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MINIMAL EMBEDDINGS OF SMALL FINITE GROUPS

ROBERT HEFFERNAN AND BRENDAN MCCANN*

Abstract. We determine the groups of minimal order in which all groups of order n can embedded for $1 \leq n \leq 15$. We further determine the minimal order of a group in which all groups of order n or less can be embedded, also for $1 \leq n \leq 15$.

1. Introduction

The question as to when a given group, or collection of groups, can be embedded in another group goes back to Cayley's Theorem [1], which states that any group of order n can be expressed as a permutation group on n symbols. Equivalently, all groups of order n (or less) can be embedded in S_n, the symmetric group of degree n. This raises two questions, namely when is $|S_n|$ minimal with respect to the embedding of all groups of order n; and when is $|S_n|$ minimal with respect to the embedding of all groups of order n or less? To give an example, both groups of order 4 can be embedded in D_4, where $D_n = \langle a, b \mid a^n = 1 = b^2, bab = a^{-1} \rangle$ denotes the dihedral group of order $2n$. Thus S_4 is not a group of minimal order in which all groups of order 4 can be embedded. However, it is easy to see that $|S_4|$ *is* minimal with respect to the embedding of all groups of order 4 or less. Here we investigate, for $1 \leq n \leq 15$, both the minimal order of a group in which all groups of order n can be embedded; and the minimal order of a group in which all groups of order n or less can be embedded. In Sections 2–5 we

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*Corresponding author.
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determine the groups of smallest order in which all groups of order n can be embedded for $1 \le n \le 15$. For the prime p, we also find the minimal order of a group in which all groups of order p^3 can be embedded. In Sections 6–8 we find groups of minimal order in which all groups of order n or less can be embedded, also for $1 \le n \le 15$. We show that the latter groups are not unique for $3 \le n \le 15$. For $6 \le n \le 15$, we further show that the orders of the respective minimal groups are less than $|S_n|$.

We note that if the p-group P has a cyclic subgroup P_1 of order p^{k_1} and an elementary abelian subgroup P_2 of rank k_2, then $|P_1 \cap P_2| \le p$, so $|P| \ge |P_1P_2| = \dfrac{|P_1||P_2|}{|P_1 \cap P_2|} \ge p^{k_1+k_2-1}$. Thus if all groups of order p^k can be embedded in the group G, then the Sylow p-subgroups of G will have order at least p^{2k-1}. This yields the following lower bounds:

Lemma 1.1. *Let p be a prime and let G be a finite group in which all groups of order p^k can be embedded. Then $|G|$ is a multiple of p^{2k-1}.*

Lemma 1.2. *Let $\{p_1, \dots, p_m\}$ denote the set of distinct primes that are less than or equal to n. For $i = 1, \dots, m$ let k_i be maximal such that $p_i^{k_i} \le n$. If G is a finite group in which all groups of order n or less can be embedded, then $|G|$ is a multiple of $p_1^{2k_1-1} \cdots p_m^{2k_m-1}$.*

For example, Lemma 1.1 shows that if G is a group of minimal order in which all groups of order 8 can be embedded, then $|G|$ is a multiple of 32; while Lemma 1.2 shows that if all groups of order 12 or less can be embedded in G, then $|G|$ is a multiple of $2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 332\,640$. In Section 2 we show that the former bound is attained, whereas Sections 6 and 7 show that the latter is not. In general, the above bounds are attainable for relatively few values of n. Indeed, Lemma 5.1 below shows that when p is an odd prime and $k \ge 3$, the lower bound given by Lemma 1.1 cannot be attained. The same result shows that the bound given by Lemma 1.2 will always be exceeded for $n \ge 27$.

Our notation is generally standard and all groups considered are finite. For clarity, we note that Q_n will denote the dicyclic group of order $4n$ $(n > 1)$, given by $\langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$. Thus Q_2 is the quaternion group of order 8 and Q_3 is the non-trivial extension of C_3 by C_4. In addition, for the prime p and the group G, the expression $p^k \top |G|$ means that p^k is the largest power of p that divides $|G|$. We note that some exploratory work was done with the aid of the computational group theory system GAP [3].

2. Embeddings of groups of order 8

In this section we show that G is a group of minimal order in which all groups of order 8 can be embedded if and only if G is isomorphic either to the semi-direct product of a cyclic group of order 8 by its automorphism group; or to the direct product of a quasi-dihedral group of order 16 and a cyclic group of order 2. We first describe certain groups in which not all groups of order 8 can be embedded.

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Lemma 2.1. *Let G be a group such that $2^5 \top |G|$ and suppose that G has an abelian subgroup H such that $|H| = 16$ and $\exp(H) \leq 4$. Then not all groups of order 8 can be embedded in G.*

Proof. By Sylow's Theorems we can assume that $|G| = 32$. Suppose that C_8 can be embedded in G and let $U \leqslant G$ be such that $U = \langle x \rangle \cong C_8$. Now $|G : H| = 2$ and H has exponent at most 4, so $G = HU$ and $H \cap U = \langle x^2 \rangle \cong C_4$. Since H and U are abelian, we have $\langle x^2 \rangle \leqslant Z(G)$. Suppose further that $C_2 \times C_2 \times C_2$ can be embedded in G and let $V \leqslant G$ be such that $V = \langle v_1, v_2, v_3 \rangle \cong C_2 \times C_2 \times C_2$. Since $\langle x^2 \rangle \leqslant Z(G)$, $V\langle x^2 \rangle$ is abelian. If $V \cap \langle x^2 \rangle = 1$, then $|V\langle x^2 \rangle| = 32$, so G is abelian and the non-abelian groups of order 8 cannot be embedded in G. Hence we may suppose that $|V \cap \langle x^2 \rangle| = 2$. Without loss of generality, we let $V \cap \langle x^2 \rangle = \langle x^4 \rangle = \langle v_3 \rangle$. Then $V\langle x^2 \rangle = \langle v_1 \rangle \times \langle v_2 \rangle \times \langle x^2 \rangle \cong C_2 \times C_2 \times C_4$. We have $|G : V\langle x^2 \rangle| = 2$, so $V\langle x^2 \rangle \trianglelefteq G$. But $V = \Omega_1(V\langle x^2 \rangle)$, so $V \trianglelefteq G$. Now, if Q_2 can be embedded in G, then we let $Q \leqslant G$ be such that $Q \cong Q_2$. By comparison of orders, we have $Q \cap V = Z(Q) \cong C_2$ and $G = QV$. But then $G/V \cong Q/Z(Q) \cong C_2 \times C_2$, so G has exponent 4, which is a contradiction. We conclude that not all groups of order 8 can be embedded in G. □

Example 2.2. We define two groups of order 32 in which all groups of order 8 can be embedded. We note that the groups of order 8 are: $C_2 \times C_2 \times C_2, C_2 \times C_4, C_8, D_4$ and Q_2. We first let $\langle x_1 \rangle \cong C_2$ and let $H_1 = \langle x_2, y \mid x_2^2 = y^8 = 1, x_2 y x_2 = y^3 \rangle$ be the quasi-dihedral (or semi-dihedral) group of order 16. We form the direct product $\langle x_1 \rangle \times H_1 \cong C_2 \times H_1$. Thus $|\langle x_1 \rangle \times H_1| = 32$. We have $\langle x_1, x_2, y^4 \rangle \cong C_2 \times C_2 \times C_2$; $\langle x_1, y^2 \rangle \cong C_2 \times C_4$; and $\langle y \rangle \cong C_8$. We further have $(y^2)^{x_2} = (y^{x_2})^2 = y^{-2}$. Thus $\langle x_2, y^2 \rangle \cong D_4$. In addition, $(x_2 y)^2 = (x_2 y x_2 y) = y^{x_2} y = y^3 y = y^4$ and $(y^2)^{x_2 y} = (y^{-2})^y = y^{-2}$. Hence $\langle y^2, x_2 y \rangle$ is non-abelian of order 8 and is generated by elements of order 4, so $\langle y^2, x_2 y \rangle \cong Q_2$. Thus all groups of order 8 can be embedded in $C_2 \times H_1$. For our second example, we let $H_2 = \langle x_1, x_2, y \mid x_1^2 = x_2^2 = y^8 = 1, [x_1, x_2] = 1, x_1 y x_1 = y^5, x_2 y x_2 = y^3 \rangle$. Thus H_2 is isomorphic to the holomorph of C_8, that is, the semi-direct product of $\langle y \rangle \cong C_8$ by $\mathrm{Aut}(C_8)$ (identified with $\langle x_1, x_2 \rangle$). As above, we see that $\langle x_1, x_2, y^4 \rangle \cong C_2 \times C_2 \times C_2$; $\langle x_1, y^2 \rangle \cong C_2 \times C_4$; $\langle y \rangle \cong C_8$; $\langle x_2, y^2 \rangle \cong D_4$; and $\langle y^2, x_2 y \rangle \cong Q_2$. Thus $C_2 \times H_1$ and H_2 are non-isomorphic groups of order 32 in which all groups of order 8 can be embedded.

By Lemma 1.1, $C_2 \times H_1$ and H_2 are groups of minimal order in which all groups of order 8 can be embedded. Next we show that, up to isomorphism, they are the only such groups.

Theorem 2.3. *Let G be a group of order 32 in which all groups of order 8 can be embedded. Then G is isomorphic either to $C_2 \times H_1$ or H_2, as in Example 2.2.*

Proof. We let G be a group of order 32 in which all groups of order 8 can be embedded. Then G has a subgroup $\langle y \rangle$ such that $\langle y \rangle \cong C_8$. In addition, G has a subgroup $\langle x_1, x_2, x_3 \rangle$ with $\langle x_1, x_2, x_3 \rangle \cong C_2 \times C_2 \times C_2$. Since $|G| = 32$, we have $\langle x_1, x_2, x_3 \rangle \cap \langle y \rangle = \langle y^4 \rangle \cong C_2$. Without loss of generality, we let $y^4 = x_3$ and see that G is the product $G = \langle x_1, x_2 \rangle \langle y \rangle$. Since G is a 2−group, $\langle y \rangle$ is a proper subgroup of $N_G(\langle y \rangle)$, so we may assume that $\langle x_1 \rangle \leqslant N_G(\langle y \rangle)$. Hence $\langle y \rangle \trianglelefteq \langle x_1, y \rangle \trianglelefteq \langle x_1, x_2, y \rangle = G$.

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Now $o(x_1) = 2$ and $x_1 \notin \langle y \rangle$ (since otherwise $|G| = 16$). Hence $\langle x_1, y \rangle$ is non-cyclic. We have $\langle y^2 \rangle = \Phi(\langle y \rangle) \leqslant \Phi(\langle x_1, y \rangle)$ and $|\langle x_1, y \rangle : \Phi(\langle x_1, y \rangle)| \geq 4$. Thus, by comparison of orders, $\langle y^2 \rangle = \Phi(\langle x_1, y \rangle) \trianglelefteq G$. Since $\mathrm{Aut}(C_4) \cong C_2$, we have $|G : C_G(\langle y^2 \rangle)| \leq 2$. If $\langle y^2 \rangle \leqslant Z(G)$, then $\langle x_1, x_2, y^2 \rangle$ is abelian and has exponent 4 and index 2. Then, by Lemma 2.1, not all groups of order 8 can be embedded in G. Thus $\langle y^2 \rangle \nleqslant Z(G)$, so $|G : C_G(\langle y^2 \rangle)| = 2$. Since $y \in C_G(\langle y^2 \rangle)$, we then have $|C_G(\langle y^2 \rangle) : \langle y \rangle| = 2$, so $\langle y \rangle \trianglelefteq C_G(\langle y^2 \rangle)$. But $C_G(\langle y^2 \rangle) = \langle y \rangle (C_G(\langle y^2 \rangle) \cap \langle x_1, x_2 \rangle)$, so we may assume, without loss of generality, that $x_1 \in C_G(\langle y^2 \rangle)$ but that $x_2 \notin C_G(\langle y^2 \rangle)$. In particular, $C_G(\langle y^2 \rangle) = \langle x_1, y \rangle \trianglelefteq G$. Four cases then arise:

Case 1. $\langle y \rangle \trianglelefteq G$ *and* $y^{x_1} = y$.

Here $x_1 \in Z(G)$ and $G = \langle x_1 \rangle \times \langle x_2, y \rangle$. Since G is non-abelian, x_2 normalises, but does not centralise, $\langle y \rangle$. Therefore $y^{x_2} = y^k$, where $k = 3$, 5 or 7. If $y^{x_2} = y^5$, then $(y^2)^{x_2} = y^{10} = y^2$, which contradicts $x_2 \notin C_G(\langle y^2 \rangle)$. If $y^{x_2} = y^7$ $(= y^{-1})$, then $(x_2 y^n)^2 = (y^n)^{x_2} y^n = y^{-n} y^n = 1$. Hence the only elements of order 4 in $\langle x_2, y \rangle$ are y^2 and y^6, so all elements of order 4 in G are in $\langle x_1, y^2 \rangle$, which is abelian. But Q_2 has non-commuting elements of order 4, so Q_2 cannot be embedded in G. Thus $y^{x_2} = y^3$, so $G = \langle x_1 \rangle \times \langle x_2, y \rangle \cong C_2 \times H_1$.

Case 2. $\langle y \rangle \trianglelefteq G$ *and* $y^{x_1} \neq y$.

Since x_1 centralises y^2, but does not centralise y, we have $y^{x_1} = y^5$. Now x_2 normalises $\langle y \rangle$, but does not centralise $\langle y^2 \rangle$. It follows that $\langle x_1, x_2 \rangle / C_{\langle x_1, x_2 \rangle}(\langle y \rangle) \cong C_2 \times C_2 \cong \mathrm{Aut}(C_8) \cong \mathrm{Aut}(\langle y \rangle)$. Hence G is the semi-direct product of $\langle y \rangle$ by $\langle x_1, x_2 \rangle$, so $G \cong H_2$.

Case 3. $\langle y \rangle \ntrianglelefteq G$ *and* $y^{x_1} = y$.

In this case $x_1 \in Z(G)$. Furthermore, $\langle x_1, y \rangle / \langle x_1 \rangle \trianglelefteq G / \langle x_1 \rangle$ and $\langle x_1, y \rangle / \langle x_1 \rangle \cong C_8$. If $[y, x_2] \in \langle x_1 \rangle$ then x_2 centralises y^2, which is ruled out. Thus $[y, x_2] \notin \langle x_1 \rangle$. But $\langle y \rangle \ntrianglelefteq G$, so $y^{x_2} = y^k x_1$, where $k \not\equiv 1 \bmod(8)$. In addition, $\langle x_1, y \rangle / \langle x_1 \rangle = \langle\ y^k x_1\ \rangle \langle x_1 \rangle / \langle x_1 \rangle \cong C_8$, so $k \equiv 3$, 5 or 7. If $k = 5$, then $(y^2)^{x_2} = (y^5 x_1)^2 = y^{10} x_1^2 = y^2$. But x_2 does not centralise y^2, so $k \neq 5$. Hence either $y^{x_2} = y^3 x_1$, or $y^{x_2} = y^7 x_1$.

We let $Q \leqslant G$ be such that $Q \cong Q_2$. By comparison of orders, $\langle y \rangle \cap Q \neq 1$. Thus $\langle y^4 \rangle \leqslant \langle y \rangle \cap Q$. Now Q_2 has a unique subgroup of order 2, so $\langle y^4 \rangle = Z(Q) = Q \cap Z(G)$. Moreover, if $g \in Q$ is such that $o(g) = 4$, then $g^2 = y^4$. In addition, since $y^4 \neq x_1$, we have $Q \cap \langle x_1 \rangle = 1$. Thus $Q \langle x_1 \rangle / \langle x_1 \rangle \cong Q \cong Q_2$, so $G / \langle x_1 \rangle$ has a subgroup isomorphic to Q_2. Now, if $y^{x_2} = y^7 x_1$ $(= y^{-1} x_1)$, then $G / \langle x_1 \rangle$ is isomorphic to D_8. But D_8 has a unique subgroup of order 4, so Q_2 cannot be embedded in $G / \langle x_1 \rangle$. Thus we cannot have $y^{x_2} = y^7 x_1$.

If $y^{x_2} = y^3 x_1$, then $(x_2 y^n)^2 = x_2 y^n x_2 y^n = (y^n)^{x_2} y^n = (y^3 x_1)^n y^n = y^{3n} x_1^n y^n = x_1^n y^{4n}$. Hence, if n is odd, we have $(x_2 y^n)^2 = x_1 y^{4n} \neq y^4$; while, if is even with $n = 2k$, we have $(x_2 y^n)^2 = x_1^{2k} y^{4(2k)} =$

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$(x_1^2)^k(y^8)^k = 1 \neq y^4$. Since $x_1 \in Z(G)$, we similarly have $(x_1x_2y^n)^2 = x_1^2(x_2y^n)^2 \neq y^4$. It follows that if $g \in G$ is such that $g^2 = y^4$, then $g \in \langle x_1, y\rangle$, which is abelian. But, from above, Q is generated by elements whose square is equal to y^4, so $Q \leqslant \langle x_1, y\rangle$, and a contradiction results. Thus Case 3 is ruled out.

Case 4. *$\langle y\rangle \ntrianglelefteq G$ and $y^{x_1} \neq y$.*

Since $\langle y\rangle \ntrianglelefteq G$, we have $\langle y\rangle/\langle y^2\rangle \ntrianglelefteq G/\langle y^2\rangle$, so $G/\langle y^2\rangle$ is non-abelian of order 8. But $\langle x_1, x_2, y^2\rangle/\langle y^2\rangle \cong C_2\times C_2$, so $G/\langle y^2\rangle \cong D_4$. Now $\langle x_1, y^2\rangle/\langle y^2\rangle \trianglelefteq G/\langle y^2\rangle$ and $|\langle x_1, y^2\rangle/\langle y^2\rangle| = 2$, so $\langle x_1, y^2\rangle/\langle y^2\rangle = Z(G/\langle y^2\rangle)$. We let $Q \leqslant G$ be such that $Q \cong Q_2$. If $\langle y^2\rangle \leqslant Q$, then $Q/\langle y^2\rangle \cong C_2$. Since $\langle x_1, y^2\rangle/\langle y^2\rangle = Z(G/\langle y^2\rangle)$, the subgroups of order 2 in $G/\langle y^2\rangle$ are $\langle x_1, y^2\rangle/\langle y^2\rangle$; $\langle x_2, y^2\rangle/\langle y^2\rangle$; $\langle x_1x_2, y^2\rangle/\langle y^2\rangle$; $\langle x_1y, y^2\rangle/\langle y^2\rangle$; and $\langle y\rangle/\langle y^2\rangle$. But y^2 is centralised by x_1, so $\langle x_1, y^2\rangle$ and $\langle x_1y, y^2\rangle$ are abelian. Moreover $\langle y\rangle$ is cyclic and, since $C_G(\langle y^2\rangle) = \langle x_1, y\rangle$, both $\langle x_2, y^2\rangle$ and $\langle x_1x_2, y^2\rangle$ are isomorphic to D_4. Hence Q cannot be identical to any of these subgroups. It follows that $\langle y^2\rangle \nleqslant Q$, so $Q \cap \langle y^2\rangle = Q \cap \langle y\rangle = \langle y^4\rangle = Z(Q)$. In particular, $Q\langle y^2\rangle/\langle y^2\rangle \cong Q/Z(Q) \cong C_2 \times C_2$.

Now $\langle x_1, x_2, y^2\rangle/\langle y^2\rangle$ and $\langle x_1, y\rangle/\langle y^2\rangle$ are the only subgroups isomorphic to $C_2 \times C_2$ in $G/\langle y^2\rangle$. Hence either $Q \leqslant \langle x_1, x_2, y^2\rangle$ or $Q \leqslant \langle x_1, y\rangle$. Since both x_2 and y^2 are centralised by x_1, we have $\langle x_1, x_2, y^2\rangle = \langle x_1\rangle \times \langle x_2, y^2\rangle \cong C_2 \times D_4$. But all elements of order 4 in $C_2 \times D_4$ commute with each other, so $Q \nleqslant \langle x_1, x_2, y^2\rangle$. Now x_1 centralises y^2 but does not centralise y, so $y^{x_1} = y^5$. Hence $(x_1y^n)^2 = x_1y^nx_1y^n = y^{5n}y^n = y^{6n}$. But then $o(x_1y^n) = 4$ if and only if $2 \nmid n$. Thus all elements of order 4 in $\langle x_1, y\rangle$ are contained in $\langle x_1, y^2\rangle$, which is abelian. Hence $Q \nleqslant \langle x_1, y\rangle$. Case 4 is thus ruled out, so our proof is complete. □

3. Embeddings of groups of order 12

We now show that there is a unique group of minimal order in which all groups of order 12 can be embedded, namely $G = S_3 \times S_4$. We first show that all groups of order 12 can be embedded in this group.

Lemma 3.1. *Let $G = S_3 \times S_4$. Then all groups of order 12 can be embedded in G.*

Proof. We let $S_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1}\rangle$ and express S_4 in terms of the usual cycle notation. Up to isomorphism, the groups of order 12 are: C_{12}, $C_2\times C_2\times C_3$, D_6, Q_3 and A_4. We note the following: $C_{12} \cong \langle a(1234)\rangle$; $C_2\times C_2\times C_3 \cong \langle a, (12)(34), (13)(24)\rangle$; $D_6 \cong \langle b, a(12)\rangle$; and $Q_3 \cong \langle a(13)(24), b(1234)\rangle$. Since $A_4 \leqslant S_4$, this confirms that all groups of order 12 can be embedded in G. □

Next we show that $|S_3 \times S_4|$ is minimal with respect to the embedding of all groups of order 12.

Lemma 3.2. *Let G be a group of minimal order in which every group of order 12 can be embedded. Then $|G| = 144$.*

Proof. By Lemma 3.1, we have $|G| \le 144$. Conversely, we note that G contains subgroups isomorphic to C_{12} and $C_2 \times C_2 \times C_3$. In particular, G has subgroups isomorphic to C_4 and $C_2 \times C_2$. We let S be a Sylow 2-subgroup of G. Then S has subgroups isomorphic to C_4 and $C_2 \times C_2$. Hence S is non-cyclic and $|S| \ge 8$. We further let P be a Sylow 3-subgroup of G. Then $|P| \ge 3$. Now suppose that $|P| = 3$. Since G has subgroups isomorphic to $C_3 \times C_4$ and $C_2 \times C_2 \times C_3$, we see that $C_G(P)$ has subgroups isomorphic to C_4 and $C_2 \times C_2$. Letting S_1 be a Sylow 2-subgroup of $C_G(P)$, we then have $|S_1| \ge 8$. In addition G has a subgroup isomorphic to D_6, so G has a 2-element that normalises, but does not centralise, P. Thus $2 \,\big|\, |N_G(P) : C_G(P)|$. Since $P \leqslant C_G(P)$, we then have $|N_G(P)| \ge 2|C_G(P)| \ge 2{\cdot}8{\cdot}3 = 48$. Moreover, G has a subgroup U with $U \cong A_4$. By conjugacy, we may assume that $P \leqslant U$. But the Sylow 3-subgroups of A_4 are self-normalizing in A_4, so $U \cap N_G(P) = P$. Hence $|G| \ge |UN_G(P)| = \dfrac{|U||N_G(P)|}{|U \cap N_G(P)|} \ge \dfrac{12 \cdot 48}{3} = 192$, which contradicts the minimality of $|G|$.

Thus $|P| \ge 9$. If $|P| \ge 27$, then $|G| \ge 8 \cdot 27 = 216$. Hence $|P| = 9$ and P is isomorphic to C_9 or $C_3 \times C_3$. If $P \cong C_9$, then P has a unique subgroup of order 3. By Sylow's Theorems all subgroups of order 3 in G are then conjugate so, as above, we derive the contradiction $|G| \ge 192$. Hence $P \cong C_3 \times C_3$. Since $|P| = 9$ and $|S| \ge 8$, we see that $|G|$ is a multiple of 72. But $|G| \le 144$, so $|G| = 144$ or $|G| = 72$.

Suppose that $|G| = 72$. Since G has a subgroup isomorphic to $C_2 \times C_2 \times C_3$, there exists $x \in G$, with $o(x) = 3$, such that $\langle x \rangle$ is centralised by a subgroup isomorphic to $C_2 \times C_2$. In addition, $\langle x \rangle$ is centralised by a Sylow 3-subgroup of G. Hence $36 \,\big|\, |C_G(x)|$. Thus either $C_G(x) = G$, and $x \in Z(G)$; or $|G : C_G(x)| = 2$, and $C_G(x) \trianglelefteq G$. In the latter case $G/C_G(x) \cong C_2$, so $\mathrm{O}^{3'}(G) \leqslant C_G(x)$. Now $U \cong A_4$, so $U \leqslant C_G(x)$. In addition $Z(U) = 1$, so $x \notin U$. By comparison of orders, we then have $C_G(x) = \langle x \rangle \times U$. But G has a subgroup isomorphic to C_{12}, so there exists $x_1 \in G$, with $o(x_1) = 3$, such that x_1 is centralised by an element of order 4, y, say. Now the Sylow 2-subgroups of $C_G(x)$ are elementary abelian, so $y \notin C_G(x)$. Since $G/C_G(x) \cong C_2$, we have $\langle x_1, y^2 \rangle \leqslant C_G(x)$. In particular y^2 is an element of order 2 in $C_G(x)$, so $C_{C_G(x)}(y^2) = \langle x \rangle \times U' \cong C_3 \times C_2 \times C_2$. But x_1 centralises y^2 and $o(x_1) = 3$, so $\langle x_1 \rangle = \langle x \rangle$. Then x is centralised by y, which is a contradiction. We thus conclude that $x \in Z(G)$.

Now G has a subgroup Q, say, with $Q \cong Q_3$. Since the Sylow 3-subgroup of Q is not central in Q, we have $Q \cap \langle x \rangle = 1$, so $Q\langle x \rangle/\langle x \rangle \cong Q_3$. We similarly have $U\langle x \rangle/\langle x \rangle \cong A_4$. Now

$$\begin{aligned} |(Q\langle x \rangle/\langle x \rangle)(U\langle x \rangle/\langle x \rangle)| &= \frac{|Q\langle x \rangle/\langle x \rangle||U\langle x \rangle/\langle x \rangle|}{|Q\langle x \rangle/\langle x \rangle \cap U\langle x \rangle/\langle x \rangle|} \\ &= \frac{|Q_3||A_4|}{|Q\langle x \rangle/\langle x \rangle \cap U\langle x \rangle/\langle x \rangle|} \\ &= \frac{144}{|Q\langle x \rangle/\langle x \rangle \cap U\langle x \rangle/\langle x \rangle|}. \end{aligned}$$

But $|G/\langle x \rangle| = 24$, so $|Q\langle x \rangle/\langle x \rangle \cap U\langle x \rangle/\langle x \rangle| \ge 6$. Since $Q\langle x \rangle/\langle x \rangle$ and $U\langle x \rangle/\langle x \rangle$ are non-isomorphic groups of order 12, we then have $|Q\langle x \rangle/\langle x \rangle \cap U\langle x \rangle/\langle x \rangle| = 6$. In particular, $U\langle x \rangle/\langle x \rangle$ has a subgroup of order 6. But $U\langle x \rangle/\langle x \rangle \cong A_4$, so a contradiction ensues. We conclude that $|G| = 144$. □

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The final result in this section shows that $S_3 \times S_4$ is the unique group of minimal order in which all groups of order 12 can be embedded.

Theorem 3.3. *Let G be a group of order 144 in which all groups of order 12 can be embedded. Then $G \cong S_3 \times S_4$.*

Proof. We let P be a Sylow 3-subgroup of G and let U_1, U_2 and U_3 be subgroups of G such that $U_1 \cong C_{12}$; $U_2 \cong C_2 \times C_2 \times C_3$; and $U_3 \cong A_4$. By Sylow's Theorems we may assume that $P \cap U_i \cong C_3$, $i = 1, 2, 3$. Since $|P| = 9$, either $P \cong C_3 \times C_3$ or $P \cong C_9$. If $P \cong C_9$, then P has a unique subgroup of order 3, P_1 say, so $U_i \cap P = P_1$ for $i = 1, 2, 3$. But U_1, U_2 and P are abelian, so $\langle U_1, U_2, P\rangle \leqslant C_G(P_1)$. Hence the Sylow 2-subgroups of $C_G(P_1)$ have subgroups isomorphic to C_4 and $C_2 \times C_2$, and thus have order at least 8. We then have $|C_G(P_1)| \geq 72$, so $|G : C_G(P_1)| \leq 2$ and $C_G(P_1) \trianglelefteq G$. Hence $\mathrm{O}^{3'}(G) \leqslant C_G(P_1)$, so $U_3 \leqslant C_G(P_1)$. But $P_1 \leqslant U_3 \cong A_4$ and $Z(A_4) = 1$, so a contradiction arises. Hence $P \cong C_3 \times C_3$.

Now G is a $\{2, 3\}$-group, so G is soluble. Hence $\mathrm{F}(G)$ is non-trivial and $\mathrm{F}(G) = \mathrm{O}_2(G) \times \mathrm{O}_3(G)$. If $|\mathrm{O}_2(G)| = 16$, then G has a normal Sylow 2-subgroup, so D_6 and Q_3 cannot be embedded in G. Thus $|\mathrm{O}_2(G)| \leq 8$. Similarly, if $|\mathrm{O}_3(G)| = 9$, then A_4 cannot be embedded in G, so $|\mathrm{O}_3(G)| \leq 3$. Since $C_3 \times C_3$ cannot be embedded in the automorphism group of a 2-group of order 8 or less, we have $C_P(\mathrm{O}_2(G)) \neq 1$. But P is abelian, so $C_P(\mathrm{O}_2(G)) \leqslant C_P(\mathrm{O}_2(G) \times \mathrm{O}_3(G)) = C_P(\mathrm{F}(G)) \leqslant \mathrm{F}(G)$. Thus $1 \neq C_P(\mathrm{O}_2(G)) \leqslant \mathrm{O}_3(G)$, so $\mathrm{O}_3(G) \cong C_3$.

If $3 \nmid |\mathrm{Aut}(\mathrm{O}_2(G))|$, then the contradiction $P \leqslant C_G(\mathrm{F}(G)) \leqslant \mathrm{F}(G)$ arises. Hence $3 \,\big|\, |\mathrm{Aut}(\mathrm{O}_2(G))|$, so $\mathrm{O}_2(G)$ is non-cyclic. If $\mathrm{O}_2(G)$ is non-abelian, then $\mathrm{O}_2(G) \cong D_4$ or $\mathrm{O}_2(G) \cong Q_2$. If $\mathrm{O}_2(G) \cong D_4$ then $3 \nmid |\mathrm{Aut}(\mathrm{O}_2(G))|$, which is ruled out. If $\mathrm{O}_2(G) \cong Q_2$, then $|\mathrm{F}(G)| = 24$. But $U_3 \cong A_4$ so, by comparison of orders, $1 \neq U_3 \cap \mathrm{F}(G)$. By minimal normality, we then have the contradiction $C_2 \times C_2 \cong U_3' \leqslant \mathrm{O}_2(G) \cong Q_2$. Hence $\mathrm{O}_2(G)$ is abelian and non-cyclic, and is thus isomorphic to $C_2 \times C_4$; $C_2 \times C_2 \times C_2$; or $C_2 \times C_2$.

If $\mathrm{O}_2(G) \cong C_2 \times C_4$ then $3 \nmid |\mathrm{Aut}(\mathrm{O}_2(G))|$, which is ruled out. If $\mathrm{O}_2(G) \cong C_2 \times C_2 \times C_2$, we see by minimal normality that $U_3' \leqslant \mathrm{O}_2(G)$. We let $\langle x\rangle$ be a Sylow 3-subgroup of U_3. By Maschke's Theorem, there exists $z \in \mathrm{O}_2(G)$ such that $\langle z\rangle$ is normalised by x and $\mathrm{O}_2(G) = \langle z\rangle \times U_3'$. But x also centralises $\mathrm{O}_3(G)$, so $\mathrm{F}(G)U_3 = \mathrm{O}_3(G) \times \langle z\rangle \times U_3 \cong C_3 \times C_2 \times A_4$. Now $|G : \mathrm{F}(G)U_3| = 2$, so $\mathrm{F}(G)U_3 \trianglelefteq G$. Since $\langle z\rangle$ is characteristic in $\mathrm{F}(G)U_3$, we have $\langle z\rangle \trianglelefteq G$, so $z \in Z(G)$. In addition $U_3' = (\mathrm{F}(G)U_3)'$, so $U_3' \trianglelefteq G$. Since D_6 can be embedded in G, there exists $y \in G$ such that $o(y) = 2$ and $y \notin \mathrm{O}_2(G)$. By Maschke's Theorem $\mathrm{O}_3(G)\,\mathrm{O}_2(G)/\mathrm{O}_2(G)$ has a complement, $W/\mathrm{O}_2(G)$, in $\mathrm{F}(G)U_3/\mathrm{O}_2(G)$ that is normalised by $\langle y\rangle\,\mathrm{O}_2(G)/\mathrm{O}_2(G)$. Then $W \trianglelefteq G$ and $W \cap \mathrm{O}_3(G) \leqslant \mathrm{O}_2(G) \cap \mathrm{O}_3(G) = 1$, so $\mathrm{F}(G)U_3 = W \times \mathrm{O}_3(G)$. Hence $W \cong \langle z\rangle \times U_3$, so we may assume that $\langle z\rangle \times U_3 \trianglelefteq G$. Then $U_3 = \mathrm{O}^{3'}(\langle z\rangle \times U_3)$, so $U_3 \trianglelefteq G$.

Now if y centralises U_3' then G has elementary abelian Sylow 2-subgroups, so C_{12} cannot be embedded in G. Thus y does not centralise U_3', so $U_3\langle y\rangle \cong S_4$. Hence $\langle z\rangle U_3\langle y\rangle \cong C_2 \times S_4$. If y centralises $\mathrm{O}_3(G)$, then $G = \mathrm{O}_3(G) \times \langle z\rangle \times U_3\langle y\rangle \cong C_3 \times C_2 \times S_4$. Since $G/\mathrm{O}_3(G)U_3$ has exponent 2 and $\mathrm{O}_3(G)U_3/U_3' \cong C_3 \times C_3$, we see that if $h \in G$ is such that $o(h) = 4$, then $h^2 \in U_3'$. Now Q_3 can be embedded in G, so there exists $g \in G$ with $o(g) = 4$ such that g normalises, but does not centralise, a subgroup of order 3 in G. But $g \in \mathrm{O}^{2'}(G) = \langle z\rangle \times U_3\langle y\rangle$, so we may assume, without loss of generality, that g normalises $\langle x\rangle$. Then g^2 centralises x and $1 \neq g^2 \in U_3'$, so a contradiction arises. We thus assume that y does not centralise $\mathrm{O}_3(G)$, so $\mathrm{O}_3(G)\langle y\rangle \cong S_3$. Hence $\mathrm{O}_3(G)\langle y\rangle \cap \mathrm{O}_2(G) = 1$, so $\mathrm{O}_3(G)\langle y\rangle \mathrm{O}_2(G)/\mathrm{O}_2(G) \cong S_3$. But $U_3\langle y\rangle \cong S_4$, so $y\,\mathrm{O}_2(G)$ inverts every element of $\mathrm{O}_3(G)\,\mathrm{O}_2(G)/\mathrm{O}_2(G) \times U_3\,\mathrm{O}_2(G)/\mathrm{O}_2(G)$. In particular, C_6 cannot be embedded in $G/\mathrm{O}_2(G)$. But $U_1 \cong C_{12}$, so $U_1 \cap \mathrm{O}_2(G) \cong C_2$, and the contradiction $U_1\,\mathrm{O}_2(G)/\mathrm{O}_2(G) \cong C_6$ arises. Hence $\mathrm{O}_2(G) \not\cong C_2 \times C_2 \times C_2$. We thus conclude that $\mathrm{O}_2(G) \cong C_2 \times C_2$.

We have $U_3 \cong A_4$, so U_3' centralises $\mathrm{O}_2(G)$. If $U_3' \cap \mathrm{O}_2(G) = 1$, then $\mathrm{O}_2(G)U_3' = \mathrm{O}_2(G) \times U_3' \cong C_2 \times C_2 \times C_2 \times C_2$, so the Sylow 2-subgroups of G are elementary abelian. But then C_{12} cannot be embedded in G. Hence $U_3' \cap \mathrm{O}_2(G) \neq 1$ so, by minimal normality, $U_3' = \mathrm{O}_2(G)$. Since $\mathrm{F}(G) = \mathrm{O}_3(G) \times \mathrm{O}_2(G) \cong C_3 \times C_2 \times C_2$, we have $\mathrm{Aut}(\mathrm{F}(G)) \cong C_2 \times S_3$. Now $\mathrm{F}(G)$ is abelian, so $C_G(\mathrm{F}(G)) = \mathrm{F}(G)$. Then, by comparison of orders, $G/\mathrm{F}(G) \cong C_2 \times S_3$. Since $G/\mathrm{F}(G)$ thus has a quotient group isomorphic to $C_2 \times C_2$, we have $|\mathrm{O}^{3'}(G)| \leq 36$. But $\mathrm{O}_3(G)U_3 \leqslant \mathrm{O}^{3'}(G)$ and $|\mathrm{O}_3(G)U_3| = 36$, so $\mathrm{O}_3(G)U_3 = \mathrm{O}^{3'}(G)$. In addition $\mathrm{O}_3(G) \cong C_3$ and $U_3 \cong A_4$, so U_3 centralises $\mathrm{O}_3(G)$. Hence $\mathrm{O}^{3'}(G) = \mathrm{O}_3(G) \times U_3 \cong C_3 \times A_4$.

Now $\mathrm{O}^{3'}(G)/\mathrm{O}_2(G) \cong C_3 \times C_3$ and $G/\mathrm{O}^{3'}(G)$ is a 2-group so, by Maschke's Theorem, we have $\mathrm{O}^{3'}(G)/\mathrm{O}_2(G) = \mathrm{O}_3(G)\,\mathrm{O}_2(G)/\mathrm{O}_2(G) \times K/\mathrm{O}_2(G)$, where $K/\mathrm{O}_2(G) \cong C_3$ and $K \trianglelefteq G$. Moreover $\mathrm{O}_3(G) \cap K \leqslant \mathrm{O}_3(G) \cap \mathrm{O}_2(G) = 1$ and $\mathrm{O}^{3'}(G) = \mathrm{O}_3(G)K$. Hence $\mathrm{O}^{3'}(G) = \mathrm{O}_3(G) \times K$ and $K \cong \mathrm{O}^{3'}(G)/\mathrm{O}_3(G) \cong A_4$. Thus we may assume, without loss of generality, that $U_3 = K \trianglelefteq G$. We let $W = C_G(\mathrm{O}_3(G))$. Since $G/\mathrm{F}(G) \cong C_2 \times S_3$, we have $\mathrm{O}_3(G) \nleqslant Z(G)$. Hence $G/W \cong C_2$ and $|W| = 72$. Now $U_3 \leqslant W$ and $U_3 \cong A_4$. Thus $C_W(U_3) \cap U_3 = 1$ and $W/C_W(U_3)$ is isomorphic either to A_4 or S_4. Since $|W| = 72$, we thus have $|C_W(U_3)| = 6$ or $|C_W(U_3)| = 3$. If $|C_W(U_3)| = 6$ then, since $\mathrm{O}_3(G)$ centralises U_3, we have $C_W(U_3) \cong C_6$. We further have $C_W(U_3) = \mathrm{O}_3(G) \times \langle h\rangle$, for a suitable $h \in W$ with $o(h) = 2$. Then $W = C_W(U_3) \times U_3 = \mathrm{O}_3(G) \times \langle h\rangle \times U_3$, so $\langle h\rangle \leqslant \mathrm{O}_2(W) \leqslant \mathrm{O}_2(G) = U_3'$, which is a contradiction. Hence $|C_W(U_3)| = 3$, so $C_W(U_3) = \mathrm{O}_3(G)$. Since $|W : \mathrm{O}_3(G) \times U_3| = 2$, there exists a 2-element $y \in W \setminus (\mathrm{O}_3(G) \times U_3)$ such that $W = (\mathrm{O}_3(G) \times U_3)\langle y\rangle$. Now $y^2 \in \mathrm{O}_3(G) \times U_3$ and y^2 is a 2-element, so $y^2 \in U_3'$. Thus $|U_3\langle y\rangle| = 24$ and $W = \mathrm{O}_3(G) \times U_3\langle y\rangle$. We further have $U_3\langle y\rangle \cong W/\mathrm{O}_3(G) = W/C_W(U_3) \cong S_4$. In addition, $U_3\langle y\rangle = \mathrm{O}^{2'}(W) \trianglelefteq G$. We let $C = C_G(U_3\langle y\rangle)$. Then $C \cap U_3\langle y\rangle = 1$ and S_4 is complete, so $G = C \times U_3\langle y\rangle$. Hence $|C| = 6$. If $C \cong C_6$, then we derive the contradiction $C_2 \cong \mathrm{O}_2(C) \leqslant \mathrm{O}_2(G) = U_3'$. Thus $C \cong S_3$, so $G = C \times U_3\langle y\rangle \cong S_3 \times S_4$. □

Example 3.4. We let $G = \langle x\rangle \times \langle y\rangle \times S_4$, where $\langle x\rangle \cong C_2$ and $\langle y\rangle \cong C_4$. We have $C_{12} \cong \langle y, (123)\rangle$; $C_2 \times C_2 \times C_3 \cong \langle x, y^2, (123)\rangle$; $D_6 \cong \langle x\rangle \times S_3$; and $Q_3 \cong \langle y(12), (123)\rangle$. Now $A_4 \leqslant S_4$, so all groups of order 12 can be embedded in G. Since $|G| = 192$, this shows that the minimal order of a group in which all groups of order n can be embedded is not, in general, a divisor of the order of every group in which these groups can be embedded.

4. Minimal embeddings of groups of order n for $1 \le n \le 15$

Having dealt with the cases $n = 8$ and $n = 12$, we present the groups of minimal order in which all groups of order n can be embedded for $1 \le n \le 15$. For reference we list the groups of order 15 or less in Table 1 and let $H_3 = \langle a, b \mid a^9 = 1 = b^3,\ b^{-1}ab = a^4 \rangle$ denote the non-abelian group of order 27 and exponent 9.

n	Groups of order n	n	Groups of order n
1	C_1	9	$C_9, C_3 \times C_3$
2	C_2	10	C_{10}, D_5
3	C_3	11	C_{11}
4	$C_4, C_2 \times C_2$	12	$C_{12}, C_2 \times C_2 \times C_3, D_6, Q_3, A_4$
5	C_5	13	C_{13}
6	C_6, D_3	14	C_{14}, D_7
7	C_7	15	C_{15}
8	$C_8, C_2 \times C_4, C_2 \times C_2 \times C_2, D_4, Q_2$		

TABLE 1. Groups of order 15 or less

Bearing Theorems 2.3 and 3.3 in mind, we have:

Theorem 4.1. *Let $1 \le n \le 15$ and let G be a group of minimal order in which all groups of order n can be embedded. Then the isomorphism class of G is as in Table 2.*

n	Group(s) of minimal order	n	Group(s) of minimal order	n	Group(s) of minimal order
1	C_1	6	D_6	11	C_{11}
2	C_2	7	C_7	12	$S_3 \times S_4$
3	C_3	8	$C_2 \times H_1,\ H_2$	13	C_{13}
4	$C_2 \times C_4,\ D_4$	9	$C_3 \times C_9,\ H_3$	14	D_{14}
5	C_5	10	D_{10}	15	C_{15}

TABLE 2. Group(s) of least order containing all groups of order n, for $1 \le n \le 15$

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5. Embeddings of groups of order p^3, p odd

Theorem 4.1 shows that the bound given by Lemma 1.1 is attained for $n = 4$, 8 and 9. Next we show that this is not the case for $n = p^k$, where p is an odd prime and $k \geq 3$.

Theorem 5.1. *Let p be an odd prime and let $k \geq 3$. If G is a group of order p^{2k-1} then not all groups of order p^k can be embedded in G.*

Proof. We suppose that C_{p^k} and the elementary abelian group of rank k can be embedded in G and let A and B be subgroups of G such that $A = \langle x \rangle \cong C_{p^k}$ and B is elementary abelian of rank k. Since A is cyclic and $|G| = p^{2k-1}$, we have $G = AB$ and $|A \cap B| = p$. By [6] Lemma 2.5, we have $B = \Omega_1(A)B \trianglelefteq G$. Then $G/B \cong A/(A \cap B) \cong C_{p^{k-1}}$. If $k \geq 4$, we let $U \leqslant G$ be such that $U \cong C_{p^2} \times C_{p^{k-2}}$. Since $U \cap B$ is elementary abelian, UB/B $(\cong U/(U \cap B))$ has a subgroup isomorphic to $C_p \times C_p$. But G/B is cyclic, so a contradiction arises. Thus for $k \geq 4$, not all groups of order p^k can be embedded in G.

If $k = 3$, then $B \cong C_p \times C_p \times C_p$ and $\mathrm{Aut}(B) \cong \mathrm{GL}(3,p)$. But p is odd, so the Sylow p-subgroups of $\mathrm{GL}(3,p)$ have exponent p. Hence $G/C_G(B)$ has exponent p, so $\langle x^p \rangle \leqslant C_G(B)$. Now $G/B \cong C_{p^2}$, so $\langle x^p \rangle B/B$ is the unique subgroup of order p in G/B. Thus all elements of order p in G are contained in $\langle x^p \rangle B$. But $x^p \in C_G(B)$, so $\langle x^p \rangle B$ is abelian. Hence the non-abelian group of order p^3 and exponent p cannot be embedded in G. □

Theorem 5.1 shows that if p is an odd prime and G is a group of minimal order in which all groups of order p^3 can be embedded, then $|G| \geq p^6$. We now show that this bound can be attained.

Example 5.2. Let p be an odd prime and let $A = \langle x \rangle$ be cyclic of order p^3. Let $B = \langle a, b, z \mid a^p = b^p = z^p = 1, [a,z] = [b,z] = 1, b^{-1}ab = az \rangle$ be the non-abelian group of order p^3 and exponent p. Form the direct product $W = A \times B$ and let $W_1 = W/\langle z^{-1}x^{p^2} \rangle$. W_1 is thus the central product of $A\langle z^{-1}x^{p^2} \rangle/\langle z^{-1}x^{p^2} \rangle$ and $B\langle z^{-1}x^{p^2} \rangle/\langle z^{-1}x^{p^2} \rangle$, in which the elements z and x^{p^2} have been identified with each other. For notational convenience, we write $\langle z^{-1}x^{p^2} \rangle = 1$, so that W_1 is the central product of A and B, with $A \cap B = \langle z \rangle$. In particular, $|W_1| = p^5$. We express W_1 in terms of generators and relations as: $W_1 = \langle a, b, x, z \mid a^p = b^p = z^p = x^{p^3} = 1, [a,z] = [b,z] = 1, b^{-1}ab = az, [a,x] = [b,x] = 1, x^{p^2} = z \rangle$. By construction, $A \cong C_{p^3}$, and B is isomorphic to the non-abelian group of order p^3 and exponent p. We can confirm that $\langle a, x^p \rangle \cong C_p \times C_{p^2}$; and that $\langle ax^p, b \rangle$ is isomorphic to the non-abelian group of order p^3 and exponent p^2. Thus, apart from $C_p \times C_p \times C_p$, all groups of order p^3 can be embedded in W_1. Now let $G = W_1 \times C_p$. Since $C_p \times C_p \cong \langle a, z \rangle \leqslant W_1$, G is a group of order p^6 in which all groups of order p^3 can be embedded.

6. Embeddings of groups of order 15 or less: the soluble case

We turn to the question of determining the least order of a group, G, in which all groups of order n or less can be embedded for $1 \leq n \leq 15$. We first consider the case where G is soluble. For $n \geq 12$ all

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groups of orders 8, 9 and 12 can be embedded in the Hall $\{2,3\}$-subgroups of G. Lemma 1.2 shows that the order of a Hall $\{2,3\}$-subgroup of G is a multiple of $2^5 \cdot 3^3 = 864$. We show that this bound is not attainable by soluble groups for $12 \leq n \leq 15$. In this section all groups considered will be soluble.

Lemma 6.1. *Let G be a group of order $2^5 \cdot 3^k$ such that $\mathrm{O}_3(G) = 1$. Then $k \leq 2$ and, if $k = 2$, then the Sylow 2-subgroups of G have exponent at most 4.*

Proof. Since $\mathrm{O}_3(G) = 1$, $\mathrm{F}(G)$ is a non-trivial 2-group. We let P be a Sylow 3-subgroup of G. Since $C_P(\mathrm{F}(G)) \leqslant C_G(\mathrm{F}(G)) \leqslant \mathrm{F}(G)$, we have $C_P(\mathrm{F}(G)) = 1$. Hence P acts faithfully on $\mathrm{F}(G)$ and thus on $\mathrm{F}(G)/\Phi(\mathrm{F}(G))$. Now $\mathrm{F}(G)/\Phi(\mathrm{F}(G))$ has rank at most 5. But $3^2 \nmid |\mathrm{GL}(5,2)|$, so $|P| \leq 9$. If $|P| = 9$, then $|\mathrm{F}(G)/\Phi(\mathrm{F}(G))| \geq 16$. But $\mathrm{F}(G)/\Phi(\mathrm{F}(G))$ is elementary abelian, so the Sylow 2-subgroups of G have exponent at most 4. □

Corollary 6.2. *Let G be a group of order $2^5 \cdot 3^k$. If C_8 can be embedded in G then $|\mathrm{O}_3(G)| \geq 3^{k-1}$.*

Proof. Let $W = \mathrm{O}_3(G)$. Then $\mathrm{O}_3(G/W) = 1_{G/W}$. Since C_8 can be embedded in G, the exponent of the Sylow 2-subgroups of G, and hence of those of G/W, is at least 8. By Lemma 6.1, $|G/W|$ is at most $2^5 \cdot 3$, so $|W| = |\mathrm{O}_3(G)| \geq 3^{k-1}$. □

Lemma 6.3. *Let G be a group of order 96 in which A_4 can be embedded. Then not all groups of order 8 can be embedded in G.*

Proof. We let $U \leqslant G$ be such that $U \cong A_4$ and let $C = C_G(U)$. Suppose first that $U = \mathrm{O}^{3'}(G)$. Since $U \trianglelefteq G$, we have $C \trianglelefteq G$ and $C \cap U = 1$. Now $\mathrm{Aut}(A_4) \cong S_4$ and $|G| = 2^5 \cdot 3$, so $|C| = 4$ or $|C| = 8$. If $|C| = 4$, then $U' \times C$ is abelian of order 16 and exponent at most 4. Hence, by Lemma 2.1, not all groups of order 8 can be embedded in G. If $|C| = 8$, then C has an abelian subgroup of order 4, C_1 say. Then $U' \times C_1$ is abelian of order 16 and exponent at most 4. Again by Lemma 2.1, not all groups of order 8 can be embedded in G. We may thus assume that U is a proper subgroup of $\mathrm{O}^{3'}(G)$.

We let $W = \mathrm{O}_2(\mathrm{O}^{3'}(G))$ and let $\langle y \rangle$ be a Sylow 3-subgroup of U. Since $\langle y \rangle \cong C_3$, we have $\mathrm{O}^{3'}(G) = W\langle y \rangle$ and $[\langle y \rangle, W] = W$. By Maschke's Theorem, either $W/\Phi(W) \cong C_2 \times C_2$ or $W/\Phi(W) \cong C_2 \times C_2 \times C_2 \times C_2$. In the latter case the Sylow 2-subgroups of G have exponent at most 4, so C_8 cannot be embedded in G. Hence we may assume that $W/\Phi(W) \cong C_2 \times C_2$.

Now suppose that $U' \nleqslant \Phi(W)$. Then, by minimal normality, $U' \cap \Phi(W) = 1$ so $W/\Phi(W) = U'\Phi(W)/\Phi(W)$. Hence $W = U'\Phi(W) = U'$ and the contradiction $U = \langle y \rangle U' = \langle y \rangle W = \mathrm{O}^{3'}(G)$ ensues. Thus $U' \leqslant \Phi(W)$, so $|\Phi(W)| \geq 4$ and $|W| \geq 16$. If $|\Phi(W)| = 8$, then W is a Sylow 2-subgroup of G. Moreover, since $C_2 \times C_2 \cong U' \leqslant \Phi(W)$, either $\Phi(\Phi(W)) = 1$ or $\Phi(\Phi(W)) \cong C_2$. If $\Phi(\Phi(W)) = 1$, then $\Phi(W)$ is elementary abelian and W has exponent at most 4, so C_8 cannot be embedded in G. If $\Phi(\Phi(W)) \cong C_2$ then, by minimal normality, $\Phi(\Phi(W)) \cap U' = 1$ and the contradiction $\Phi(W) = U'\Phi(\Phi(W)) = U'$ arises.

We may thus assume that $|\Phi(W)| = 4$, so $\Phi(W) = U'$. Now $|W/\Phi(W)| = 4$, so $|W| = 16$. Since $\mathrm{Aut}(U') \cong S_3$ and $\mathrm{O}^{3'}(G)$ has no proper $3'$-factor groups, we have $W \leqslant C_G(U')$, so $U' \leqslant Z(W)$. In addition W/U' is elementary abelian, so W is either abelian or has class 2. Since $W/\Phi(W)$ has rank 2, we let $W = \langle x, y\rangle$, for suitable $x, y \in W$. Then $[x,y] \in \Phi(W) = U' \leqslant Z(W)$. Hence if W is non-abelian, we have $W' = \langle [x,y]\rangle \cong C_2$. By minimal normality, we derive the contradiction $U' = W' \cong C_2$. Thus W is abelian. Since $W/\Phi(W) \cong C_2 \times C_2 \cong \Phi(W) = U'$, we see that W has exponent at most 4. Hence, by Lemma 2.1, not all groups of order 8 can be embedded in G. □

Corollary 6.4. *Let G be a group of order $2^5 \cdot 3^k$ in which A_4 can be embedded. Then not all groups of order 8 can be embedded in G.*

Proof. Let $U \leqslant G$ be such that $U \cong A_4$. By Corollary 6.2, if C_8 can be embedded in G, then $|\mathrm{O}_3(G)| \geq 3^{k-1}$, so $|G/\mathrm{O}_3(G)|$ is a divisor of $2^5 \cdot 3 = 96$. But $U \cap \mathrm{O}_3(G) \leqslant \mathrm{O}_3(U) = 1$, so $3 \,\big|\, |G/\mathrm{O}_3(G)|$. Hence $|G/\mathrm{O}_3(G)| = 96$ and $U\,\mathrm{O}_3(G)/\mathrm{O}_3(G) \cong A_4$. Since the Sylow 2-subgroups of $G/\mathrm{O}_3(G)$ are isomorphic to those of G, we apply Lemma 6.3 to see that not all groups of order 8 can be embedded in G. □

As a consequence of Lemma 1.2 and Corollary 6.4 the following result gives a lower bound for the order of a soluble group in which all groups of order n or less can be embedded for $12 \leq n \leq 15$. In Section 8 we show that the bound can be attained in these cases.

Theorem 6.5. *Let G be a finite soluble group.*

(a) If all groups of order 12 or less can be embedded in G, then $|G|$ is a multiple of $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 665\,280$.

(b) If all groups of order n or less can be embedded in G, where $n = 13, 14$ or 15, then $|G|$ is a multiple of $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 8\,648\,640$.

7. Embeddings of groups of order 15 or less: the non-soluble case

Turning to embeddings in non-soluble groups, we prove a result analogous to Theorem 6.5 in the case where G is non-soluble. We begin with some results concerning normal quasisimple subgroups and central products.

Lemma 7.1. *Let A and B be subgroups of the group G such that $[A,B]$ is abelian and $B \leqslant C_G([A,B])$. Then $B' \leqslant C_G(A)$.*

Proof. Let $a \in A$ and let $b_1, b_2 \in B$. Since $B \leqslant C_G([A,B])$, we have $a^{b_1 b_2} = (a[a,b_1])^{b_2} = a^{b_2}[a,b_1] = a[a,b_2][a,b_1]$. We similarly obtain $a^{b_2 b_1} = a[a,b_1][a,b_2]$. Now $[A.B]$ is abelian, so $a^{b_1 b_2} = a^{b_2 b_1}$. Thus $a^{b_1 b_2 b_1^{-1} b_2^{-1}} = a^{[b_1^{-1},\, b_2^{-1}]} = a$ for all $b_1, b_2 \in B$. Hence $B' \leqslant C_G(A)$. □

Corollary 7.2. *Let G be a group and let $N \trianglelefteq G$ be such that $N = N'$. Then $C_G(N/Z(N)) = C_G(N)$.*

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Proof. Let $C/Z(N) = C_{G/Z(N)}(N/Z(N))$. Then $C = C_G(N/Z(N))$ and $[C,N] \leqslant Z(N)$. In particular we have $N \leqslant C_G([C,N])$. By Lemma 7.1, $N = N' \leqslant C_G(C)$, so $C \leqslant C_G(N)$. Since the reverse inclusion is evident, we conclude that $C_G(N/Z(N)) = C_G(N)$. □

Lemma 7.3. *Let G be a finite group and let W be a non-abelian finite simple group. Let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be a chief series of G and suppose that G_k/G_{k-1} is a chief factor such that $G_k/G_{k-1} \cong W$. Suppose in addition that, for $i = 1,\dots,k-1$, there do not exist subgroups K and H in $\mathrm{Aut}(G_i/G_{i-1})$ with $K \trianglelefteq H$ and $H/K \cong W$. Then there exists $N \trianglelefteq G$ such that $N' = N$ and $N/Z(N) \cong W$.*

Proof. We use induction on $|G|$. For $G_1 = G_k$, the result is trivial. If $G_1 \neq G_k$, then $1_{G/G_1} = G_1/G_1 \trianglelefteq G_2/G_1 \trianglelefteq \cdots \trianglelefteq G_n/G_1 = G/G_1$ is a chief series of G/G_1 that satisfies our hypotheses. By induction, there exists $N_1 \trianglelefteq G$ with $G_1 \leqslant N_1$ and $(N_1/G_1)' = N_1/G_1$ such that, for $Z_1/G_1 = Z(N_1/G_1)$, we have $N_1/Z_1 \cong (N_1/G_1)/(Z_1/G_1) \cong W$.

We let $C_1 = C_{N_1}(G_1) \trianglelefteq G$. If $C_1 \leqslant Z_1$ then, for $H = N_1/C_1$ and $K = Z_1/C_1$, H and K can be identified with subgroups of $\mathrm{Aut}(G_1)$ such that $H/K \cong W$. But this is excluded, so $1 \neq C_1Z_1/Z_1 \trianglelefteq N_1/Z_1$. Now N_1/Z_1 is simple, so $C_1Z_1/Z_1 = N_1/Z_1$. In particular $N_1 = C_1Z_1$ and $N_1/G_1 = (N_1/G_1)' = (C_1Z_1/G_1)' = (C_1'G_1/G_1)(Z_1'G_1/G_1)[C_1,Z_1]G_1/G_1$. But $Z_1'G_1/G_1 = 1_{G/G_1}$, so $N_1/G_1 = (C_1'G_1/G_1)[C_1,Z_1]G_1/G_1 \leqslant C_1G_1/G_1$. Hence $N_1 = C_1G_1$.

Since G_1 is a chief factor of G, either $G_1 = U_1 \times \cdots \times U_s$, where $U_j \cong U$ $(j = 1,\dots,s)$ and U is a suitable non-abelian finite simple group; or G_1 is elementary abelian. In the first case $Z(G_1) = 1$, so $G_1 \cap C_1 = 1$. Thus $N_1 = G_1 \times C_1$ and $C_1 \cong N_1/G_1$. Then $N = C_1 \trianglelefteq G$ has the required properties. In the second case G_1 is abelian, so $N_1 = C_1G_1 = C_1$ and $G_1 \leqslant Z(N_1)$. Since G_1 is minimal normal in G, either $G_1 \cap N_1' = 1$ or $G_1 \leqslant N_1'$. If $G_1 \cap N_1' = 1$, then $N_1/G_1 = (N_1/G_1)' = N_1'G_1/G_1 \cong N_1'/(N_1' \cap G_1) \cong N_1'$. Hence $N = N_1'$ satisfies our requirements. If $G_1 \leqslant N_1'$, then $N_1 = N_1'G_1 = N_1'$. Now $Z_1/G_1 = Z(N_1/G_1)$, so $[N_1,Z_1] \leqslant G_1 \leqslant Z(N_1)$. Thus $N_1 \leqslant C_G([N_1,Z_1])$ and $[N_1,Z_1]$ is abelian. By Lemma 7.1, N_1' centralises Z_1. But $N_1' = N_1$, so $Z_1 = Z(N_1)$. Taking $N = N_1$, we then have $N = N'$ and $N/Z(N) \cong W$, as desired. □

Corollary 7.4. *Let G be a finite group and let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be a chief series of G. Suppose that G_{k_1}/G_{k_1-1} and G_{k_2}/G_{k_2-1} are chief factors of G with $k_1 \neq k_2$ such that $G_{k_i}/G_{k_i-1} \cong W_i$, where W_i is a non-abelian simple group $(i = 1,2)$. Suppose in addition that, for $i = 1,2$ and $j = 1,\dots,k_i-1$, there do not exist subgroups K and H in $\mathrm{Aut}(G_j/G_{j-1})$ with $K \trianglelefteq H$ and $H/K \cong W_i$. Then there exist $N_1,N_2 \trianglelefteq G$ with $[N_1,N_2] = 1$ such that $N_i' = N_i$ and $N_i/Z(N_i) \cong W_i$ $(i = 1,2)$.*

Proof. By Lemma 7.3, there exist $N_1,N_2 \trianglelefteq G$ such that $N_i' = N_i$ and $N_i/Z(N_i) \cong W_i$ $(i = 1,2)$. Hence N_1 and N_2 are normal quasisimple subgroups of G. Assuming that $k_1 < k_2$, we see that W_2 cannot be embedded in $\mathrm{Aut}(W_1) \cong \mathrm{Aut}(G_{k_1}/G_{k_1-1})$. In particular, $N_1/Z(N_1) \cong W_1 \ncong W_2 \cong N_2/Z(N_2)$. Thus N_1 and N_2 are distinct components of G. Hence, by [5] Theorem 9.4, we have $[N_1,N_2] = 1$. □

Remark 7.5. We note the following properties of N_1 and N_2, as in Corollary 7.4.

(i) N_1N_2 is the central product of N_1 and N_2.

(ii) Let $Z = Z(N_1N_2)$. By simplicity, $N_i \cap Z = Z(N_i)$ $(i = 1,2)$. Hence $N_iZ/Z \cong N_i/Z(N_i) \cong W_i$ $(i = 1,2)$.

(iii) If $N_1Z/Z \cap N_2Z/Z \neq 1_{G/Z}$ then, by simplicity, $N_1Z/Z = N_2Z/Z$. But this contradicts $W_1 \ncong W_2$. Thus $N_1Z/Z \cap N_2Z/Z = 1_{G/Z}$, so $N_1N_2/Z = N_1Z/Z \times N_2Z/Z \cong W_1 \times W_2$.

In the remainder of this section we show that not all groups of order 12 or less can be embedded in a non-soluble group whose order is a divisor of $2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. We make use of Table 3, abstracted from [2] and [7], which lists the non-abelian simple groups whose orders divide $2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. The outer automorphism group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ and the Schur multiplier M of each group are also listed.

Order	Factorised Order	Group	M	$\mathrm{Aut}(G)/\mathrm{Inn}(G)$
60	$2^2 \cdot 3 \cdot 5$	A_5	C_2	C_2
168	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSL}(3,2) \cong \mathrm{GL}(3,2)$	C_2	C_2
360	$2^3 \cdot 3^2 \cdot 5$	A_6	C_6	$C_2 \times C_2$
504	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSL}(2,8)$	1	C_3
660	$2^2 \cdot 3 \cdot 5 \cdot 11$	$\mathrm{PSL}(2,11)$	C_2	C_2
1092	$2^2 \cdot 3 \cdot 7 \cdot 13$	$\mathrm{PSL}(2,13)$	C_2	C_2
2520	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	A_7	C_6	C_2
5616	$2^4 \cdot 3^3 \cdot 13$	$\mathrm{PSL}(3,3)$	1	C_2
6048	$2^5 \cdot 3^3 \cdot 7$	$\mathrm{U}_3(3)$	1	C_2
7920	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	M_{11}	1	1
9828	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	$\mathrm{PSL}(2,27)$	C_2	C_6

TABLE 3. Non-abelian finite simple groups with orders divising $2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$

Our next five results concern particular groups in which not all groups of order 8 or 9 can be embedded.

Lemma 7.6. *Let G be a finite group whose order is a divisor of $2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. Suppose that $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ is a chief series of G for which at least two distinct chief factors are not soluble. Then not all groups of order 8 can be embedded in G.*

Proof. We let G_{k_1}/G_{k_1-1} and G_{k_2}/G_{k_2-1}, $k_1 \neq k_2$, be distinct non-soluble chief factors of the chief series. For $i = 1,2$ we have $G_{k_i}/G_{k_i-1} \cong W_{i_1} \times \cdots \times W_{i_{n_i}}$, where $W_{i_j} \cong W_i$ for $j = 1,\ldots,n_i$ and W_i is a non-abelian simple group. As all $\{2,3\}$-groups are soluble, there exists a prime $p_i \in \{5,7,11,13\}$ such that p_i is a divisor of $|W_i|$ $(i = 1,2)$. If $n_i \geq 2$ then p_i^2 is a divisor of $|G_{k_i}/G_{k_i-1}|$ and hence of $|G|$. But this is

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excluded, so $n_i = 1$ $(i = 1, 2)$. Thus $G_{k_i}/G_{k_i-1} \cong W_i$ $(i = 1, 2)$. Now $|W_1||W_2| = |G_{k_1}/G_{k_1-1}||G_{k_2}/G_{k_2-1}|$, so $|W_1||W_2|$ divides $|G|$. In particular, $p_1p_2 \,\big|\, |G|$, so $p_1 \neq p_2$. Hence $W_1 \ncong W_2$.

From Table 3 we see that $2^2 \cdot 3 \cdot p_i$ is a divisor of $|W_i|$ $(i = 1, 2)$. Since $|W_1||W_2|$ is a divisor of $|G|$, the remaining chief factors have orders dividing $2 \cdot 3 \cdot r_1 \cdot r_2$, where $\{r_1, r_2\} = \{5, 7, 11, 13\} \setminus \{p_1, p_2\}$. But no group in Table 3 has an order that divides $2 \cdot 3 \cdot r_1 \cdot r_2$. Thus the remaining chief factors, if any, must be of order 2, 3, 5, 7, 11 or 13. These are all primes, so the automorphism groups of the remaining chief factors are cyclic. We note further from Table 3 that $p_1 \nmid |\mathrm{Aut}(W_2)|$ and that $p_2 \nmid |\mathrm{Aut}(W_1)|$. Hence, by Corollary 7.4, there exist $N_1, N_2 \trianglelefteq G$ such that $[N_1, N_2] = 1$; $N_i' = N_i$; and $N_i/Z(N_i) \cong W_i$ $(i = 1, 2)$. Letting $Z = Z(N_1N_2)$, we see by Remark 7.5 that $N_1N_2/Z = N_1Z/Z \times N_2Z/Z \cong W_1 \times W_2$. Since $4 \,\big|\, |W_i|$ $(i = 1, 2)$, we then have $16 \,\big|\, |N_1N_2/Z|$.

We let S_i be a Sylow 2-subgroup of N_i $(i = 1, 2)$. We have $[N_1, N_2] = 1$, so S_1S_2 a Sylow 2-subgroup of N_1N_2. In addition, $S_1 \cap S_2 \leqslant Z \cap S_1S_2$. Now S_iZ/Z is a Sylow 2-subgroup of N_iZ/Z and $N_iZ/Z \cong W_i$. Thus $4 \,\big|\, |S_i/(S_i \cap Z)|$ $(i = 1, 2)$. We have $S_1 \cap S_2 \leqslant Z$, so $4 \,\big|\, |S_i/(S_1 \cap S_2)|$. Hence $|S_i| \geq 4|S_1 \cap S_2|$, so $|S_1S_2| \geq \dfrac{4|S_1 \cap S_2| \cdot 4|S_1 \cap S_2|}{|S_1 \cap S_2|} = 16|S_1 \cap S_2|$. But $2^5 \top |G|$, so $|S_1 \cap S_2| \leq 2$.

If $|S_1 \cap S_2| = 2$, then $|S_1S_2| = 32$ and $|S_i| = 8$ $(i = 1, 2)$. In particular, S_1S_2 is a Sylow 2-subgroup of G. If both S_1 and S_2 have exponent at most 4, then the central product S_1S_2 has exponent at most 4 and C_8 cannot be embedded in G. If, say, S_1 has exponent 8, then $S_1 \cong C_8$ and $S_1 \leqslant Z(S_1S_2)$. If $C_2 \times C_2 \times C_2$ can be embedded in G then S_1S_2 contains a subgroup $V \cong C_2 \times C_2 \times C_2$. By comparison of orders, S_1S_2 is the product of the central subgroup S_1 and the abelian subgroup V, so S_1S_2 is abelian and the non-abelian groups of order 8 cannot be embedded in G. If $|S_1 \cap S_2| = 1$, then $S_1S_2 = S_1 \times S_2$. If, say, $|S_1| = 8$, then $|S_2| = 4$, so S_2 is abelian. In addition, $S_1 \times S_2$ is a Sylow 2-subgroup of G. If C_8 can be embedded in G, then $S_1 \cong C_8$, so S_1S_2 is abelian. Again, the non-abelian groups of order 8 cannot be embddded in G. Finally if $|S_1 \cap S_2| = 1$ and $|S_1| = |S_2| = 4$, then S_1S_2 is abelian of order 16 and exponent at most 4. Hence, by Lemma 2.1, not all groups of order 8 can be embedded in G. □

Lemma 7.7. *Let $H = D\langle y \rangle$ be a group of order 16 for subgroups D and $\langle y \rangle$ such that $D \cong D_4$ and $\langle y \rangle \cong C_2$. Suppose, in addition, that $C_H(D) = Z(D)$. Then H has a unique cyclic subgroup of order 4. In particular, Q_2 cannot be embedded in H.*

Proof. Since $|H : D| = 2$, we have $D \trianglelefteq H$. We let $D = \langle a, b \mid a^2 = b^4 = 1, aba = b^{-1} \rangle$. Since $\mathrm{Aut}(D_4) \cong D_4$, we have $H/C_H(D) = H/\langle b^2 \rangle \cong D_4$. Since $\langle b \rangle$ is characteristic in D, we have $\langle b \rangle \trianglelefteq H$. Now $o(y) = o(a) = 2$ and $y \notin \langle a, b \rangle$, so $y\langle b \rangle$ and $a\langle b \rangle$ are distinct elements of order 2 in $H/\langle b \rangle$. In addition, $|H/\langle b \rangle| = 4$, so $H/\langle b \rangle \cong C_2 \times C_2$. Thus $\Phi(H) \leqslant \langle b \rangle$. Since $H/\langle b^2 \rangle \cong D_4$, we see that $\Phi(H) \neq \langle b^2 \rangle$. Thus $\Phi(H) = \langle b \rangle$.

Now $H = \langle a, b, y \rangle$, so $H = \langle a, y \rangle$. Since $\Phi(H)$ is cyclic, we have $H' = \langle [a, y] \rangle$. If $[a, y] \in \langle b^2 \rangle$ then $H/\langle b^2 \rangle$ is abelian, which contradicts $H/\langle b^2 \rangle \cong D_4$. Hence either $[a, y] = b$, or $[a, y] = b^{-1}$. Thus either $a^y = ab$ or $a^y = ab^{-1}$. Since $\langle b \rangle \trianglelefteq H$ and $\langle b \rangle \cong C_4$, we have either $b^y = b$ or $b^y = b^{-1}$. If $b^y = b$ and $a^y = ab$, then the contradiction $a = a^{y^2} = (ab)^y = a^y b^y = abb = ab^2 \neq a$, arises. Similarly if $b^y = b$ and $a^y = ab^{-1}$, we derive the contradiction $a = a^{y^2} = ab^{-2}$. We thus conclude $b^y = b^{-1}$.

We now determine the elements of order 4 in H. If $h \in H$ and $o(h) = 4$, then h^2 is an element of order 2 in $\Phi(H) = \langle b \rangle$. Thus $o(h) = 4$ if and only if $h^2 = b^2$. We let $h = a^\alpha b^\beta y^\gamma$, for suitable α, β and γ. If $\gamma \equiv 0 \pmod 2$, then $h = a^\alpha b^\beta \in \langle a, b \rangle \cong D_4$, so $h \in \langle b \rangle$. If $\gamma \equiv 1 \pmod 2$ and $\alpha \equiv 0 \pmod 2$, then $h = b^\beta y$. In this case $o(h) = 4$ if and only if $b^2 = h^2 = (b^\beta y)^2 = (b^\beta y)(b^\beta y) = b^\beta (b^\beta)^y = b^\beta b^{-\beta} = 1$, which is a contradiction.

If $\gamma \equiv 1 \pmod 2$ and $\alpha \equiv 1 \pmod 2$, then $h = ab^\beta y$, so $o(h) = 4$ if and only if $b^2 = (ab^\beta y)^2 = ab^\beta y a b^\beta y = ab^\beta a^y (b^\beta)^y = ab^\beta a^y b^{-\beta}$. If, in addition, $a^y = ab$, then $b^2 = ab^\beta abb^{-\beta} = b^{-\beta} bb^{-\beta} = bb^{-2\beta}$, and the contradiction $b = b^2 b^{2\beta} \in \langle b^2 \rangle$ ensues. Finally, if $\gamma \equiv \alpha \equiv 1 \pmod 2$ and $a^y = ab^{-1}$, then $o(h) = 4$ if and only if $b^2 = ab^\beta ab^{-1} b^{-\beta} = b^{-\beta} b^{-1} b^{-\beta} = b^{-1} b^{-2\beta}$, so $b^{-1} = b^2 b^{2\beta} \in \langle b^2 \rangle$, which is again a contradiction. Thus b and b^{-1} are the only elements of order 4 in H, so $\langle b \rangle$ is the unique cyclic subgroup of order 4 in H. □

Corollary 7.8. *Let G be a group of order 32 such that $G = (D \times \langle x \rangle)\langle y \rangle$ for subgroups D, $\langle x \rangle$ and $\langle y \rangle$ such that $D \cong D_4$; $\langle x \rangle \cong C_2$; $\langle x \rangle \trianglelefteq G$; and $y^2 \in \langle x \rangle$. If Q_2 can be embedded in G, then G has exponent 4. In particular, C_8 cannot be embedded in G.*

Proof. We let $Q \leqslant G$ be such that $Q \cong Q_2$. Since $|D \times \langle x \rangle| = 16$ and $|G| = 32$, we have $|Q \cap (D \times \langle x \rangle)| \geq 4$. Now $D_4 \times C_2$ has exactly four elements of order 4 and Q_2 has six elements of order 4. Hence $Q \nleqslant D \times \langle x \rangle$, so $|Q \cap (D \times \langle x \rangle)| = 4$. We let $Q \cap (D \times \langle x \rangle) = \langle x_1 \rangle$, where $o(x_1) = 4$. Now $\Phi(D \times \langle x \rangle) = D' \cong C_2$, so $\langle x_1^2 \rangle = D'$. Hence $Q \cap \langle x \rangle = 1$. Now $D\langle x \rangle/\langle x \rangle \cong D_4$ and $\langle y, x \rangle/\langle x \rangle \cong C_2$. Hence, by comparison of orders, $D\langle x \rangle/\langle x \rangle \cap \langle y, x \rangle/\langle x \rangle = 1_{G/\langle x \rangle}$. We have $Q\langle x \rangle/\langle x \rangle \cong Q$, so Q_2 can be embedded in $G/\langle x \rangle$. Hence, by Lemma 7.7, $Z(D\langle x \rangle/\langle x \rangle)$ is a proper subgroup of $C_{G/\langle x \rangle}(D\langle x \rangle/\langle x \rangle)$. Now $|D\langle x \rangle/\langle x \rangle \cap C_{G/\langle x \rangle}(D\langle x \rangle/\langle x \rangle)| = |Z(D\langle x \rangle/\langle x \rangle)| = 2$ so, by comparison of orders, we have $|C_{G/\langle x \rangle}(D\langle x \rangle/\langle x \rangle)| = 4$. Thus $G/\langle x \rangle$ is the central product of $D\langle x \rangle/\langle x \rangle$ and $C_{G/\langle x \rangle}(D\langle x \rangle/\langle x \rangle)$, so $G/\langle x \rangle$ has exponent 4. If G does not have exponent 4, then we let $g \in G$ be such that $o(g) = 8$. Since $|G : D \times \langle x \rangle| = 2$, we have $g^2 \in D \times \langle x \rangle$. Hence $\langle g^4 \rangle = \Phi(D \times \langle x \rangle) = D'$. But $G/\langle x \rangle$ has exponent 4, so $g^4 \in \langle x \rangle$ and the contradiction $g^4 \in D' \cap \langle x \rangle = 1$ ensues. □

Lemma 7.9. *Let G be a finite group such that $2^5 \top |G|$ and such that G has a subgroup U with $U \cong A_4$. Suppose, in addition, that G has a normal subgroup, N, with the following properties:*

(a) $N = N'$;

(b) $Z(N/Z(N)) = 1$;

(c) $4 \,\big|\, |N/Z(N)|$;

(d) $4 \nmid |Z(N)|$;

(e) $16 \nmid |\mathrm{Aut}(N/Z(N))|$;

(f) $3 \nmid |\mathrm{Aut}(N/Z(N))/\mathrm{Inn}(N/Z(N))|$.

Then not all groups of order 8 *can be embedded in* G.

Proof. We let $Z = Z(N)$ and have $Z = A \times B$, where A is a 2-group and B is a $2'$-group. Then $B \trianglelefteq G$ and $U \cap B \leqslant \mathrm{O}_{2'}(U) = 1$. In addition, the Sylow 2-subgroups of G/B are isomorphic to those of G. Since $Z(N/Z) = 1$, we have $Z(N/B) = Z/B$. Hence $Z((N/B)/Z(N/B)) = Z((N/B)/(Z/B)) \cong Z(N/Z)) = 1$. Thus we may work "modulo B" and assume, without loss of generality, that $B = 1$. We let $C/Z = C_{G/Z}(N/Z)$. Then $C \trianglelefteq G$. Since $Z(N/Z) = 1$, we have $C \cap N = Z$. By Corollary 7.2, we further have $C = C_G(N)$. We let S_1 be a Sylow 2-subgroup of N and let S_2 be a Sylow 2-subgroup of C. Then S_1S_2 is a central product, with $S_1 \cap S_2 = Z$. In addition $N/Z = N/(N \cap C) \cong NC/C \cong \mathrm{Inn}(N/Z)$. Hence $G/NC \cong (G/C)/(NC/C)$ is isomorphic to a subgroup of $\mathrm{Aut}(N/Z)/\mathrm{Inn}(N/Z)$. But $3 \nmid |\mathrm{Aut}(N/Z)/\mathrm{Inn}(N/Z)|$, so $3 \nmid |G/NC|$ and $\mathrm{O}^{3'}(G) \leqslant NC$. In particular $U \leqslant NC$. Since $2^5 \top |G|$ and $16 \nmid |\mathrm{Aut}(N/Z)|$, we have $4 \big| |C|$. In addition, since $4 \big| |N/Z|$, we see that either $2^2 \top |C|$ or $2^3 \top |C|$.

Suppose first that $2^2 \top |C|$. Then $|S_2| = 4$. If $Z = 1$, then $S_1S_2 = S_1 \times S_2$. Since $4 \big| |N/Z|$, S_1 has a subgroup of order 4, H, say. Then $H \times S_2$ is abelian and has order 16 and exponent at most 4. By Lemma 2.1, not all groups of order 8 can be embedded in G. If $Z \neq 1$ then, since $4 \nmid |Z|$, we have $Z \cong C_2$. In particular, $2 \top |C/Z|$. If $U' \cap N = 1$, then $C_2 \times C_2 \cong U' \cong U'N/N \leqslant NC/N \cong C/(N \cap C) = C/Z$. But $2 \top |C/Z|$, so a contradiction arises. Thus $U' \cap N \neq 1$ so, by minimal normality, $U' \leqslant N$. In particular, $U'Z \leqslant N$. Since $U' \cap Z \trianglelefteq U$ and $Z \cong C_2$, it again follows by minimal normality that $U' \cap Z = 1$. Hence $U'Z = U' \times Z \cong C_2 \times C_2 \times C_2$. Since $|S_2| = 4$ and $S_2 \cap (U'Z) \leqslant S_1 \cap S_2 = Z \cong C_2$, we have $|S_2(U'Z)| = 16$. In addition $S_2(U'Z)$ is a central product of abelian groups of exponent at most 4. By Lemma 2.1 we then see that not all groups of order 8 can be embedded in G.

We now assume that $2^3 \top |C|$. Then $|S_2| = 8$. If $Z = 1$, then $S_1S_2 = S_1 \times S_2$. But $4 \big| |N/Z|$, so $|S_1| = 4$ and $S_1 \times S_2$ is a Sylow 2-subgroup of G. If C_8 can be embedded in G, then $S_2 \cong C_8$. Thus $S = S_1 \times S_2$ is abelian, so the non-abelian groups of order 8 cannot be emebdded in G. Hence we can assume that $Z \neq 1$, so $Z \cong C_2$. Suppose, in addition that both $U' \cap N = 1$ and $U' \cap C = 1$. As in the preceding paragraph, we see that $C_2 \times C_2$ can be embedded in C/Z; whilst a similar argument shows that $C_2 \times C_2$ can also be embedded in N/Z. By comparison of orders, $S_1S_2/Z = S_1/Z \times S_2/Z \cong C_2 \times C_2 \times C_2 \times C_2$. But $Z \cong C_2$, so the Sylow 2-subgroups of G have exponent at most 4 and C_8 cannot be embedded in G.

We may thus assume that either $U' \cap C \neq 1$ or $U' \cap N \neq 1$. Suppose that $U' \cap C \neq 1$. Then, by minimal normality, $U' \leqslant C$. As above, $U'Z = U' \times Z \cong C_2 \times C_2 \times C_2$. Hence $S_2 = U'Z$. Letting $H \leqslant S_1$ be such that $|H| = 4$, we see as above that HS_2 is an abelian group of order 16 and exponent at most 4. By Lemma 2.1, we see that not all groups of order 8 can be embedded in G. Finally if $U' \cap N \neq 1$, a similar argument again shows that not all groups of order 8 can be embedded in G. ☐

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Lemma 7.10. *Let G be a finite group such that $3^3 \nmid |G|$. Suppose that H is a subgroup of G such that $3 \,\big|\, |Z(H)|$ and such that $H/Z(H)$ has a subgroup isomorphic to A_6. Then C_9 cannot be embedded in G.*

Proof. We let $K \leqslant H$ be such that $Z(H) \leqslant K$ and $K/Z(H) \cong A_6$. Since $9 \,\big|\, |A_6|$ we have $27 \,\big|\, |K|$. Thus the Sylow 3-subgroups of G are isomorphic to those of K. We suppose that C_9 can be embedded in G, and let $U \leqslant K$ be such that $U \cong C_9$. We identify $K/Z(H)$ with A_6 and assume, without loss of generality, that $Z(H) \cong C_3$. Since the Sylow 3-subgroups of A_6 are isomorphic to $C_3 \times C_3$, we have $Z(H) \leqslant U$. Consideration of the possible cycle types of order three in A_6 shows that we can identify $U/Z(H)$ with $\langle(123)\rangle$ or $\langle(123)(456)\rangle$. Now, $\langle(123),(12)(45)\rangle$ and $\langle\ (123)(456),(12)(45)\rangle$ are both isomorphic to S_3. Hence there exists $x \in K$ such that $\langle x, U\rangle/Z(H) \cong S_3$. Since $Z(H) \cong C_3$, we may assume that $o(x) = 2$. Then conjugation by x induces an automorphism of order 2 on $U/Z(H) = U/\Phi(U)$, and hence on U. It follows that conjugation by x inverts every element of U. But $\Phi(U) = Z(H)$, so a contradiction arises. We conclude that C_9 cannot be embedded in G. □

We use our next result in the proof of Proposition 7.13, the main result of this section.

Lemma 7.11. *There exists an element $y \in \mathrm{Aut}(\mathrm{GL}(3,2))$ with $o(y) = 2$ such that $\mathrm{Aut}(\mathrm{GL}(3,2)) = \mathrm{Inn}(\mathrm{GL}(3,2))\langle y\rangle$.*

Proof. We let $W = \mathrm{GL}(3,2)$ and identify W with $\mathrm{Inn}(W)$. From Table 3 we have $|\mathrm{Aut}(W) : W| = 2$. We let R be a Sylow 7-subgroup of W and have $R \ntrianglelefteq W$. Hence, by Sylow's Theorems, $|W : N_W(R)| = 8$, so $|N_W(R)| = 21$. Thus there is a non-trivial homomorphism from W into S_8. But W is simple, so W can be embedded in S_8.

We let P be a Sylow 3-subgroup of $N_W(R)$. Then P is also a Sylow 3-subgroup of W. We have $P \cong C_3$ and $N_W(R) = PR$. If $N_W(R)$ is abelian then $N_W(R) \cong C_{21}$, so W possesses an element of order 21. But W can be embedded in S_8, so a contradiction arises. Thus $N_W(R)$ is non-abelian, so P normalises, but does not centralise, R. Hence P has exactly 7 conjugates in $N_W(R)$. Then, by Sylow's Theorems, P has either 7 or 28 conjugates in W. If P has 7 conjugates in W, then $\mathrm{O}^{3'}(W) = \langle P^r \mid r \in R\rangle = N_W(R) \trianglelefteq W$. But this contradicts the simplicity of W. Thus P has 28 conjugates in W, so $|W : N_W(P)| = 28$ and $|N_W(P)| = 6$. Now S_3 can be embedded in $\mathrm{GL}(3,2)$ and S_3 has a normal Sylow 3-subgroup. But $|N_W(P)| = 6$, so $N_W(P) \cong S_3$. Letting $\langle x\rangle$ be a Sylow 2-subgroup of $N_W(P)$, we have $\langle x\rangle \cong C_2$ and $N_W(P) = \langle x\rangle P$.

Since W is identified with $\mathrm{Inn}(W)$, we see by the Frattini argument that $\mathrm{Aut}(W) = WN_{\mathrm{Aut}(W)}(P)$. It follows that $|N_{\mathrm{Aut}(W)}(P) : N_W(P)| = |N_{\mathrm{Aut}(W)}(P) : W \cap N_{\mathrm{Aut}(W)}(P)| = |WN_{\mathrm{Aut}(W)}(P) : W| = |\mathrm{Aut}(W) : W| = 2$. Thus $|N_{\mathrm{Aut}(W)}(P)| = 2|N_W(P)| = 12$. We let S be a Sylow 2-subgroup of $N_{\mathrm{Aut}(W)}(P)$ such that $\langle x\rangle \leqslant S$. Then $|S| = 4$. Since x does not centralise P, we have $|C_S(P| = 2$ and

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$S = \langle x\rangle \times C_S(P)$. Now $C_S(P) \cap W \leqslant S \cap C_W(P) = S \cap P = 1$. We let $C_S(P) = \langle y\rangle$. Then $o(y) = 2$ and $y \notin W$. Thus $\mathrm{Aut}(W) = W\langle y\rangle$, with $o(y) = 2$, as desired. □

Corollary 7.12. *Let S be a Sylow 2-subgroup of $\mathrm{Aut}(\mathrm{GL}(3,2))$. Then there exists an element $y \in \mathrm{Aut}(\mathrm{GL}(3,2))$ such that $o(y) = 2$ and $S = (S \cap \mathrm{Inn}(\mathrm{GL}(3,2)))\langle y\rangle$, where $S \cap \mathrm{Inn}(\mathrm{GL}(3,2)) \cong D_4$ and $(S \cap \mathrm{Inn}(\mathrm{GL}(3,2))) \cap \langle y\rangle = 1$.*

Proof. Let $\langle y\rangle$ be as in Lemma 7.11 and let S be a Sylow 2-subgroup of $\mathrm{Aut}(\mathrm{GL}(3,2))$ with $y \in S$. Then $S \cap \mathrm{Inn}(\mathrm{GL}(3,2))$ is a Sylow 2-subgroup of $\mathrm{Inn}(\mathrm{GL}(3,2))$. Now the Sylow 2-subgroups of $\mathrm{GL}(3,2)$ are isomorphic to D_4, so $S \cap \mathrm{Inn}(\mathrm{GL}(3,2)) \cong D_4$. Since $o(y) = 2$ and $y \notin \mathrm{Inn}(\mathrm{GL}(3,2))$, it follows, by comparison of orders, that $S = (S \cap \mathrm{Inn}(\mathrm{GL}(3,2)))\langle y\rangle$ and $(S \cap \mathrm{Inn}(\mathrm{GL}(3,2))) \cap \langle y\rangle = 1$. □

Proposition 7.13. *Let G be a finite group whose order is a divisor of $2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. Suppose that $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ is a chief series of G for which exactly one chief factor, G_k/G_{k-1}, is not soluble. Then not all groups of order 12 or less can be embedded in G.*

Proof. As in the proof of Lemma 7.6, we have $G_k/G_{k-1} \cong W$, where W is a non-abelian finite simple group. By Lemma 7.6 we may also assume that the remaining chief factors are elementary abelian. From Table 3 we see that $2^2 \,\big|\, |W|$ and that, where $2^2 \top |W|$, then either 5, 11 or 13 is also a divisor of $|W|$. If $2^2 \top |W|$, then the maximum possible rank of a 2-chief factor is $r = 3$. Moreover if G_i/G_{i-1} is a 2-chief factor of rank 3, then $\mathrm{Aut}(G_i/G_{i-1}) \cong \mathrm{GL}(3,2)$, so $|W| \nmid |\mathrm{Aut}(G_i/G_{i-1})|$. In addition $3 \,\big|\, |W|$, so the maximum possible rank of a 3-chief factor is $r = 2$. We further note that $\mathrm{Aut}(C_3 \times C_3) \cong \mathrm{GL}(2,3)$ is soluble. Thus the conditions for Lemma 7.3 are satisfied, so there exists $N \trianglelefteq G$ such that $N = N'$ and $N/Z(N) \cong W$. We let $C/Z(N) = C_{G/Z(N)}(N/Z(N))$. By Corollary 7.2, $C = C_G(N)$, so NC is the central product of N and C. In particular, $C \cap N = Z(N)$. Letting S_1, P_1, S_2 and P_2 be Sylow 2- and 3-subgroups of N and C respectively, we see that the central products S_1S_2 and P_1P_2 are Sylow 2- and 3-subgroups respectively of NC. We deal with the possible isomorphism types for W in ascending order of $|W|$. We let M denote the Schur multiplier of W and note that $Z(N)$ will always be isomorphic to some factor group of M (see [2], p.xvii).

Case 1. $|W| = 60 = 2^2 \cdot 3 \cdot 5$; $W \cong A_5$; $M \cong C_2$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$.

Here Z(N) is isomorphic to a factor group of $M \cong C_2$, so $4 \nmid |Z(N)|$. In addition, we have $16 \nmid |\mathrm{Aut}(N/Z(N))|$ and $3 \nmid |\mathrm{Aut}(N/Z(N))/\mathrm{Inn}(N/Z(N))|$. Hence, by Lemma 7.9, if A_4 can be embedded in G, then not all groups of order 8 can be embedded in G.

Case 2. $|W| = 168 = 2^3 \cdot 3 \cdot 7$; $W \cong \mathrm{PSL}(3,2) \cong \mathrm{GL}(3,2)$; $M \cong C_2$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$.

Since $M \cong C_2$, either $Z(N) = 1$ or $Z(N) \cong C_2$. In addition, $Z(N) = S_1 \cap S_2$. Since $Z(N)$ is a 2-group, we have $|P_1| = 3$. Now G/C is isomorphic to a subgroup of $\mathrm{Aut}(W)$ and $3 \top |\mathrm{Aut}(W)|$, so $3^2 \top |C|$ and

$|P_2| = 9$. In addition $P_1 \cap P_2 \leqslant N \cap C = Z(N)$. But $|Z(N)| \leq 2$, so $P_1 \cap P_2 = 1$ and $|P_1P_2| = |P_1||P_2| = 27$. Hence $P_1P_2 = P_1 \times P_2$ is a Sylow 3-subgroup of G. Now $NC \trianglelefteq G$, so $\mathrm{O}^{3'}(G) \leqslant NC$.

We suppose that A_4 can be embedded in G and let $U \leqslant G$ be such that $U \cong A_4$. Then $U \leqslant \mathrm{O}^{3'}(G) \leqslant NC$. If $U \cap N = 1$, then $UN/N \cong U$. But $CN/N \cong C/(C \cap N) = C/Z(N)$, so $C/Z(N)$ has a subgroup isomorphic to U. In particular, $C/Z(N)$ has a subgroup isomorphic to $U' \cong C_2 \times C_2$. Now $S_1/Z(N)$ is isomorphic to a Sylow 2-subgroup of W, so $S_1/Z(N) \cong D_4$. But $U' \cong C_2 \times C_2$ can be embedded in $S_2/Z(N)$, so by comparison of orders, $S_1 \cap S_2 = Z(N) = 1$ and the Sylow 2-subgroups of G are isomorphic to $S_1 \times S_2 \cong D_4 \times C_2 \times C_2$. Then the Sylow 2-subgroups of G have exponent 4, so C_8 cannot be embedded in G.

We may now assume that $U \cap N \neq 1$ so, by minimal normality, $U' \leqslant N$. Now there exists $x \in U$ such that $o(x) = 3$ and $[x, U'] = U'$. Without loss of generality, we assume that $x \in P_1 \times P_2$ and let $x = x_1x_2$, with $x_1 \in P_1$ and $x_2 \in P_2$. Note that $o(x_i) = 1$ or $o(x_i) = 3$ $(i = 1, 2)$. If $o(x_1) = 1$ then $x = x_2$, so $U' = [x_2, U'] \leqslant C$. Then $U' \leqslant C \cap N = Z(N)$. But $|Z(N)| \leq 2$, so a contradiction arises. Hence $o(x_1) = 3$. Now $x_2 \in C$, so x_2 centralises N. In particular, x_2 centralises U'. But x_2 also centralises $x = x_1x_2$, so x_2 centralises $U = \langle x_1x_2, U'\rangle$. Moreover $[x_1, U'] = [x_1x_2, U'] = U'$, so x_1 normalises U'. Since $Z(N)$ is a 2-group and x_2 is a 3-element such that $\langle x_2\rangle \cap U'\langle x_1\rangle \leqslant C \cap N = Z(N)$, we further have $\langle x_2\rangle \cap U'\langle x_1\rangle = 1$. Hence $U\langle x_2\rangle = U'\langle x_1x_2, x_2\ \rangle = U'\langle x_1\rangle\langle x_2\rangle = U'\langle x_1\rangle \times \langle x_2\rangle$. Thus $U'\langle x_1\rangle \cong U\langle x_2\rangle/\langle x_2\rangle \cong U \cong A_4$, so A_4 can be embedded in N. Therefore, without loss of generality, we assume that $U \leqslant N$.

Suppose now that $Z(N) = 1$. Then S_1 is isomorphic to a Sylow 2-subgroup of $\mathrm{GL}(3,2)$, so $S_1 \cong D_4$. In particular, S_1 has exponent 4. Now $|\operatorname{Aut}(W) : \operatorname{Inn}(W)| = 2$, so $2^4 \top |\operatorname{Aut}(W)|$. Since $2^3 \top |W|$, it follows that either $2^2 \top |C|$ or $2 \top |C|$. If $2^2 \top |C|$ then $|S_2| = 4$. But $Z(N) = 1$, so $S_1S_2 = S_1 \times S_2$ is a Sylow 2-subgroup of G and the Sylow 2-subgroups of G have exponent 4. Thus C_8 cannot be embedded in G. If $2 \top |C|$, then $S_2 \cong C_2$ and $S_1S_2 \cong D_4 \times C_2$. Letting S be a Sylow 2-subgroup of G that contains S_1S_2, we have $S_2 = S \cap C = C_S(N) \trianglelefteq S$. Since $2^4 \top |\operatorname{Aut}(\mathrm{GL}(3,2))|$, we see that S/S_2 is isomorphic to a Sylow 2-subgroup of $\operatorname{Aut}(\mathrm{GL}(3,2))$. We identify S/S_2 with a Sylow 2-subgroup of $\operatorname{Aut}(\mathrm{GL}(3,2))$ and S_1S_2/S_2 with a Sylow 2-subgroup of $\operatorname{Inn}(\mathrm{GL}(3,2))$. By Corollary 7.12, there exists $y \in S$ with $\langle y\rangle S_2/S_2 \cong C_2$ such that $S/S_2 = (S_1S_2/S_2)\langle y\rangle S_2/S_2$. In particular, $y^2 \in S_2$. Then, by Corollary 7.8, not all groups of order 8 can be embedded in G.

We may thus assume that $Z(N) \cong C_2$. Now $Z(N) \leqslant C$ and, as above, either $2^2 \top |C|$ or $2 \top |C|$. If $2^2 \top |C|$, then $|S_2| = 4$, so S_2 is abelian. Since $U \leqslant N$, we have $U' \cap C \leqslant Z(U) = 1$ and $[U', C] = 1$. In particular, $U' \cap S_2 = 1$ and $[U', S_2] = 1$. Thus $U'S_2 = U' \times S_2$. But $U' \cong C_2 \times C_2$ and $|S_2| = 4$, so $U'S_2$ is abelian of order 16 and exponent at most 4. By Lemma 2.1, not all groups of order 8 can be embedded in G. We thus assume that $2 \top |C|$. Then $Z(N) = S_1 \cap S_2 = S_2$ is a Sylow 2-subgroup of C. In particular, $S_2 \trianglelefteq G$. Since $U \leqslant N$, we may assume, without loss of generality, that $U' \leqslant S_1$. By minimal

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normality, we have $U \cap S_2 = U' \cap S_2 = 1$, so $US_2/S_2 \cong U \cong A_4$. Now $|U'S_2| = |U'||S_2| = 8$ and $|S_1| = 16$, so $|S_1 : U'S_2| = 2$ and $U'S_2 \trianglelefteq S_1$. Thus $\langle U, S_1\rangle \leqslant N_N(U'S_2)$. Since $3 \,\big|\, |U|$ we have $48 \,\big|\, |N_N(U'S_2)|$. Then $|N/S_2 : N_N(U'S_2)/S_2| \,\big|\, 7$. But $N/S_2 \cong \mathrm{GL}(3,2)$ so, by simplicity, $|N_N(U'S_2)| = 48$. By comparison of orders, we further see that $N_N(U'S_2) = US_1$.

Now $|US_2| = 24$, so $|US_1 : US_2| = 2$ and $US_2 \trianglelefteq US_1$. Since $US_2 \cong A_4 \times C_2$, we have $U = \mathrm{O}^{3'}(US_2) \trianglelefteq US_1$. In particular, $U' \trianglelefteq US_1$. We let $y_1 \in S_1 \setminus U'S_2$. Then y_1 centralises S_2 and normalises U'. Now $U'S_2 = U' \times S_2 \cong C_2 \times C_2 \times C_2$, so $U'S_2$ is abelian. Thus $S_1' = [U'S_2, \langle y_1\rangle] = [U', \langle y_1\rangle][S_2, \langle y_1\rangle] = [U', \langle y_1\rangle] \leqslant U'$. Since S_1 is 2-group, $[U', \langle y_1\rangle]$ is a proper subgroup of U'. But S_1/S_2 is isomorphic to a Sylow 2-subgroup of $\mathrm{GL}(3,2)$, so $S_1/S_2 \cong D_4$. Hence S_1 is non-abelian, so $S_1' = [U', \langle y_1\rangle] \cong C_2$. We let $S_1' = \langle y_2\rangle$ and let S be a Sylow 2-subgroup of G that contains S_1. Since $|S : S_1| = 2$, we have $S_1 \trianglelefteq S$, and $\langle y_2\rangle = S_1' \trianglelefteq S$. If $o(g) \geq 4$ for all $g \in S_1 \setminus U'S_2$, then $U'S_2 = \langle x \in S_1 \mid x^2 = 1\rangle = \Omega_1(S_1) \trianglelefteq S$. If Q_2 can be embedded in G, then we let $Q \leqslant S$ be such that $Q \cong Q_2$. Since Q_2 has a unique element of order 2, we have $|Q \cap U'S_2| = 2$ and $S = QU'S_2$. Then $S/U'S_2 = QU'S_2/U'S_2 \cong Q/(Q \cap U'S_2) \cong Q_2/Z(Q_2) \cong C_2 \times C_2$. Thus $S/U'S_2$ has exponent 2. But $U'S_2$ also has exponent 2, so S has exponent at most 4. Hence C_8 cannot be embedded in G.

We may finally assume that there exists $g \in S_1 \setminus U'S_2$ such that $o(g) = 2$. If $\langle g\rangle U' \cap S_2 \neq 1$, then $S_2 \leqslant \langle g\rangle U'$, so $S_1 = \langle g\rangle U'S_2 = \langle g\rangle U'$ and the contradiction $|S_1| = 8$ arises. We may thus assume that $\langle g\rangle U' \cap S_2 = 1$, so $\langle g\rangle U' \cong \langle g\rangle U'S_2/S_2 = S_1/S_2 \cong D_4$. Since $S_2 \leqslant Z(N)$, we have $S_1 = \langle g\rangle U' \times S_2 \cong D_4 \times C_2$. Now S/S_2 is isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}(\mathrm{GL}(3,2))$ so, by Corollary 7.12, there exists $y \in S$ such that $S/S_2 = (S_1/S_2)(\langle y\rangle S_2/S_2)$ and $\langle y\rangle S_2/S_2 \cong C_2$. In particular, $y^2 \in S_2$. Hence, by Corollary 7.8, not all groups of order 8 can be embedded in G.

Case 3. $|W| = 360 = 2^3 \cdot 3^2 \cdot 5$; $W \cong A_6$; $M \cong C_6$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2 \times C_2$.

If $3 \,\big|\, |Z(N)|$ then, by Lemma 7.10, C_9 cannot be embedded in G. If $3 \nmid |Z(N)|$ then, since $3^2 \,\big|\, |W|$ and $3 \nmid |\mathrm{Aut}(W)/\mathrm{Inn}(W)|$, we have $3 \top |C|$. Hence $P_2 \cong C_3$. But the Sylow 3-subgroups of A_6 are isomorphic to $C_3 \times C_3$, so $P_1 \cong C_3 \times C_3$. Thus $P_1P_2 = P_1 \times P_2 \cong C_3 \times C_3 \times C_3$ is a Sylow 3-subgroup of G, and again C_9 cannot be embedded in G.

Case 4. $|W| = 504 = 2^3 \cdot 3^2 \cdot 7$; $W \cong \mathrm{PSL}(2,8)$; $M = 1$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_3$.

Since $M = 1$ and $2^3 \top |W|$, we have $Z(N) = 1$ and $|S_1| = 8$. Now $2 \nmid |\mathrm{Aut}(W)/\mathrm{Inn}(W)|$, so $|S_2| = 4$. Then $S_1S_2 = S_1 \times S_2$ has order 32, so S_1S_2 is a Sylow 2-subgroup of G. We let $K \leqslant S_1$ be such that $|K| = 4$. Then $K \times S_2$ is abelian and has order 16 and exponent at most 4. Hence, by Lemma 2.1, not all groups of order 8 can be embedded in G.

Case 5. $|W| = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$; $W \cong \mathrm{PSL}(2,11)$; $M \cong C_2$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$.

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Since $M \cong C_2$ and $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$, we see by Lemma 7.9 that if A_4 can be embedded in G, then not all groups of order 8 can be embedded in G.

Case 6. $|W| = 1092 = 2^2 \cdot 3 \cdot 7 \cdot 13$; $W \cong \mathrm{PSL}(2,13)$; $M \cong C_2$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$.

The result for Case 6 follows as in Case 5.

Case 7. $|W| = 2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$; $W \cong A_7$; $M \cong C_6$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$.

As in Case 3, we see that C_9 cannot be embedded in G.

Case 8. $|W| = 5616 = 2^4 \cdot 3^3 \cdot 13$; $W \cong \mathrm{PSL}(3,3)$; $M = 1$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$.

By comparison of orders, the Sylow 3-subgroups of G are isomorphic to those of $N/Z(N) \cong \mathrm{PSL}(3,3)$. But the Sylow 3-subgroups of $\mathrm{PSL}(3,3)$ are isomorphic to those of $\mathrm{GL}(3,3)$, so the Sylow 3-subgroups of G have exponent 3. Hence C_9 cannot be embedded in G.

Case 9. $|W| = 6048 = 2^5 \cdot 3^3 \cdot 7$; $W \cong \mathrm{U}_3(3)$; $M = 1$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_2$.

Since $2^5 \top |N|$, we have $\mathrm{O}^{2'}(G) \leqslant N$. We see that $5 \nmid |\mathrm{U}_3(3)|$ and $5 \nmid |\mathrm{Aut}(\mathrm{U}_3(3))|$, so C contains a Sylow 5-subgroup of G. Hence $\mathrm{O}^{5'}(G) \leqslant C$, so $\mathrm{O}^{2'}(G) \cap \mathrm{O}^{5'}(G) \leqslant N \cap C = 1$ and $\mathrm{O}^{2'}(G)\,\mathrm{O}^{5'}(G) = \mathrm{O}^{2'}(G) \times \mathrm{O}^{5'}(G)$. Thus all 2-elements of G commute with all 5-elements of G. In particular, D_5 cannot be embedded in G.

Case 10. $|W| = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$; $W \cong \mathrm{M}_{11}$; $M = 1$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) = 1$.

By [8] Table 5.1, we see that A_6 can be embedded in a maximal subgroup of M_{11}. Now the Sylow 3-subgroups of A_6 are isomorphic to $C_3 \times C_3$ so, by comparison of orders, the Sylow 3-subgroups of M_{11} are isomorphic to $C_3 \times C_3$. Since $M = 1$, we have $N \cong \mathrm{M}_{11}$. Thus $P_1 \cong C_3 \times C_3$. Since $Z(N) = 1$, we have $N \cap C = 1$. Moreover $\mathrm{Aut}(\mathrm{M}_{11}) = \mathrm{Inn}(\mathrm{M}_{11})$, so $3 \top |C|$ and $P_2 \cong C_3$. Hence $P_1 P_2 = P_1 \times P_2 \cong C_3 \times C_3 \times C_3$ is a Sylow 3-subgroup of G, so C_9 cannot be embedded in G.

Case 11. $|W| = 9828 = 2^2 \cdot 3^3 \cdot 7 \cdot 13$; $W \cong \mathrm{PSL}(2,27)$; $M \cong C_2$; $\mathrm{Aut}(W)/\mathrm{Inn}(W) \cong C_6$.

Since $3^3 \top |\mathrm{GL}(2,27)|$, the Sylow 3-subgroups of $\mathrm{PSL}(2,27)$ are isomorphic to those of $\mathrm{GL}(2,27)$. But the Sylow 3-subgroups of $\mathrm{GL}(2,27)$ are isomorphic to the additive group of the field with 27 elements and are thus elementary abelian. Hence C_9 cannot be embedded in G. □

8. Minimal embeddings of groups of order n or less for $n \leq 15$

We finally construct examples to show that the bound given by Lemma 1.2 can be attained for $n = 1, \ldots, 11$, and that the bounds given by Theorem 6.5 are attained in both the soluble and non-soluble cases for $n = 12, \ldots, 15$. Using Theorem 4.1, we can verify that groups in Table 4 are minimal with respect to the embedding of all groups of order n or less for $1 \le n \le 11$. Table 4 shows, in particular, that the bound given by Lemma 1.2 can be achieved by more than one group for $n = 3, \ldots, 11$. We recall that H_1 refers to the quasi-dihedral (or semi-dihedral) group of order 16.

n	Examples of groups of minimal order in which all groups of order n or less can be embedded
1	C_1
2	C_2
3	$C_6,\ D_3$
4	$C_4 \times D_3,\ S_4$
5	$C_4 \times C_5 \times D_3,\ S_5$
6	$C_4 \times C_5 \times D_3,\ S_5$
7	$C_4 \times C_5 \times C_7 \times D_3,\ C_7 \times S_5$
8	$C_5 \times C_7 \times D_3 \times H_1,\ C_7 \times D_{15} \times H_1$
9	$C_5 \times C_7 \times C_9 \times D_3 \times H_1,\ C_7 \times C_9 \times D_{15} \times H_1$
10	$C_7 \times C_9 \times D_{15} \times H_1,\ C_9 \times D_{105} \times H_1$
11	$C_7 \times C_9 \times C_{11} \times D_{15} \times H_1,\ C_9 \times C_{11} \times D_{105} \times H_1$

Table 4. Some groups of minimal order in which all groups of order n or less can be embedded, for $1 \le n \le 11$

Table 4 shows that, for $n = 5$, 6 and 7, both soluble and insoluble minimal groups can be found. We now construct examples of groups of minimal order in which all groups of order n can be embedded for $12 \le n \le 15$. The groups in question are subgroups of the direct product of extensions of certain groups of small order by suitable transpositions.

Example 8.1. We define the groups $K_1, \ldots, K_7$ by:

$$K_1 = \langle a_1, a_2 \mid a_1^4 = a_2^4 = 1, a_1^2 = a_2^2, a_1^{-1}a_2a_1 = a_2^3\rangle \cong Q_2;$$

$$K_2 = \langle b_1, b_2, b_3 \mid b_1^2 = b_2^2 = b_3^3 = 1, [b_1, b_2] = 1, b_3^{-1}b_1b_3 = b_2, b_3^{-1}b_2b_3 = b_1b_2\rangle \cong A_4;$$

$$K_3 = \langle c\rangle \cong C_9;\ K_4 = \langle d\rangle \cong C_5;\ K_5 = \langle e\rangle \cong C_7;\ K_6 = \langle f\rangle \cong C_{11};\ K_7 = \langle g\rangle \cong C_{13}.$$

For each group K_i we define an automorphism θ_i (of order 2) as follows:

$$a_1^{\theta_1} = a_2,\ a_2^{\theta_1} = a_1;$$

$$b_1^{\theta_2} = b_2,\ b_2^{\theta_2} = b_1,\ b_3^{\theta_2} = b_3^{-1};$$

$$c^{\theta_3} = c^{-1};\quad d^{\theta_4} = d^{-1};\quad e^{\theta_5} = e^{-1};\quad f^{\theta_6} = f^{-1};\quad g^{\theta_7} = g^{-1}.$$

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We extend K_i by $\langle\theta_i\rangle$ to form the semi-direct product $K_i\langle\theta_i\rangle$ for $i = 1, \dots, 7$. Expressing the symmetric group $S_5 = A_5\langle(12)\rangle$ in cycle notation, we form the direct product:

$$G = K_1\langle\theta_1\rangle \times \cdots \times K_7\langle\theta_7\rangle \times A_5\langle(12)\rangle.$$

Note that G is a faithful extension of the direct product $Q_2 \times A_4 \times C_9 \times C_5 \times C_7 \times C_{11} \times C_{13} \times A_5$ by $\langle\theta_1, \dots, \theta_7, (12)\rangle$, which is an elementary abelian 2-group of rank 8. The following isomorphisms provide embeddings in G of those groups of order 15 or less that are not already subgroups of direct factors of $Q_2 \times A_4 \times C_9 \times C_5 \times C_7 \times C_{11} \times C_{13} \times A_5$.

$$D_3 \cong \langle c^3, \theta_3\theta\rangle,\ \theta \in \langle\theta_1, \theta_2, \theta_4, \dots, \theta_7, (12)\rangle;$$

$$D_3 \cong \langle(123), (12)\theta\rangle,\ \theta \in \langle\theta_1, \dots, \theta_7\rangle;$$

$$C_8 \cong \langle a_1\theta_1\theta\rangle,\ \theta \in \langle\theta_2, \dots, \theta_7, (12)\rangle;$$

$$C_2 \times C_4 \cong \langle a_1, b_1\rangle \cong \langle a_1, (12)(34)\rangle;$$

$$C_2 \times C_2 \times C_2 \cong \langle a_1^2, b_1, b_2\rangle \cong \langle a_1^2, (12)(34), (14)(32)\rangle;$$

$$D_4 \cong \langle a_1a_2, \theta_1\theta\rangle,\ \theta \in \langle\theta_2, \dots, \theta_7, (12)\rangle;$$

$$C_3 \times C_3 \cong \langle b_3, c^3\rangle \cong \langle c^3, (123)\rangle;$$

$$C_{10} \cong \langle a_1^2 d\rangle \cong \langle a_1^2(12345)\rangle;$$

$$D_5 \cong \langle(12345), (15)(24)\rangle \cong \langle\ d, \theta_4\theta\rangle,\ \theta \in \langle\theta_1, \theta_2, \theta_3, \theta_5, \theta_6, \theta_7\rangle;$$

$$C_2 \times C_2 \times C_3 \cong C_2 \times C_6 \cong \langle b_1, b_2c^3\rangle \cong \langle(12)(34), (14)(32)c^3\rangle;$$

$$C_{12} \cong \langle a_1c^3\rangle;$$

$$D_6 \cong \langle a_1^2 b_3, \theta_2\theta\rangle,\ \theta \in \langle\theta_1, \theta_3, \dots, \theta_7\rangle;$$

$$D_6 \cong \langle\ a_1^2(123),\ (12)\theta\ \rangle,\ \theta \in \langle\ \theta_1, \dots, \theta_7\ \rangle;$$

$$Q_3 \cong \langle b_1\theta_2\theta_3\theta, c^3\rangle,\ \theta \in \langle\theta_1, \theta_4, \dots, \theta_7\rangle;$$

$$Q_3 \cong \langle(14)(32)(12)\theta_3\theta, c^3\rangle,\ \theta \in \langle\theta_1, \theta_2, \theta_4, \dots, \theta_7\rangle;$$

$$C_{14} \cong \langle a_1^2 e\rangle;$$

$$D_7 \cong \langle\ e, \theta_5\theta\rangle,\ \theta \in \langle\theta_1, \dots, \theta_4, \theta_6, \theta_7, (12)\rangle;$$

$$C_{15} \cong \langle c^3 d\rangle \cong \langle c^3(12345)\rangle.$$

Bearing in mind the above embeddings, the isomorphism types of $K_1, \dots, K_7$ and the results of Theorem 4.1, we see that the following families of subgroups of G will attain the bounds given by Corollary 6.5 for the respective values of n. The subgroups in question are extensions of subgroups of

$K_1 \times \cdots \times K_7 \times A_5$ (referred to as "base groups") by suitable subgroups of order 2 in $\langle \theta_1, \ldots, \theta_7, (12) \rangle$. The order of the extension and the isomorphism type of the "base group" is given in each case.

$n = 12$ (soluble)

Extensions: $(K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6)\langle \theta_1\theta_2\theta_3\theta_4\theta \rangle$, $\theta \in \langle \theta_5, \theta_6 \rangle$

Order: $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 665\ 280$

Base Group: $Q_2 \times A_4 \times C_9 \times C_5 \times C_7 \times C_{11}$

$n = 12$ (non-soluble)

Extensions: $(K_1 \times K_3 \times K_5 \times K_6 \times A_5)\langle \theta_1\theta_3(12)\theta \rangle$, $\theta \in \langle \theta_5, \theta_6 \rangle$

Order: $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 665\ 280$

Base Group: $Q_2 \times C_9 \times C_7 \times C_{11} \times A_5$

$n = 13, 14, 15$ (soluble)

Extensions: $(K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \times K_7)\langle \theta_1\theta_2\theta_3\theta_4\theta_5\theta \rangle$, $\theta \in \langle \theta_6, \theta_7 \rangle$

Order: $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 8\ 648\ 640$

Base Group: $Q_2 \times A_4 \times C_9 \times C_5 \times C_7 \times C_{11} \times C_{13}$

$n = 13, 14, 15$ (non-soluble)

Extensions: $(K_1 \times K_3 \times K_5 \times K_6 \times K_7 \times A_5)\langle \theta_1\theta_3\theta_5(12)\theta \rangle$, $\theta \in \langle \theta_6, \theta_7 \rangle$

Order: $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 8\ 648\ 640$

Base Group : $Q_2 \times C_9 \times C_7 \times C_{11} \times C_{13} \times A_5$

These examples show that, for $n = 12, \ldots, 15$, the bound given by Theorem 6.5 is attained by both soluble and insoluble groups. Proposition 7.13 shows that the orders of the respective groups are also minimal in the non-soluble case. Our final result presents the minimal order of a group in which all groups of order n or less can be embedded, for $1 \leq n \leq 15$.

Theorem 8.2. *Let G be a group of minimal order in which all groups of order n or less can be embedded. Then, for $3 \leq n \leq 15$, G is not unique and $|G|$ is as in Table 5.*

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n	Minimal order	n	Minimal order
1	1	9	$30\ 240 = 2^5 \cdot 3^3 \cdot 5 \cdot 7$
2	2	10	$30\ 240$
3	$6 = 2 \cdot 3$	11	$332\ 640 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
4	$24 = 2^3 \cdot 3$	12	$665\ 280 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
5	$120 = 2^3 \cdot 3 \cdot 5$	13	$8\ 648\ 640 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
6	120	14	$8\ 648\ 640$
7	$840 = 2^3 \cdot 3 \cdot 5 \cdot 7$	15	$8\ 648\ 640$
8	$3\ 360 = 2^5 \cdot 3 \cdot 5 \cdot 7$		

Table 5. Minimal order of a group in which all groups of order n or less can be embedded, for $1 \leq n \leq 15$

Theorem 8.2 shows, in particular, that $|S_n|$ is not minimal with respect to the embedding of all groups of order n or less for $6 \leq n \leq 15$. It is an open question as to whether this holds in general for $n \geq 6$. We suggest that the related problem of determining the minimal order of a group in which all groups of order n or less can be embedded will present a considerable challenge for large values of n.

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Robert Heffernan

Department of Mathematics, Cork Institute of Technology, Bishopstown, Cork, Ireland

Email: robert.heffernan@cit.ie

Brendan McCann

Department of Computing and Mathematics, Waterford Institute of Technology, Cork Road, Waterford, Ireland

Email: bmccann@wit.ie

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