. To do this, let

$$\begin{aligned} S' &= \Big\{(g,\, h) \ \mid \ g \in D_G,\, h \in S \text{ or } r_2(h|S) = (2,\, 2, \ldots,\, 2) \text{ or } r_2(h|S) = (1,\, 1, \ldots,\, 1)\Big\} \\ &\quad \cup \Big\{(g, h) \ \mid \ g \in V(G) \setminus D_G, h \in Y \cup Y' \cup (V(H) \setminus S)\Big\}, \end{aligned}$$

where D_G is a smallest dominating set for G, and S is a 2-basis of H.

We now prove that S' is a resolving set for W. Take two distinct vertices $(g_1,\, h_1)$ and $(g_2,\, h_2)$ in $V(W) \setminus S'$, because $(g,\, h) \in S'$ is the unique vertex of W for which $d_W\Big((g,\, h),\, (g,\, h)\Big) = 0$. We now consider the following three cases:

Case 1.: $d_G(g_1,\, g_2) \geq 2$. If $g_1 \in D_G$ $\Big(g_1 \in V(G) \setminus D_G\Big)$, then there exists $h \in S$ $\Big(h \in Y \cup Y' \cup (V(H) \setminus S)\Big)$ such that $hh_1 \in E(H)$. Thus $d_W\Big((g_1,\, h_1), (g_1,\, h)\Big) = 1 < 2 \leq d_W\Big((g_2,\, h_2), (g_1,\, h)\Big)$ and so $r\Big((g_1,\, h_1)|S'\Big) \neq r\Big((g_2,\, h_2)|S'\Big)$.

Case 2.: $d_G(g_1,\, g_2) = 1$. By the structure of S', it is clear that $r\Big((g_1,\, h_1)|S'\Big) \neq r\Big((g_2,\, h_2)|S'\Big)$ when $g_1,\, g_2 \in D_G$ or $g_1,\, g_2 \in V(G) \setminus D_G$. So, suppose that $g_1 \in D_G$ and $g_2 \in V(G) \setminus D_G$. Since $m > 2$, then there exists $g \in V(G)$ such that $gg_1 \in E(G)$ or $gg_2 \in E(G)$. If g is adjacent to either g_1 or g_2, but not both, then it is trivial that $r\Big((g_1,\, h_1)|S'\Big) \neq r\Big((g_2,\, h_2)|S'\Big)$. Otherwise, g is adjacent to both vertices g_1 and g_2. Therefore, $d_W\Big((g_1,\, h_1), (g,\, h_2)\Big) = 1 < 2 = d_W\Big((g_2,\, h_2), (g,\, h_2)\Big)$ when $g \in D_G$, and $d_W\Big((g_1,\, h_1), (g,\, h_1)\Big) = 2 > 1 = d_W\Big((g_2,\, h_2), (g,\, h_1)\Big)$ when $g \in V(G) \setminus D_G$.

Case 3.: $g_1 = g_2$. If $g_1 \in D_G$, then it is clear that $r\Big((g_1,\, h_1)|S'\Big) \neq r\Big((g_2,\, h_2)|S'\Big)$ because $r_2(h_1|S) \neq r_2(h_2|S)$, where $r_2(h_1|S) \neq (1,\, 1, \ldots,\, 1) \neq r_2(h_2|S)$. Otherwise, $g_1 \in V(G) \setminus D_G$. Then $h_1,\, h_2 \in S \setminus (Y \cup Y')$. So, $d_W\Big((g_1,\, h_1), (g,\, h_1)\Big) = 2 \neq 1 = d_W\Big((g_2,\, h_2), (g,\, h_1)\Big)$ where $g \in D_G$ and $gg_1 \in E(G)$. Consequently, $r\Big((g_1,\, h_1)|S'\Big) \neq r\Big((g_2,\, h_2)|S'\Big)$.

Therefore, we have $r\Big((g_1,\, h_1)|S'\Big) \neq r\Big((g_2,\, h_2)|S'\Big)$ in each case, and so S' is a resolving set for W. Next we suppose that $\dim_2(H) < \frac{n}{2}$ and provide a resolving set for W of size

$$m \dim_2(H) + \sigma_1\Big(m - \alpha(G)\Big) + \sigma_2 m.$$

To do this, we set

$$\begin{aligned} S'' &= \Big\{(g,\, h) \ \mid \ g \in V(G),\, h \in S \text{ or } r_2(h|S) = (2,\, 2, \ldots,\, 2)\Big\} \\ &\quad \cup \Big\{(g,\, h) \ \mid \ g \in V(G) \setminus X,\, r_2(h|S) = (1,\, 1, \ldots,\, 1)\Big\}, \end{aligned}$$

where S is a 2-basis of H and X is an independent set of largest size for G. A repeated argument for S'' implies that S'' is a resolving set for W, which completes our proof. □

DOI: http://dx.doi.org/10.22108/toc.2019.115674.1622

Theorem 2.3. *Let G be a bipartite graph with $m > 2$ vertices, H be a graph with n vertices, S be a 2-basis of H and $W = G[H] - nG$. Then*

$$\dim(W) \leq \begin{cases} \Big(2\gamma(G) - m\Big)\dim_2(H) + \Big(n + |Y|\Big)\Big(m - \gamma(G)\Big) + \sigma\gamma(G) & \text{if } \dim_2(H) \geq \frac{n}{2}, \\ m\dim_2(H) + m\sigma & \text{otherwise,} \end{cases}$$

where $\sigma = 1$ if there exists $h \in V(H)$ such that $r_2(h|S) = (2, 2, \ldots, 2)$ and $\sigma = 0$ otherwise, and $Y = \Big\{h \in V(H) \mid r_2(h|S) = (2, 2, \ldots, 2)$ or $\Big(h \in S$ and $r_2\Big(h|(V(H) \setminus S)\Big) = (2, 2, \ldots, 2)\Big)\Big\}$.

Proof. We first assume that $\dim_2(H) \geq \frac{n}{2}$. Let

$$\begin{aligned} S' \;=\; & \Big\{(g, h) \;\mid\; g \in D_G, h \in S \text{ or } r_2(h|S) = (2, 2, \ldots, 2)\Big\} \cup \Big\{(g, h) \;\mid\; g \in V(G) \setminus D_G, \\ & h \in Y \cup (V(H) \setminus S)\Big\}, \end{aligned}$$

where D_G is a smallest dominating set for G, and S is a 2-basis of H.

We prove that S' is a resolving set for W. Take two distinct vertices (g_1, h_1) and (g_2, h_2) in $V(W) \setminus S'$. We now consider the following three cases:

(1) $d_G(g_1, g_2) \geq 2$. This case can be proved by a similar argument as Case 1 in the proof of Theorem 2.2.

(2) $d_G(g_1, g_2) = 1$. Since G is a bipartite graph with more than two vertices, there exist $g \in V(G)$ such that g is adjacent to either g_1 or g_2, but not both. Without loss of generality, we can assume that $gg_1 \in E(G)$. Let $g \in D_G$ $(g \in V(G) \setminus D_G)$. Then $d_W\Big((g_1, h_1), (g, h)\Big) = 1 < 2 \leq d_W\Big((g_2, h_2), (g, h)\Big)$, where $h \in V(H) \setminus S$ and $r_2(h|S) \neq (2, 2, \ldots, 2)$ $\Big(h \in S$ or $r_2(h|S) = (2, 2, \ldots, 2)\Big)$. Therefore, $r\Big((g_1, h_1)|S'\Big) \neq r\Big((g_2, h_2)|S'\Big)$.

(3) $g_1 = g_2$. This case can be proved by a similar argument as Case 3 in the proof of Theorem 2.2.

Therefore, $r\Big((g_1, h_1)|S'\Big) \neq r\Big((g_2, h_2)|S'\Big)$ in each case and so S' is a resolving set for W. Now suppose that $\dim_2(H) < \frac{n}{2}$ and

$$S'' = \Big\{(g, h) \;\mid\; g \in V(G), h \in S \text{ or } r_2(h|S) = (2, 2, \ldots, 2)\Big\},$$

where S is a 2-basis of H. Repeating the above argument for S'' implies that S'' is a resolving set for W, which completes our proof. □

DOI: http://dx.doi.org/10.22108/toc.2019.115674.1622

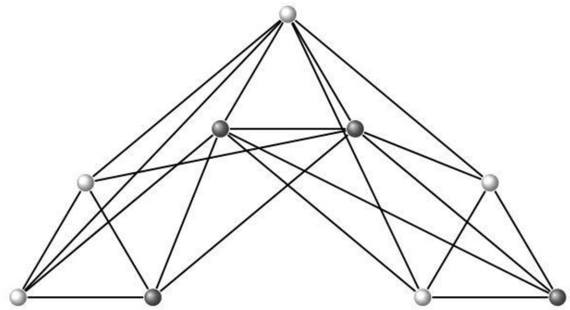

FIGURE 2. The black vertices are elements of S' in $P_3[K_3] - 3P_3$.

It is not difficult to check that the inequality of Theorem 2.3 is sharp. To see this, we assume that $G \cong P_m$ and $H \cong K_n$. By inserting $\gamma(P_m) = \lfloor \frac{m+2}{3} \rfloor$ and $\dim_2(K_n) = n-1$, $|Y| = 0$ and $\sigma = 0$ in Theorem 2.3, we have

$$\begin{aligned}\dim\Big(P_m[K_n] - nP_m\Big) &\leq \Big(2\gamma(P_m) - m\Big)\dim_2(K_n) + \Big(n + |Y|\Big)\Big(m - \gamma(P_m)\Big) + \sigma\gamma(P_m)\\ &= \Big(2\left\lfloor \frac{m+2}{3} \right\rfloor - m\Big)(n-1) + (n+0)\Big(m - \left\lfloor \frac{m+2}{3} \right\rfloor\Big) + 0\\ &= (n-2)\left\lfloor \frac{m+2}{3} \right\rfloor + m.\end{aligned}$$

Thus $\dim(P_3[K_3] - 3P_3) \leq 4$. The black vertices in Figure 2 show the elements of S' in $P_3[K_3] - 3P_3$. On the other hand, the exact value of $\dim(P_3[K_3] - 3P_3)$ is equal to 4 which shows that the inequality of Theorem 2.3 is sharp.

Theorem 2.4. *Let G and H be two graphs with $|V(G)| > 1$ and $|V(H)| = n > 1$. Then*

$$a_k\Big(G[H] - nG\Big) \leq n\, a_p(G),$$

where $p = \left\lceil \frac{k-\delta_H}{n-1} \right\rceil$.

Proof. Let $W = G[H] - nG$. If G does not contain defensive p-alliance set, where $p = \left\lceil \frac{k-\delta_H}{n-1} \right\rceil$, then by the definition, we have $a_p(G) = \infty$. Hence $a_k\Big(W\Big) \leq \infty = n\, a_p(G)$. Otherwise, let A be a smallest defensive p-alliance set in G, where $p = \left\lceil \frac{k-\delta_H}{n-1} \right\rceil$. Then $a_p(G) = |A| > 0$. By the definition of defensive p-alliance set, we have $\Big|N_A^G(g_i)\Big| \geq \Big|N_{\bar{A}}^G(g_i)\Big| + p$ for any vertex $g_i \in V(G)$. Thus we have

$$\Big|N_A^G(g_i)\Big| - \Big|N_{\bar{A}}^G(g_i)\Big| \geq p = \left\lceil \frac{k-\delta_H}{n-1} \right\rceil. \tag{2.1}$$

Set

$$A' = A \times V(H).$$

DOI: http://dx.doi.org/10.22108/toc.2019.115674.1622

We now prove that A' is a defensive alliance set in W. To do this, we suppose that $(g_i,\, h_j)$ is a vertex in W, where $g_i \in A$ and $h_j \in V(H)$. Then

$$\left|N^{W}_{A'}(g_i,\, h_j)\right| = \left|N^{G}_{A}(g_i)\right|(n-1) + deg_H(h_j)$$

$$\text{and} \qquad \left|N^{W}_{\bar{A}'}(g_i,\, h_j)\right| = \left|N^{G}_{\bar{A}}(g_i)\right|(n-1).$$

From these results with (2.1), we have

$$\begin{aligned}\left|N^{W}_{A'}(g_i,\, h_j)\right| - \left|N^{W}_{\bar{A}'}(g_i,\, h_j)\right| &= (n-1)\Big(\left|N^{G}_{A}(g_i)\right| - \left|N^{G}_{\bar{A}}(g_i)\right|\Big) + deg_H(h_j) \\ &\geq (n-1)\left\lceil \frac{k-\delta_H}{n-1} \right\rceil + \delta_H \geq k.\end{aligned}$$

Therefore, A' is a defensive k-alliance set in W. Thus we have

$$a_k\Big(W\Big) \leq |A'| = |A|\,|V(H)| = n\,a_p(G).$$

This completes the proof of the theorem. □

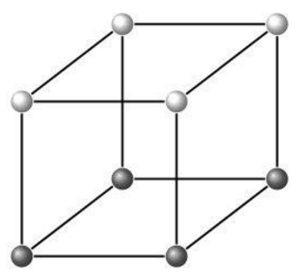

FIGURE 3. The 3-cube Q_3 with a strong defensive alliance.

The n-cube Q_n $(n \geq 1)$ is the graph whose vertex set is the set of all $n-$tuples of 0s and 1s, where two $n-$tuples are adjacent if they differ in precisely one coordinate. Consider Q_3 that is shown in Figure 3. This graph is isomorphic to $C_4[P_2] - 2C_4$. It is not so difficult to check that $a_0(Q_3) = 4$. On the other hand, by Theorem 2.4, $a_0(Q_3) = a_0(C_4[P_2] - 2C_4) \leq 4$. It shows that the inequality of Theorem 2.4 is sharp. In Figure 3, the elements of a strong defensive alliance in $C_4[P_2] - 2C_4$ are colored black.

Acknowledgments

The first author was supported by the National Research Foundation of the Korean government with grant No. 2017R1D1A1B03028642.

References

[1] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara and D. R. Wood, On the metric dimension of cartesian products of graphs, *SIAM J. Discrete Math.* **21** (2007) 423–441.

[2] A. Estrada-Moreno, C. García-Gómez, Y. Ramírez-Cruz and J. A. Rodríguez-Velázquez, The Simultaneous Strong Metric Dimension of Graph Families, *Bull. Malays. Math. Sci. Soc.*, **39** (2016) 175–192.

[3] J. Feigenbaum and A. A. Schäffer, Recognizing composite graphs is equivalent to testing graph isomorphism, *SIAM J. Comput.* **15** (1986) 619–627.

[4] B. Frelih and Š. Miklavič, Edge regular graph products, *Electron. J. Combin.* **20** (2013) pp. 62.

[5] C. E. Go, S. R. Canoy, Jr., Domination in the corona and join of graphs, *Int. Math. Forum*, **6** (2011) 763–771.

[6] F. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914.

[7] T. W. Haynes, D. Knisley, E. Seier and Y. Zou, A quantitative analysis of secondary RNA structure using domination based parameters on trees, *BMC Bioinformatics*, **7** (2006) Pages/record No. 108.

[8] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.*, **70** (1996) 217–229.

[9] P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi, Alliances in graphs, *J. Combin. Math. Combin. Comput.*, **48** (2004) 157–177.

[10] J. Quadras and S. M. M. Albert, Domination Parameters in Coronene Torus Network, *Math. Comput. Sci.*, **9** (2015) 169–175.

[11] J. A. Rodríguez-Velázquez, C. G. Gómez, G. A. Barragán-Ramírez, On the Local Metric Dimension of Corona Product, *Bull. Malays. Math. Sci. Soc.*, **39** (2016) 157–173.

[12] S. W. Saputro, R. Simanjuntak, S. Uttunggadewa, H. Assiyatun, E. T. Baskoro, A. N. M. Salman and M. Bača, The metric dimension of the lexicographic product of graphs, *Discrete Math.*, **313** (2013) 1045–1051.

[13] K. H. Shafique and R. D. Dutton, On satisfactory partitioning of graphs, *Congr. Numer.*, **154** (2002) 183–194.

[14] P. J. Slater, Leaves of trees, *Congr. Numer.*, **14** (1975) 549–559.

[15] M. Tavakoli, F. Rahbarnia nd A. R. Ashrafi, Distribution of some graph invariants over hierarchical product of graphs, *Appl. Math. Comput.*, **220** (2013) 405–413.

[16] I. G. Yero, D. Kuziak and A. R. Aguilar, Coloring, location and domination of corona graphs, *Aequat. Math.* **86** (2013) 1–21.

[17] I. G. Yero, J. A. Rodrı́guez-Velázquez, Boundary defensive k-alliances in graphs, *Discrete Appl. Math.*, **158** (2010)12051211.

Kinkar Chandra Das

Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea

`Email: kinkardas2003gmail.com`

Mostafa Tavakoli

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P.O.Box 1159, Mashhad 91775, Iran

`Email: m_tavakolium.ac.ir`

DOI: http://dx.doi.org/10.22108/toc.2019.115674.1622