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ON DERIVABLE TREES

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Abstract. This paper defines the concept of partitioned hypergraphs, and enumerates the number of these hypergraphs and discrete complete hypergraphs. A positive equivalence relation is defined on hypergraphs, this relation establishes a connection between hypergraphs and graphs. Moreover, we define the concept of (extended) derivable graph. Then a connection between hypergraphs and (extended) derivable graphs was investigated. Via the positive equivalence relation on hypergraphs, we show that some special trees are derivable graph and complete graphs are self derivable graphs.

1. Introduction

In 1736, Euler introduced the concept of graph theory. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operation research, optimization, economics, networking routing, transportation and computer science.

It was natural to try and generalize the concept of a graph, in order to attack additional combinatorial problems. The notion of hypergraph has been introduced by Berge as a generalization of graph around 1960 and one of the initial concerns was to extend some classical results of graph theory and the notion of hypergraph has been considered as a useful tool to analyze the structure of a system. Further materials regarding graph and hypergraph are available in the literature, too [2, 6]. Hypergraphs makes it possible to more compactly describe many proofs in graph theory and may also unify

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several theorems in ordinary graph theory. Today, some features of hypergraphs are used in computer science, notably in machine learning, and there has been a lot of research about using hypergraphs in relational databases, which might be viewed as a sort of data mining. There is also research about networks where matroids and hypergraphs are used together in the demonstrations[5].

Recently, M. Hamidi et al. redefined the concept of hypergraphs via the concept of hypergroupoid and considered a connection between of hypergraphs and hypergroupoids. Furthermore, they defined the concept of fundamental graphs and via a fundamental relation considered a relation between fundamental group and fundamental graph [8].

One of our main motivation from this paper, is considering some applications in graph theory. Graph theoretical concepts are widely used to study and model various applications, in different areas. They include, study of molecules, construction of bonds in chemistry, the study of atoms, sociology (for example to measure actors prestige or to explore diffusion mechanisms), in biology and conservation efforts (where a vertex represents regions where certain species exist and the edges represent migration path or movement between the regions, this information is important when looking at breeding patterns or tracking the spread of disease, parasites and to study the impact of migration that affect other species), in operations research(For example, the traveling salesman problem, the shortest spanning tree in a weighted graph, obtaining an optimal match of jobs and men and locating the shortest path between two vertices in a graph), in modeling transport networks, activity networks and theory of games[1, 4, 9, 10, 12]. But there is some problems in graph theory, whence there is connection between more than two points, so we apply the concept of hypergraphs for solving of these problems. Also we need to link up our demands to graph theory and so for solving our problems in graph theory have to link up to hypergrap theory. Indeed, this paper connect hypergraph theory to graph theory via a positive relation between them and so with this regards, solves some what any problems.

Regarding these points, the aim of this paper is to introduce the notation of partitioned hypergraphs and to compute the number of partitioned hypergraphs and discrete complete hypergraphs constructed on every arbitrary set via the inclusion–exclusion principle. It is a natural question what is the relationships between hypergraphs and graphs. It is a motivation for us to study derivable graphs. So we have to consider a relation on hypergraphs which is not necessarily a transitive relation and consider some conditions in such a way that it becomes an equivalence relation and consequently, construct quotient hypergraphs. Indeed, we consider how some graphs are derivable graph and it was introduced that the concepts of extended derivable graphs and self derivable graphs. We show that for every $n \neq 4$, cyclic graphs are extended derivable graphs and only self derivable graphs are complete graphs. Finally, it is shown that complete bigraphs are self derivable graphs from partitioned hypergraphs.

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2. Preliminaries

Definition 2.1. [3] *A graph G is a finite nonempty set V of objects called vertices (the singular is vertex) together with a set E of 2-element subsets of V called edges and is shown with $G = (V, E)$. Two graphs G and H are isomorphic (they have the same structure) if there exists a bijective function $\varphi : V(G) \longrightarrow V(H)$ so that two vertices u and v are adjacent in G if and only if $\varphi(u)$ and $\varphi(v)$ are adjacent in H. The function φ is then called an isomorphism. If G and H are isomorphic, we write $G \cong H$.*

Definition 2.2. [7] *Let $G = (V, E)$ be a graph. An elementary subdivision of G results when an edge $e = \{u, w\}$ is removed from G and then the edges $\{u, v\}, \{v, w\}$ are added to $G \setminus e$, where $v \notin V$. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they are isomorphic or if they can both are obtained from the same graph $H = (V_3, E_3)$ by a sequence of elementary subdivision.*

Definition 2.3. [2] *Let $G = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A hypergraph on G is a family $H = (G, E_1, E_2, \ldots, E_m) = (G, \{E_i\}_{i=1}^{m})$ of subsets of G such that*

- *(i) for all $1 \leq i \leq m, E_i \neq \emptyset$;*
- *(ii) $\bigcup_{i=1}^{m} E_i = G$. A simple hypergraph (Sperner family) is a hypergraph $H = (G, E_1, E_2, \ldots, E_m)$ in such a way that*
- *(iii) $E_i \subset E_j \Longrightarrow i = j$.*

The elements $x_1, x_2, \ldots, x_n$ of G are called vertices, and the sets $E_1, E_2, \ldots, E_m$ are the edges (hyperedges) of the hypergraph. For every $1 \leq k \leq m$ if $|E_k| \geq 2$, then E_k is represented by a solid line surrounding its vertices, if $|E_k| = 1$ by a cycle on the element (loop). If for all $1 \leq k \leq m$ $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph.

In every hypergraph, two vertices x and y are said to be adjacent if there exists a hyperedge E_i which contains the two vertices ($x, y \in E_i$). The degree of a vertex is the number of hyperedges which contains the vertex and is shown by $deg(x)$ ($deg(x) = |\{E_i \mid x \in E_i\}|$).

Definition 2.4. [8] *Let $H = (G, \{E_x\}_{x \in G})$ be a hypergraph, where for $x \in G, E_x$ is listed as a hyperedge that vertex x is appeared in it. Then H is called a complete hypergraph, if for every $x, y \in G$ there exits a hyperedge $E_{x,y}$ such that $\{x, y\} \subseteq E_{x,y}$ and is shown a complete hypergraph with n elements by K_n^*. Let $H = (G, \{E_i\}_{i=1}^{n+1})$ be a complete hypergraph.*

- *(i) H is called a joint complete hypergraph, if for every $1 \leq i \leq n$, $|E_i| = i, E_i \subseteq E_{i+1}$ and $|E_{n+1}| = n$;*
- *(ii) H is called a discrete complete hypergraph, if for every $1 \leq i \neq j \leq n, |E_i| = |E_j|, E_i \cap E_j = \emptyset$ and $|E_{n+1}| = n$;*
- *(iii) H is called an ordinary complete hypergraph, if $n = 1$ and $E_i = G$.*

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Definition 2.5. [8] *Let $H = (G, \{E_x\}_{x\in G})$ be a hypergraph. Then define a binary relation ρ on G as follows: for every integer $n \geq 1$, ρ_n is defined as follows:*

$$x\rho_n y \Longleftrightarrow |E_x^m| = |E_y^m|, \text{ where } |E_x^m| = \min\{|E_t|;\ x \in E_t\} \text{ or } |E_x^m| \leq |E_x|$$

and $n = \min\{deg(x), deg(y)\}$.

Obviously the relation $\rho = \bigcup_{n\geq 1} \rho_n$ is an equivalence relation on G. We denote the set of all equivalence classes of ρ by G/ρ. Hence $G/\rho = \{\rho(x) \mid x \in G\}$.

Theorem 2.6. [8] *Let G be a set and $\mathcal{H}g(G)$ be set of all hypergraphs are constructed on G. If $|G| = n$, then $|\mathcal{H}g(G)| = \sum_{k=0}^{n}(-1)^k \binom{n}{k} 2^{\mathbf{2^{(n-k)}-1}}$.*

3. Discrete complete hypergraphs and partitioned hypergraphs

In this section, we define the concept of partitioned hypergraphs and compute the number of this class of hypergraps via the Stirling numbers of the second kind.

Let $H' = (H, \{E_i\}_{i=1}^k)$ be a discrete complete hypergraph, $m \in \mathbb{N}$ and $1 \leq j, j' \leq k$. Denote the set of all discrete complete hypergraphs with $|E_j| = |E_{j'}| = m$ on H (m–hyperedges), by $\mathcal{D}_c^{(m)}(H)$ and the set of all discrete complete hypergraphs on H, by $\mathcal{D}_c(H)$.

Theorem 3.1. *Let $H' = (H, \{E_i\}_{i=1}^k)$ be a discrete complete hypergraph, $|H| = n$ and $k \mid n$. Then*

(i) $|\mathcal{D}_c^{(n/k)}(H)| = \frac{1}{k!}\prod_{r=0}^{n(k-1)/k}\binom{n-r}{n/k} = \frac{n!}{k!((n/k)!)^k}$;

(ii) $|\mathcal{D}_c(H)| = \sum_{k\mid n}\frac{1}{k!}\prod_{r=0}^{n(k-1)/k}\binom{n-r}{n/k} = \sum_{k\mid n}\frac{n!}{k!((n/k)!)^k}$.

Proof. (i) Let $H = \{a_1, a_2, \ldots, a_n\}$. Because $H = E_1 \cup E_2 \cup \cdots E_k$ and $H' = (H, \{E_i\}_{i=1}^k)$ is a discrete complete hypergraph, so $|E_1| + |E_2| + \cdots + |E_k| = n$. Thus for every $1 \leq j \leq k, |E_j| = n/k$(because of $k \mid n$). n/k elements from n elements can be selected for E_1, n/k elements from $n - n/k$ elements can be selected for E_2 and consequently n/k elements from $n - (j-1)n/k$ elements can be selected for E_j, where $1 \leq j \leq k$. Moreover for $E_1, E_2, \ldots, E_k$ there exist $k!$ permutations, hence $|\mathcal{D}_c^{(n/k)}(H)| = \frac{1}{k!}\prod_{r=0}^{n(k-1)/k}\binom{n-r}{n/k}$.

(ii) By (i), for every $1 \leq j \leq k, |E_j| \in D(n)$, so

$$|\mathcal{D}_c(H)| = \sum_{k\mid n}\frac{1}{k!}\prod_{r=0}^{n(k-1)/k}\binom{n-r}{n/k} = \sum_{k\mid n}\frac{n!}{k!((n/k)!)^k}.$$

□

Definition 3.2. *Let $|H| = n$ and $k, n \in \mathbb{N}$. Then $H' = (H, \{E_i\}_{i=1}^k)$ is called a partitioned hypergraph, if $\mathcal{P} = \{E_1, E_2, \ldots, E_k\}$ is a partition of H(for every $1 \leq i \neq j \leq k, E_i \cap E_j = \emptyset$). We will denote the*

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set of partitioned hypergraphs with $|\mathcal{P}| = k$ *on* H, *by* $\mathcal{P}_h^{(k)}(H)$ *and the set of all partitioned hypergraphs on* H, *by* $\mathcal{P}_h(H)$.

Example 3.3. *Let* $H = \{a, b, c, d, e, f, g\}$. *Consider the hypergraph in Figure 1.*

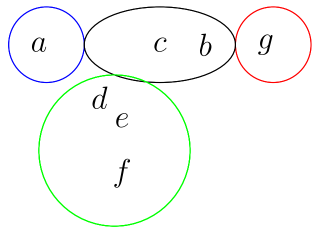


FIGURE 1. Hypergraph $H' = (H, E_1, E_2, E_3, E_4)$

Clearly $H' = (H, E_1, E_2, E_3, E_4)$ *is a partitioned hypergraph, where* $E_1 = \{a\}, E_2 = \{b, c\}, E_3 = \{g\}$ *and* $E_4 = \{d, e, f\}$ *and so* $H' \in \mathcal{P}_h^{(4)}(H)$.

Corollary 3.4. *Let* H *be a non–empty set. Then* $\mathcal{P}_h(H) \cap \mathcal{D}_c(H) = \mathcal{D}_c(H)$.

Lemma 3.5. *Let* $H = \{b_1, b_2, \ldots, b_n\}$ *and* $\mathcal{A}_j = \{E_j \mid b_j \notin E_j\}$. *Then*

(i) *for every* $r \in \mathbb{N}$, *we have* $|\bigcap_{j=1}^{r} \mathcal{A}_j| = (n-r)^n$;

(ii) *for every* $1 \leq r \neq s \leq n$ *we have* $\mathcal{P}_h^{(r)}(H) \cap \mathcal{P}_h^{(s)}(H) = \emptyset$.

Proof. (i) Let $r \in \mathbb{N}$. Then $\bigcap_{j=1}^{r} \mathcal{A}_j = \bigcap_{j=1}^{r} \{E_j \mid b_j \notin E_j\} = \{E_j \mid b_1, b_2, \ldots, b_r \notin E_j\}$. Thus $|\bigcap_{j=1}^{r} \mathcal{A}_j| = (n-r)^n$.

(ii) It follows immediately. □

Corollary 3.6. *Let* $n, k \in \mathbb{N}$ *and* $|H| = n$. *Then by Stirling numbers of the second kind,*

(i) $|\mathcal{P}_h^{(k)}(H)| = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n$;

(ii) $|\mathcal{P}_h(H)| = \sum_{k=1}^{n} |\mathcal{P}_h^{(k)}(H)| = \sum_{k=1}^{n} \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n$;

(iii) *if* $k \mid n$, *then* $|\mathcal{P}_h^{(k)}(H)| \geq |\mathcal{D}_c^{(k)}(H)|$;

(iv) *if* $k \mid n$, *then* $\frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n \geq \frac{n!}{k!((n/k)!)^k}$;

(v) *if* n *is a prime and* $k \mid n$, *then* $|\mathcal{P}_h^{(k)}(H)| = |\mathcal{D}_c^{(k)}(H)|$;

(vi) *if* n *is a prime and* $k \mid n$, *then* $\frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n = \frac{n!}{k!((n/k)!)^k}$;

(vii) $|\mathcal{D}_c(H)| \leq |\mathcal{P}_h(H)|$;

(x) $\sum_{k \mid n} \frac{n!}{k!((n/k)!)^k} \leq \sum_{k=1}^{n} \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n$.

4. Positive equivalence relation η^*

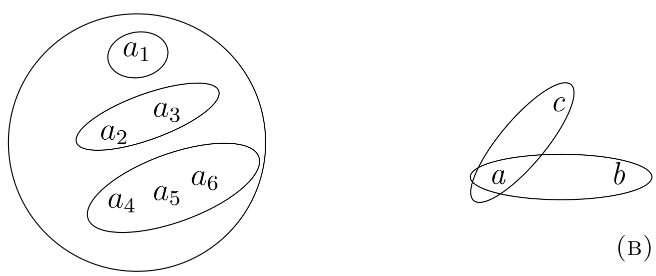
In this section, we define a reflexive and symmetric relation, η, on hypergraph H', which is not necessarily a transitive relation. Later on we consider η^* be the *transitive closure* of η (the smallest transitive relation such that contains η). Furthermore, we construct quotient, H'/η^*, and show that it is a graph.

Definition 4.1. *Let $H' = (H, \{E_i\}_{i=1}^n)$ be a hypergraph. Define a binary relation η on H as follows: $\eta_1 = \{(x,x) \mid x \in H\}$ and for every integer $k > 1$,*

$$x\ \eta_k\ y \Longleftrightarrow \exists\ \ E_i^s,\ \text{such that } \{x,y\} \subseteq E_i^s,\ \text{where } k = |E_i^s| = \min\{|E_t|;\ x,y \in E_t\}$$

and for all $1 \leq i,j \leq n$, there is no $E_i \neq E_i^s$, or $E_j \neq E_i^s$, such that $x \in E_i, y \in E_j$ and $|E_i| < k, |E_j| < k$. Obviously $\eta = \bigcup\limits_{k\geq 1} \eta_k$ is a reflexive and symmetric relation on H.

Example 4.2. *Let $H_1 = \{a,b,c\}$ and $H_2 = \{a_1,a_2,a_3,a_4,a_5,a_6\}$. Consider the hypergraphs in (Figure 2).*



(A) H_2 (B) H_1

FIGURE 2. Hypergaphs on H_1 and H_2

(i) In the hypergraph on H_2, It is easy to see that $\eta_1 = \{(a_i,a_i) \mid 1 \leq i \leq 6\}, \eta_2 = \{(a_2,a_3),(a_3,a_2)\}, \eta_3 = \{(a_i,a_j) \mid 4 \leq i \neq j \leq 6\}$ and $\eta = \eta_1 \cup \eta_2 \cup \eta_3$. Clearly η is an equivalence relation and so $\eta(a_1) = \{a_1\}, \eta(a_2) = \eta(a_3) = \{a_2,a_3\}, \eta(a_4) = \eta(a_5) = \eta(a_6) = \{a_4,a_5,a_6\}$.
(ii) In the hypergraph on H_1, clearly $\eta_1 = \{(a,a),(b,b),(c,c)\}, \eta_2 = \{(c,a),(a,c),(a,b),(b,a)\}$ and $\eta = \eta_1 \cup \eta_2$. Since $(c,b) \notin \eta$, so η is not a transitive relation.

The relation η is not necessarily transitive. Let η^* be the *transitive closure* of η (the smallest transitive relation such that contains η). In the following we create conditions so that $\eta^* = \eta$. Clearly η^* is an equivalence relation and we denote the set of all equivalence classes of η^* by H/η^* or H'/η^*. Hence $H/\eta^* = \{\eta^*(x) \mid x \in H\}$.

Theorem 4.3. *Let $H' = (H, \{E_i\}_{i=1}^n)$ be a hypergraph. If for every $1 \leq i \neq j \leq n, E_i \cap E_j = \emptyset$, then $\eta^* = \eta$.*

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Proof. Let $(x,y)\in\eta^*$. Then there exist $x=a_0,a_1,a_2,\ldots,a_n=y\in G$, non–distinct $k_1,k_2,\ldots,k_{n-1}\in\mathbb{N}$ and $E_i^{s_i}$, such that $\{a_i,a_{i+1}\}\subseteq E_i^{s_i}$, where $k_i=|E_i^{s_i}|=\min\{|E_t|;\ a_i,a_{i+1}\in E_t\}$ and for all $1\leq i,j\leq n$, there is no $E_i\neq E_i^{s_i}$, or $E_j\neq E_i^{s_i}$, such that $a_i\in E_i, a_{i+1}\in E_j$ and $|E_i|<k_i,|E_j|<k_i$. Since for every $1\leq i\neq j\leq n$, $E_i\cap E_j=\emptyset$, we obtain that there is unique $E_i^{s_i}$, such that $\{a_i,a_{i+1}\}\subseteq E_i^{s_i}$, where $k_i=|E_i^{s_i}|=\min\{|E_t|;\ a_i,a_{i+1}\in E_t\}$. It follows that $x=a_0,a_1=y$ or $\mathcal{P}_h(H)=\mathcal{P}_h^1(H)$, so in two cases we get that $(x,y)\in\eta$. Thus $\eta^*=\eta$. □

Example 4.4. *Let $H=\{a,b,c\}$. Consider the hypergraph in Figure 3.*

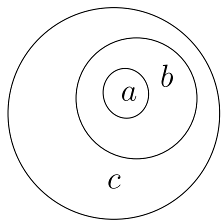

FIGURE 3. Joint complete hypergraph

Obviously, $\eta^*=\eta$ but for any $1\leq i\neq j\leq 3$, $E_i\cap E_j\neq\emptyset$ so the converse of Theorem 4.3, is not necessarily true.

Corollary 4.5. *Let $H'=(H,\{E_i\}_{i=1}^n)$ be a hypergraph.*

(i) *If H' is a partitioned hypergraph, then $\eta^*=\eta$.*
(ii) *If H' is a discrete complete hypergraph, then $\eta^*=\eta$.*

Example 4.6. *Let $H=\{a,b,c\}$. Consider the hypergraph in Figure 4.*

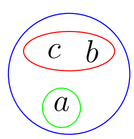

FIGURE 4. Hypergraph

Obviously, $\eta^*=\eta$ but $H'=(H,\{E_1,E_2,E_3\})$ is not a partitioned hypergraph and discrete complete hypergraph so the converse of Corollary 4.5, is not necessarily true.

Theorem 4.7. *Let $H'=(H,\{E_i\}_{i=1}^{n+1})$ be a joint complete hypergraph. Then $\eta^*=\eta$.*

Proof. Let $(x,y)\in\eta^*$. Then there exist $x=a_0,a_1,a_2,\ldots,a_n=y\in G$, non–distinct $k_1,k_2,k_3,\ldots,k_{n-1}\in\mathbb{N}$ and $E_i^{s_i}$, such that $\{a_i,a_{i+1}\}\subseteq E_i^{s_i}$, where $k_i=|E_i^{s_i}|=\min\{|E_t|;\ a_i,a_{i+1}\in E_t\}$ and for all $1\leq i,j\leq n$, there is no $E_i\neq E_i^{s_i}$, or $E_j\neq E_i^{s_i}$, such that $a_i\in E_i, a_{i+1}\in E_j$ and $|E_i|<k_i,|E_j|<k_i$. Since $H'=(H,\{E_i\}_{i=1}^n)$ is a joint complete hypergraph, for every $1\leq i\neq j\leq n+1, |E_i|<|E_j|$ implies that $E_i\subseteq E_j$. It follows that for all $a_i,a_j\in H, a_i\eta a_j$ and so $(x,y)\in\eta$. Thus $\eta^*=\eta$. □

Example 4.8. *Let $H=\{a,b,c\}$. Consider the hypergraph of Figure 5,*

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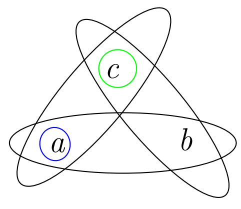

FIGURE 5. Complete hypergraph

Then clearly H is a complete hypergraph, but $\eta^* \neq \eta$.

Theorem 4.9. *Let $H' = (H, \{E_x\}_{x\in H})$ be a hypergraph, $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\eta = \eta^*$. Then for every $i \in \mathbb{N}^*$ there exists a well–defined binary relation $*_i$ on H'/η such that $(H'/\eta, *_i)$ is a graph.*

Proof. Let $i \in \mathbb{N}^*$. For every $\eta(x), \eta(y) \in H/\eta$, define a binary relation " $*_i$ " on H/η by

$$\eta(x) *_i \eta(y) = \begin{cases} \widehat{\eta(x), \eta(y)} & \text{if } \big||\eta(x)| - |\eta(y)|\big| = i, \\ \widehat{\emptyset} & \text{otherwise,} \end{cases}$$

where for every $x, y \in H, (\widehat{\eta(x), \eta(y)})$ is represented as an ordinary (simple) edge and $\widehat{\emptyset}$ means that there is no edge. Moreover, for every $x \in H$ define $\eta(x) *_i \eta(x) = \eta(x)$ and will denote by $\eta(x)$ as a single point. We just show that for every $i \in \mathbb{N}^*$, $*_i$ on H/η is a well-defined binary relation . Let $x_1\eta x_2$ and $y_1\eta y_2$, where $x_1, x_2, y_1, y_2 \in H$. Then there exist $k, k' \geq 2, E^s_{i_k}$, such that $\{x_1, x_2\} \subseteq E^s_{i_k}$, where $k = |E^s_i| = \min\{|E_t|;\ x_1, x_2 \in E_t\}$ and $E^s_{i_{k'}}$, such that $\{y_1, y_2\} \subseteq E^s_{i_{k'}}$, where $k' = |E^s_i| = \min\{|E_t|;\ y_1, y_2 \in E_t\}$ so that for all $1 \leq i, j \leq n$, there is no $E_i \neq E^s_{i_k}$, or $E_j \neq E^s_{i_k}$, such that $x_1 \in E_i, x_2 \in E_j$ and $|E_i| < k, |E_j| < k$ and for all $1 \leq i, j \leq n$, there is no $E_i \neq E^s_{i_{k'}}$, or $E_j \neq E^s_{i_{k'}}$, such that $y_1 \in E_i, y_2 \in E_j$ and $|E_i| < k', |E_j| < k'$. If $\big||\eta(x_1)| - |\eta(y_1)|\big| = i$, then $\big||\eta(x_2)| - |\eta(y_2)|\big| = i$ and so $\eta(x_1) *_i \eta(y_1) = \eta(\widehat{x_1), \eta(}y_1) = \eta(\widehat{x_2), \eta(}y_2) = \eta(x_2) *_i \eta(y_2)$ and if $\big||\eta(x_1)| - |\eta(y_1)|\big| \neq i$, then $\big||\eta(x_2)| - |\eta(y_2)|\big| \neq i$ and so $\eta(x_1) *_i \eta(y_1) = \widehat{\emptyset} = \eta(x_2) *_i \eta(y_2)$. It is easy to see that $(H/\eta, *_i)$ is a graph. □

Definition 4.10. *Let $H' = (H, \{E_x\}_{x\in H})$ be a hypergraph, $i \in \mathbb{N}^*$ and $\eta = \eta^*$. The graph $(H'/\eta, *_i)$ that is corresponding to the hypergraph H' and is obtained by the relation $*_i$ is called an $(i+1)^{th}$ η^*-graph.*

Remark: *For every $i \in \mathbb{N}^*$ and with respect to every arbitrary hypergraph H', the $(i+1)^{th}$ η^*-graph is obtained. Next sections will determine that graphs can be derived in this way.*

Example 4.11. *Let $H = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. Consider the hypergraph in Figure 6.*

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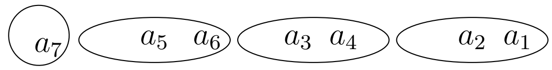

FIGURE 6. Hypergraph

Since $E_1^s = \{a_1, a_2\}, E_2^s = \{a_3, a_4\}, E_3^s = \{a_5, a_6\}$ *and* $E_4^s = \{a_7\}$, *by Theorem 4.9, we get that* $\eta^* = \eta$ *and* $\eta(a_1) = \eta(a_2) = \{a_1, a_2\}, \eta(a_3) = \eta(a_4) = \{a_3, a_4\}, \eta(a_5) = \eta(a_6) = \{a_5, a_6\}$, *and* $\eta(a_7) = \{a_7\}$. *Now, for* $i = 0$, *we obtained* 1^{th} η^**-graph in Figure 7.*

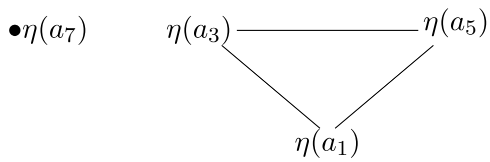

FIGURE 7. 1^{th} η^*-graph H'/η obtained from hypergraph Figure 6

H/η *is a un–connected graph with* 4 *vertices and* 3 *edges. For* $i = 1$, *we obtained* 2^{th} η^**-graph* $H'/\eta \cong K_{1,3}$ *as Figure 8.* H'/η *is a connected graph with* 4 *vertices and* 3 *edges. Moreover, for every* $i \geq 2$, $(i+1)^{th}$ η^**-graph* H'/η *is isomorphic to null graph* $\overline{K}_4$.

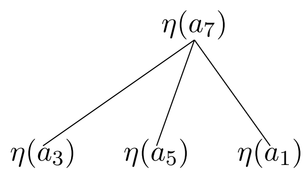

FIGURE 8. 2^{th} η^*-graph H'/η obtained from hypergraph Figure 6

Theorem 4.12. *Let* $H' = (H, \{E_x\}_{x\in G})$ *be a discrete complete hypergraph.*

(*i*) *If* $*_i = *_0$, *then there exists* $p \in \mathbb{N}$ *such that* $|H'/\eta^*| = K_p$.

(*ii*) *If* $*_i \neq *_0$, *then there exists* $p \in \mathbb{N}$ *in such a way that,* $|H'/\eta^*| = \overline{K}_p$.

Proof. (*i*) Let $H = \{a_1, a_2, \ldots, a_n\}$. Then there exists $m \in \mathbb{N}$ in such a way that $m \mid n$. For every $1 \leq j \leq n/m$, consider $E_j = \{a_{(j-1)m+1}, a_{(j-1)m+2}, \ldots, a_{jm}\}$. A simple computation shows that $H' = (H, \overline{E} = \{E_j\}_{j=1}^{n/m})$ is a discrete complete hypergraph and for every $1 \leq j \leq n/m, \eta^*(a_{(j-1)m+1}) = E_j$. Thus $H'/\eta^* = \{E_1, E_2, \ldots, E_{n/m}\}$, since for $1 \leq j, j' \leq n/m, |E_j| = |E_{j'}|$ and $*_i = *_0$, we get that for every $a_j \in E_j, a_{j'} \in E'_j, \eta^*(a_j) *_i \eta^*(a_{j'}) = \widehat{\eta^*(a_j), \eta^*}(a_{j'})$ and $|\overline{E}/\eta^*| = (n^2 - mn)/2m^2$. It follows that $H'/\eta^* \cong K_{n/m}$.

(*ii*) By (*i*) there exists $m \in \mathbb{N}$ such that $H'/\eta^* = \{E_1, E_2, \ldots, E_{n/m}\}$. Since for every $1 \leq j, j' \leq n/m, |E_j| = |E_{j'}|$ and $*_i \neq *_0$, we get that for every $a_j \in E_j, a_{j'} \in E'_j, \eta^*(a_j) *_i \eta^*(a_{j'}) = \widehat{\emptyset}$, $|H'/\eta^*| = n/m$ and $|\overline{E}/\eta^*| = 0$. It follows that $H'/\eta^* \cong \overline{K}_{n/m}$. □

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Consider $\mathcal{P}_h^{(k)}(H)$, where $k \in \mathbb{N}$. We denote $\mathbb{P}_h^{(k)}(H)$ as the set of all partitioned hypergraphs with hyperedges of same order.

Corollary 4.13. *Let $H' = (H, \{E_i\}_{i=1}^n)$ be a hypergraph and $k \mid |H|$.*

(i) If $*_i = *_0$ and $H' \in \mathbb{P}_h^{(k)}(H)$, then $H'/\eta^* \cong K_k$.

(ii) If $*_i \neq *_0$ and $H' \in \mathbb{P}_h^{(k)}(H)$, then $H'/\eta^* \cong \overline{K}_k$.

Example 4.14. *Let $H = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$. Consider the hypergraph in Figure 9.*

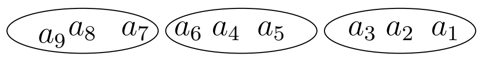

FIGURE 9. Hypergraph

Since $E_1^s = \{a_1, a_2, a_3\}, E_2^s = \{a_4, a_5, a_6\}$ and $E_3^s = \{a_7, a_8, a_9\}$, by Theorem 4.9, we get that $\eta^* = \eta$ and $\eta(a_1) = \{a_1, a_2, a_3\}, \eta(a_4) = \{a_4, a_5, a_6\}$ and $\eta(a_7) = \{a_7, a_8, a_9\}$. Now, for $i = 0$, we obtained 1^{th} η^*-graph in Figure 10. So by Corollary 4.13 must $k = 3$.

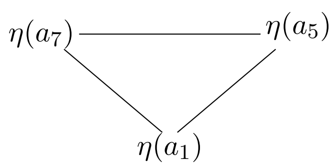

FIGURE 10. 1^{th} η^*-graph H'/η obtained from hypergraph Figure 9

Theorem 4.15. *Let $H' = (H, \{E_x\}_{x \in H})$ be a hypergraph and $\eta^* = \eta$. If $H' = (H, \{E_x\}_{x \in H})$ is a joint complete hypergraph, then for all $i \in \mathbb{N}^*$, we have $|H'/\eta^*| = 1$.*

Proof. Let $H = \{a_1, a_2, \ldots, a_n\}$. For every $1 \leq j \leq n$, consider $E_j = \{a_1, a_2, \ldots, a_j\}$. It is easy to see that $H' = (H, \{E_j\}_{j=1}^n)$ is a joint complete hypergraph. Since for every $1 \leq j \leq n, a_j \in E_j, \{a_j, a_{j+1}\} \subseteq E_{j+1}$ and $|E_i| < |E_{j+1}|$, we get that $\eta^*(a_j) = \eta^*(a_{j+1})$. It follows that $H'/\eta^* = \{\eta^*(a_1)\}$ and so $H'/\eta^* \cong K_1$. □

Example 4.16. *Let $H = \{1, 2, 3, 4, 5\}$. Consider a hypergraph in Figure 11:*

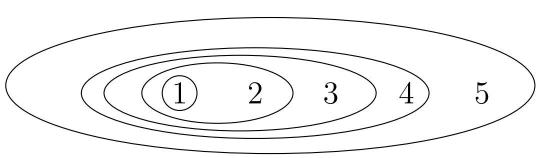

FIGURE 11. Joint complete hypergraph

Obviously, H is a joint complete hypergraph and for every $1 \leq j \leq 5, \eta^*(j) = \{1, 2, 3, 4, 5\}$. Thus $H/\eta^* = \{\eta^*(1)\}$.

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5. (Extended) derivable graphs

In this section, we introduce the concept of derivable graphs via the equivalence relation η^* on hypergraphs. We show that every graph is not necessarily a derivable graph and prove it under some conditions. Furthermore, we show that some trees and all complete bigraphs are derivable graph and complete graph are self derivable graphs.

Definition 5.1. *A graph $G = (V, E)$ is said to be:*

(i) a derivable graph if there exists a nontrivial hypergraph $H' = (H, \{E_k\}_{k=1}^n)$ such that $(H, \{E_k\}_{k=1}^n)$ $/\eta^* \cong G = (V, E)$ and H' is called an associated hypergraph with graph G. In other words, it is equal to the quotient of nontrivial hypergraph on η^* up to isomorphic;

(ii) a self derivable graph, if it is a derivable graph by itself.

Example 5.2. *Let $H = \{a_1, a_2, \ldots, a_n\}$ and $i = 0$. Consider the hypergraph in Figure 12.*

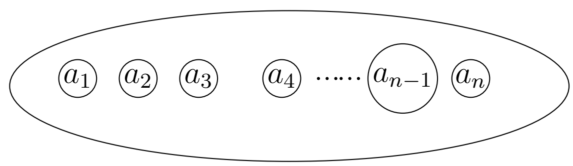

FIGURE 12. Discrete complete hypergraph K_n^*

Then it is easy to see that $K_n^* = (H, \{E_j\}_{j=1}^{n+1})$ is a discrete complete hypergraph, where for every $1 \leq j \leq n$, $E_j = \{a_j\}$ and $E_{n+1} = H$. Clearly for every $1 \leq j \leq n$, $E_{a_j}^s = \{a_j\}$. Hence $K_n^*/\eta^* = \{\eta(a_j) \mid 1 \leq j \leq n\}$ and so $K_n^*/\eta^* \cong K_n$. Therefore for every $n \in \mathbb{N}, K_n$ is a derivable graph.

Theorem 5.3. *Let $G = (V, E)$ be a derivable graph by a hypergraph $H' = (H, \overline{E} = \{E_k\}_{k=1}^n)$. Then*

(i) $|H/\eta| = |V|$ and $|\overline{E}| = |E|$;

(ii) $|H| \geq |V|$;

(iii) if $G = (V, E)$ is a connected graph and $|H| = |V|$, then $*_i = *_0$.

Proof. (iii) Let $|H| = |V|$. Then for every $x, y \in H, \eta(x) \neq \eta(y)$. Since $G = (V, E)$ is a connected graph, we get that for every $x, y \in H, |\eta(x)| = |\eta(y)|$ and so $*_i = *_0$.

□

Theorem 5.4. *Let $G = (V, E)$ be a connected graph. G is a self derivable graph if and only if G is a complete graph.*

Proof. Let $V = \{a_1, a_2, \ldots, a_n\}$. If G is a complete graph, then by Example 5.2, G is a self derivable graph.

Conversely, let G be a self derivable graph. Then by Theorem 5.3, we get $*_i = *_0$ and so for every $x, y \in G, \eta(x) *_i \eta(y) = \eta\widehat{(x), \eta(}y)$. Thus $G = (V, E)$ is a complete derivable graph. □

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6. Derivable cycle graph

In this section, we consider derivable cycle graph and show that C_n is a derivable cycle graph if and only if $n \in \{3, 4\}$.

Example 6.1. *Consider the cycle graph C_4 as follows(Figure 13):*

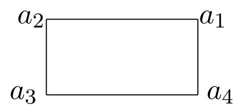

FIGURE 13. Cycle graph C_4

Consider $H = (\overline{V}, \{E_j\}_{j=1}^{4})$, where $\overline{V} = \{a_1, a_2, a_3, a_4, b_1, b_2\}, E_1 = \{a_1\}, E_2 = \{a_3\}, E_3 = \{a_2, b_1\}$ and $E_4 = \{a_4, b_2\}$ and $b_1, b_2 \notin V$. Clearly $H = (\overline{V}, \{E_j\}_{j=1}^{4})$ is a hypergraph, $\eta = \eta^*$, $\eta(a_1) = \{a_1\}, \eta(a_2) = \{a_1, b_1\}, \eta(a_3) = \{a_3\}$ and $\eta(a_4) = \{a_4, b_2\}$. By Theorem 4.9 and for $i = 1$, we obtain that $H/\eta \cong C_4$ is an 2^{th} η^*-graph.

Lemma 6.2. *C_5 is not a derivable graph.*

Proof. Consider the cycle graph C_5 as follows(Figure 14).

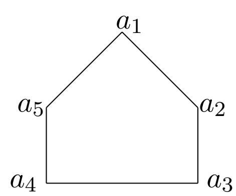

FIGURE 14. Cycle graph C_5

Let C_5 be a derivable graph and $H = (\overline{V}, \{E_j\}_{j=1})$ be an associated hypergraph with graph C_5. Since $\eta(a_1) *_i \eta(a_2) = \eta(\widehat{a_1), \eta(}a_2)$ and $\eta(a_1) *_i \eta(a_5) = \eta(\widehat{a_1), \eta(}a_5)$, we get $k \in \mathbb{N}$, $E_1, E_2, E_3 \subseteq H$ so that $a_1 \in E_1, |E_1| = k, a_2 \in E_2, |E_2| = k + i, a_5 \in E_3$ and $|E_3| = k + i$. On the other hand, $\eta(a_3) *_i \eta(a_2) = \eta(\widehat{a_3), \eta(}a_2)$, $\eta(a_3) *_i \eta(a_4) = \eta(\widehat{a_3), \eta(}a_4)$ and $\eta(a_3) *_i \eta(a_1) = \widehat{\emptyset}$, implies that there exist $E_4, E_5 \subseteq H$ so that $a_3 \in E_4, |E_4| = |E_1|$, $a_4 \in E_5$ and $|E_2| = |E_3| = |E_5| = k + i$. Since $\big||\eta(a_4)| - |\eta(a_5)|\big| \neq i$, $\big||\eta(a_3)| - |\eta(a_5)|\big| = i$ and $\big||\eta(a_4)| - |\eta(a_1)|\big| = i$, we get that $\eta(a_4) *_i \eta(a_5) = \widehat{\emptyset}$, $\eta(a_3) *_i \eta(a_5) = \eta(\widehat{a_3), \eta(}a_5)$ and $\eta(a_4) *_i \eta(a_1) = \eta(\widehat{a_4), \eta(}a_1)$. But $H = (V, \overline{E})$, where $\overline{E} = \{E_1, E_2, E_3, E_4, E_5\}$, $H/\eta^* \cong C_5$ and $|\overline{E}/\eta^*| = 6$, which is a contradiction. Therefore C_5 can not be a derivable graph. □

Proposition 6.3. *Let $6 \leq n \in \mathbb{N}$. Then C_n is not a derivable graph.*

Proof. Since for every $6 \leq n \in \mathbb{N}$, C_n is homeomorphic to C_5, by Lemma 6.2, C_n is not a derivable graph. □

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Theorem 6.4. *Let $C_n = (V, E)$ be a cycle graph. Then C_n is a derivable graph if and only if $n = 3$ and $n = 4$.*

Proof. If $n = 3$, then $C_3 \cong K_3$ and so by Theorem 5.4, C_3 is a derivable graph. If $n = 4$, then by Example 6.1, C_4 is a derivable graph. By Proposition 6.3, the converse, is obtained. □

Corollary 6.5. *Let $G = (V, E)$ be a non-complete graph in which has cycle. Then $G = (V, E)$ is a non-derivable graph if and only if*

(i) *$G \ncong C_3$ and $G \ncong C_4$;*

(ii) *it contains a subgraph that is homeomorphic to C_n, where $3, 4 \neq n$.*

Example 6.6. *Consider the cycle graph C_6 in Figure 15.*

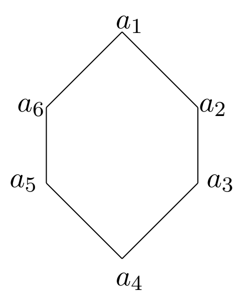

FIGURE 15. Cycle graph C_6

By Proposition 6.3, C_6 is not a derivable graph. Now we add some edges to C_6 as follows(Figure 16).

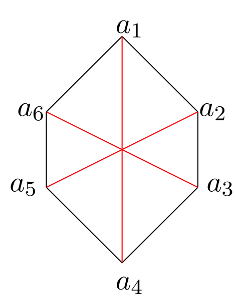

FIGURE 16. Graph $C_6^{\nearrow}$

Hence consider the hypergraph $\overline{G} = (\overline{V}, \{\overline{E_i}\}_{i=1}^n)$ in Figure 17.

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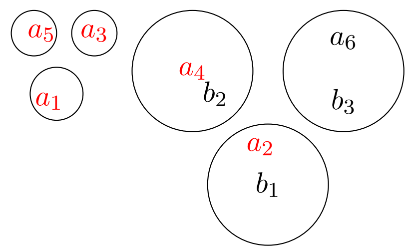

Figure 17. Associated hypergraph with graph Figure 16

Clearly $\eta^*(a_1)=\{a_1\},\eta^*(a_2)=\{a_2,b_1\},\eta^*(a_3)=\{a_3\},\eta^*(a_4)=\{a_4,b_2\},\eta^*(a_5)=\{a_5\},\eta^*(a_6)=\{a_6,b_3\}$ *and it is easy to compute that* $H/\eta^*\cong C_6^{\nearrow}$.

In Example 6.12, we saw that C_6 was not a derivable graph, while we added some edges to this graph and converted to a derivable graph. Due to this problem we will have the following definition.

Definition 6.7. *Let* $G=(V,E)$ *be a non derivable graph and* $i\neq 0$. *We will call non complete graph* $G^{\nearrow}=(V,E^{\nearrow})$ *is an extended derivable graph of* G, *if* $E^{\nearrow}$ *is obtained by adding the least number of edges to* E *such that* $G^{\nearrow}$ *be a derivable graph. Also we will say* $G=(V,E)$ *is an extended derivable graph.*

Example 6.8. *By Example 6.12, cycle graph* C_6 *is an extended derivable graph.*

Theorem 6.9. C_5 *is not an extended derivable graph.*

Proof. Since $i\geq 1$, by Lemma 6.2, for every hypergraph $H=(V,\overline{E})$, where $\overline{E}=\{E_1,E_2,E_3,E_4,E_5\}$, we get that H/η^* has a cycle of maximum length 4. By adding every edges to C_5, since $|V|$ is an odd again, due to Lemma 6.2, we get that $|\overline{E}/\eta^*|=6$, else all edges in H/η^* be connected that implies $i=0$, which is a contradiction. □

Corollary 6.10. *Let* $k\in\mathbb{N}$. *Then* C_{2k+1} *is not an extended derivable graph.*

Theorem 6.11. *Let* $k\in\mathbb{N}$. *Then*

(i) $C_{2k}=(V,E^{\nearrow})$ *is an extended derivable graph;*

(ii) $|E^{\nearrow}|=k^2$.

Proof. (i,ii) Let $V=V_1\cup V_2$, where $V_1=\{a_1,a_3,a_5,\ldots,a_{2k-1}\}$, $V_2=\{a_2,a_4,a_6,\ldots,a_{2k}\}$ and for every $1\leq j\leq 2k, e_j=\widehat{a_j,a_{j+1}}$. Now for every $j\in\{1,3,5,\ldots,a_{2k-1}\}$ consider E_j so that $a_j\in E_j$, $|E_j|=k_j$, $a_{j+1}\in E_{j+1}$ and $|E_{j+i}|=k_j+i$, where $k_j\in\mathbb{N}$. A simple computation shows that $H=(\overline{V},\overline{E})$ is a hypergraph, where $\overline{V}=V\cup W$ and W is every set so that $|W|=i(n/2)$. Moreover, by definition of η^*, we can see that $\eta^*(a_j)=E_j$ and $\eta^*(a_j)*_i\eta^*(a_{j+1})=\widehat{\eta^*(a_j),\eta^*(a_{j+1})}$, whence $1\leq j\leq 2k$. Since $|V_1|=|V_2|=n/2$, for every $j\neq 1,\eta^*(a_j)*_i\eta^*(a_{j+1})=\widehat{\eta^*(a_j),\eta^*(a_{j+1})},\eta^*(a_j)*_i\eta^*(a_{j-1})=\widehat{\eta^*(a_j),\eta^*(a_{j-1})}$, $\eta^*(a_1)*_i\eta^*(a_2)=\widehat{\eta^*(a_1),\eta^*(a_2)}$ and $\eta^*(a_1)*_i\eta^*(a_{2k})=\widehat{\eta^*(a_1),\eta^*(a_{2k})}$, we get that $|E^{\nearrow}|=(2k/2)((2k)/2-2)+2k=k^2$. □

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Example 6.12. *Consider a graph $G = (\{a, b, c, d, e\}, E)$ in Figure 18. It is easy to see that G not a derivable graph. So we obtain the graph $G^{\nearrow} = (V, E^{\nearrow})$ in Figure 19, where is a derivable graph, while for all $k \in \mathbb{N}, G \ncong C_{2k}$. Thus the converse of Theorem 6.11, necessarily is not true.*

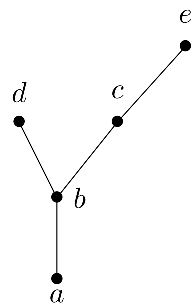

FIGURE 18. Non derivable graph G

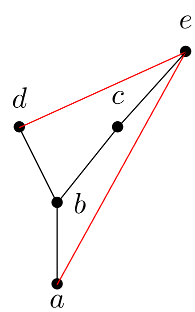

FIGURE 19. Derivable graph $G^{\nearrow} = (V, E^{\nearrow})$ from graph 18

Theorem 6.13. *Let $G = (V, E)$ be a graph and $|V| = n$. If G is an extended derivable graph, then $|E| < \frac{(n-1)(n-2)}{2}$.*

Proof. If $|E| \geq \frac{(n-1)(n-2)}{2}$, then $G \cong K_n$ or $|E| = \frac{(n-1)(n-2)}{2}$. If $G \cong K_n$, then we conclude that $i = 0$ which is a contradiction. If $|E| = \frac{(n-1)(n-2)}{2}$, because G is an extended derivable graph, we have to add at most an edge to E. It follows that $|E^{\nearrow}| = \frac{(n-1)(n-2)}{2} + 1 = \frac{n(n-1)}{2}$ and so $i = 0$. Thus in any cases we imply that $i = 0$ which it is a contradiction and so must $|E| < \frac{(n-1)(n-2)}{2}$. □

7. Derivable trees

In this section, we study derivable trees and introduce Y–tree T_6 which is not a derivable tree.

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Lemma 7.1. *Y–tree T_6 is not a derivable graph.*

Proof. Let $V = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $E = \{e_1, e_2, e_3\}$. Consider tree $T = (V, E)$ in Figure 20.

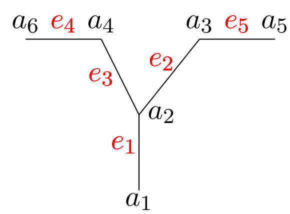

FIGURE 20. Y–Tree T_6

Let Y–tree T_6(Figure 20), be a derivable graph and $H = (\overline{V}, \{E_j\}_{j=1})$ be an associated hypergraph with graph Y–tree T_6. Since for every $1 \leq j \leq 4$, $\eta(a_2) *_i \eta(a_j) = \widehat{\eta(a_2), \eta(a_j)}$, $\eta(a_3) *_i \eta(a_5) = \widehat{\eta(a_3), \eta(a_5)}$ and $\eta(a_4) *_i \eta(a_6) = \widehat{\eta(a_4), \eta(a_6)}$, we get $k \in \mathbb{N}$, $E_1, E_2, E_3, E_4 \subseteq H$ so that $a_1 \in E_1, |E_1| = k, a_2 \in E_2, |E_2| = k + i, a_3, a_4 \in E_3$, $|E_3| = k + 2i$ $a_5, a_6 \in E_4$ and $|E_4| = k + 3i$. It follows that, $\eta(a_3) *_i \eta(a_6) = \widehat{\eta(a_3), \eta(a_6)}$ and $\eta(a_4) *_i \eta(a_5) = \widehat{\eta(a_4), \eta(a_5)}$, which is a contradiction. Therefore Y–tree T_5 can not be a derivable graph. □

Theorem 7.2. *Let $T = (V, E)$ be a tree and for every $v \in V, deg(v) \leq 2$. Then T is a derivable graph.*

Proof. Let $i \in \mathbb{N}, G = (V, E, *')$ be a graph with no cycle, in such a way that $V = \{a_1, a_2, \ldots, a_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, where $m \leq n$. Suppose that for every $1 \leq j \neq j' \leq m$, $e_{jj'} = \{a_j, a_{j'}\} = a_j *' a_{j'}$. Define a hypergraph $\overline{G} = (\overline{V}, \{\overline{E_i}\}_{i=1}^{n})$ as follows:

$$\overline{E_j} = \{a_j\} \cup A_j$$

such that $|A_1| = 1$, for every $1 < l \leq n$, $|A_{l+1}| - |A_l| = i$ and for every $1 \leq j, j' \leq n$ we have $A_j \cap A_{j+1} = \emptyset$. It is easy to see that for every $1 < j, j' \leq n$, $|\overline{E_{j+1}}| - |\overline{E_j}| = i$ and $\overline{E_j} \cap \overline{E_{j'}} = \emptyset$. A simple computation shows that $\overline{V} = \bigcup\limits_{i=1}^{n} A_i \cup V$ and $(\overline{V}, \{\overline{E_i}\}_{i=1}^{n})$ is a hypergraph. Clearly for every $1 \leq j \leq n, \eta(a_j) = \overline{E_j}$ and since $\overline{E_j} \cap \overline{E_{j'}} = \emptyset$, we get that $G/\eta = \{\eta(a_j) \mid 1 \leq j \leq n\}$ and so for every $i \in \mathbb{N}^*$ obtain

$$\eta(a_r) *_i \eta(a_s) = \begin{cases} \widehat{\eta(a_r), \eta(a_s)} & \text{if } |r - s| = i, \\ \widehat{\emptyset} & \text{if } |r - s| \neq i. \end{cases}$$

Now, define a map $\varphi : (G/\eta, *i) \longrightarrow G = (V, E)$ by $\varphi(\eta(a_j)) = a_j$ and $\varphi((\widehat{\eta(a_j), \eta(a_{j'})}))$ $= e_{jj'}$. Let $a_j, a_{j'} \in \overline{V}$. If $\eta(a_j) = \eta(a_{j'})$, then $|\overline{E_j}| = |\overline{E_{j'}}|$ and so $\overline{E_j} = \overline{E_{j'}}$. Thus $\varphi(\eta(a_j)) = \varphi(\eta(a_{j'}))$. Since for every $1 \leq j \neq j' \leq n$,

$$\varphi(\eta(a_j) *_i \eta(a_{j'})) = \varphi((\widehat{\eta(a_j), \eta(a_{j'})})) = e_{jj'} = a_j *' a_{j'} = \varphi(\eta(a_j)) *' \varphi(\eta(a_{j'})),$$

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in other words, if $\eta(a_j)$ and $\eta(a_{j'})$ in G/η are adjacent, then $\varphi(\eta(a_j))$ and $\varphi(\eta(a_{j'}))$ in G are adjacent. So φ is a homomorphism. It is easy to see that φ is a bijection and so is an isomorphism. It follows that every graph is a derivable graph. □

Corollary 7.3. *Let $n \leq 5$. Then T_n is a derivable graph.*

Example 7.4. *Let $V = \{a_1, a_2, a_3, a_4\}$, $E = \{e_1, e_2, e_3\}$. Consider tree T_4 in Figure 21.*

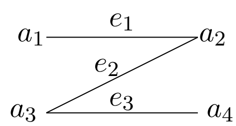

FIGURE 21. Tree T_4

For every arbitrary set $B = \bigcup_{i=1}^{10} b_i$ where for all j, j' have $a_j \neq b'_j$, define a hypergraph $\overline{G} = (\overline{V}, \{\overline{E_j}\})$ in such a way that $\overline{E_1} = \{a_1\} \cup \{b_1\}, \overline{E_2} = \{a_2\} \cup \{b_2, b_3\}, \overline{E_3} = \{a_3\} \cup \{b_4, b_5, b_6\}$ and $\overline{E_4} = \{a_4\} \cup \{b_7, b_8, b_9, b_{10}\}$. Hence consider the hypergraph $\overline{G} = (\overline{V}, \{\overline{E_i}\}_{i=1}^{n})$ in Figure 22.

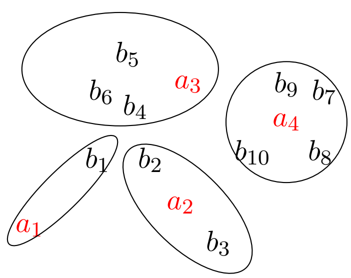

FIGURE 22. Associated hypergraph of graph Figure 21

By Theorem 7.2, for $*_i = *_1$ and every $1 \leq j \leq 4, \eta(a_j) = \overline{E_j}$, so

$$\overline{G}/\eta = (\{\eta(a_1), \eta(a_2), \eta(a_3), \eta(a_4)\}, \{\{\eta(a_1), \eta(a_2)\}, \{\eta(a_2), \eta(a_3)\}, \{\eta(a_3), \eta(a_4)\}\})$$

is a tree in Figure 23.

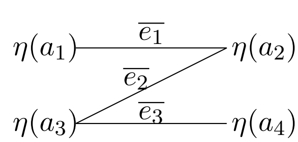

FIGURE 23. 2^{th} η^*–Tree obtained by hypergraph of Figure 22

It is easy to see that $T_4 \cong \overline{G}/\eta$ and so $T_4 = (V, E)$ is a derivable graph. Moreover, for every $2 \leq i \in \mathbb{N}$, we can construct another associated hypergraphs of graph G.

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Corollary 7.5. *Let (G, V, E) be a derivable graph. If*

$$\mathcal{A} = \{H \mid H \text{ is an associated hypergraph of graph } G\}.$$

Then $|\mathcal{A}| = |\mathbb{N}^*|$.

Theorem 7.6. *Let $T_n = (V, E)$ be a tree which is not contain a subtree, that is homeomorphic to Y–tree T_6 and $n \geq 6$. Then T_n is a derivable tree.*

Proof. Let $V = \{a_1, a_2, \ldots, a_n\}$. We rearrange the tree T by $n-1 \geq d_n \geq \cdots \geq d_3 \geq d_2 \geq d_1$, where $d_i = deg(a_i)$. Since $deg(a_1) = d_1$, we get that $a_{j1}, a_{j2}, \ldots, a_{jd_1-1}, E_1, E_2$ so that $a_1 \in E_1, a_{j1}, a_{j2}, \ldots, a_{jd_1-1} \in E_2$ and $\big||E_1| - |E_2|\big| = i$. By Lemma 7.1, for every $1 \leq j' \leq d_1 - 1$, $deg(a_{jj'}) = 1$ and by rearrangement $a_{jd_1} = a_2$. If $deg(a_{jd_1}) = 1$ the proof is obtained. If $deg(a_{jd_1}) > 1$ in a similar way, there exist $a_{r1}, a_{r2}, \ldots, a_{jd_2-1}, E_3, E_4$ so that $a_2 \in E_3, a_{r1}, a_{r2}, \ldots, a_{rd_1-1} \in E_4$ and $\big||E_3| - |E_4|\big| = i$. By Lemma 7.1, for every $1 \leq r' \leq d_2 - 1$, $deg(a_{rr'}) = 1$ and by rearrangement $a_{rd_2} = a_3$. If $deg(a_{rd_2}) = 1$ the proof is obtained. If $deg(a_{rd_2}) > 1$ we can continue. Since $|V| < \infty$, then this process stops. A simple computation shows that $H = (V \cup V', \{E_i\}_i)$ is a hypergraph and similar to Theorem 7.2, $H/\eta^* \cong T$. □

Corollary 7.7. *Let $T = (V, E)$ be a tree. Then $T = (V, E)$ is a non–derivable graph if and only if it contains a subtree that is homeomorphic to Y–tree T_6.*

Theorem 7.8. *Let $T = (V, E)$ be a tree and $i \in \mathbb{N}^*$. If $(V, \{E_j\}_j)$ is an associated hypergraph of tree T, then $\Delta(V/\eta^*) = |E| - i$.*

Proof. Let $V = \{a_1, a_2, \ldots, a_n\}$, for every $1 \leq j \leq n, deg(a_j) = d_j$ and $\Delta(G) = d_p$, where $1 \leq p \leq n$. Set $E_1 = \{a_p\}, E_2 = \{a_j \mid j \neq p \text{ and } \widehat{a_j, a_p} \neq \widehat{\emptyset}\}$ such that $|E_2| = i+1, E_k = \{a_k \mid a_k \notin E_1 \cup E_2\}$ and $|E_k| = 1$. A simple calculation shows that $(V, \{E_i\}_i)$ is a hypergraph. Since $|V| = n$, we get $\big|\{E_j;\ |E_j| = 1\}\big| = n - (i+1)$ and so hypergraph $(V, \{E_i\}_i)$ has $(n-i)$ hyperedges. It is easy to see that

$$V/\eta^* = \{\eta^*(a_p), \eta^*(a_k), \eta^*(a_{k'}) \mid a_k \in E_k, a_{k'} \notin E_1 \cup E_2\},$$

$|\eta^*(a_p)| = 1$, for every $a_k \in E_k$ we have $|\eta^*(a_k)| = i+1$ and for every $a_{k'} \notin E_1 \cup E_2, |\eta^*(a_{k'})| = 1$. So for every $r \neq r'$,

$$\eta(a_r) *_i \eta(a_{r'}) = \begin{cases} \widehat{\emptyset} & \text{if } \{a_r, a_{r'}\} \subseteq E_1 \cup E_k \cup E_{k'}, \\ \widehat{\eta(a_r), \eta(a_{r'})} & \text{otherwise.} \end{cases}$$

Since $|V| = n$ and T is a tree, we get that $|E| = n-1$ and so $\Delta(V/\eta^*) = n - i - 1 = |E| - i$. □

Example 7.9. *Let $V = \{1, 2, 3, 4, 5, 6, 7\}$. Consider tree $T = (V, E)$ in Figure 24.*
Since $\Delta(T) = 4$, we obtain $*_i = *_3$ and the hypergraph in Figure 25.
Hence V/η^* is a 4^{th} η^*–tree with 4 vertices and 3 edges such that $\Delta(V/\eta^*) = 3$ (Figure 26).

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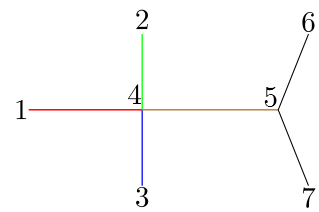

FIGURE 24. Tree T

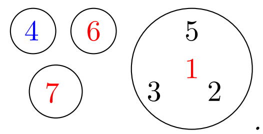

FIGURE 25. Obtained hypergraph from graph Figure 24

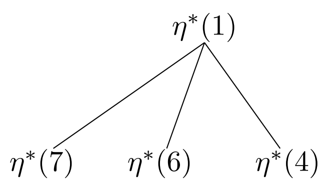

FIGURE 26. 4^{th} η^*–graph G/η^* obtained from hypergraph Figure 25

Corollary 7.10. *Let $n \in \mathbb{N}$.*

(i) *If $n \geq 2$, then any tree T_n is not a self derivable graph;*

(ii) *C_n is not a self derivable graph.*

Proof. (i) Let $|V| = n$, $T_n = (V, E)$ and T_n be a self derivable graph. By Theorem 7.8, T_n is a self derivable graph if and only if $|E| - i = |E|$ if and only if $i = 0$. By Theorem 5.4, must $T_n \cong K_n$ where is a contradiction.

(ii) Since C_n has cycle, by Theorem 5.4, it is clear. □

Definition 7.11. *Let $H, H' \in \mathcal{H}g$ and $i \in \mathbb{N}^*$. Define "$\sim_i$" on $\mathcal{H}g$ as follows:*

$$H \sim_i H' \Longleftrightarrow (H/\eta^*, *_i) \cong (H/\eta^*, *_i).$$

It is clear that $\sim_i$ is an equivalence relation.

Corollary 7.12. *Let $G, G' \in \mathcal{P}_h(H)$ and $G/\eta^* \cong T_n$. Then*

(i) *$G \sim_i G'$ if and only if $i \neq 0$;*

(ii) *$G \sim_i G'$ if and only if $n \mid \big||G'| - |G|\big|$;*

(iii) *$|[G]| = |\mathbb{N}|$.*

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8. Derivable complete graphs

In this section, we consider complete graph, complete bigraphs and investigate their associated hypergraphs.

Theorem 8.1. *Let $G = (V, E)$ be a derivable complete graph via associated hypergraph H and $|V| = n$. Then*

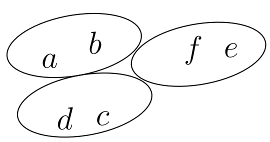
- (i) *if H is a discrete complete hypergraph, where their hyperedges are $2 \leq k$-hyperedge, then $*_i = *_0$;*
- (ii) *if H is a discrete complete hypergraph, where their hyperedges are $2 \leq k$-hyperedge, then $|H| = kn$.*

Proof. Clearly $H/\eta \cong G$.

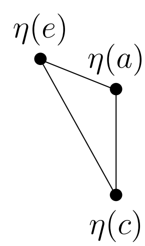
(i) Since H/η is a complete graph, we get that for any $\eta(x), \eta(y) \in H/\eta, \eta(x) *_i \eta(y) = \widehat{\eta(x), \eta(y)}$ and so $\big||\eta(x)| - |\eta(y)|\big| = i$. On the other hand, H is a discrete complete hypergraph, where $2 \leq k$-hyperedge, then for every $\eta(x), \eta(y) \in H/\eta, |\eta(x)| = |\eta(x)| = k$. It follows that $i = \big||\eta(x)| - |\eta(y)|\big| = |k - k| = 0$.

(ii) Since for every $\eta(x), \eta(y) \in H/\eta, |\eta(x)| = |\eta(x)| = k$ and $n = |H/\eta| = |H|/|\eta|$, we get that $|H| = n|\eta| = nk$. □

Example 8.2. *Let $H = \{a, b, c, d\}$. Consider the complete graph K_3 in Figure 27.*



(A) Hypergraph H'



(B) Complete graph K_3

FIGURE 27. Hypergraph H' and complete graph K_3

Define a hypergraph H' in Figure 27. Obviously, $\eta^* = \eta, \eta(a) = \{a, b\}, \eta(c) = \{c, d\}, \eta(e) = \{e, f\}$ and so for $*_i = *_0$ we have $H'/\eta \cong K_3$. Since H' is not a discrete complete hypergraph, we get that the converse of Theorem 8.1, is not necessarily true.

Theorem 8.3. *Let $n, r \in \mathbb{N}, H' = (H, \{E_i\}_{i=1}^n)$ be a discrete complete hypergraph, where for all $1 \leq i \leq n, |E_i| = r$ and $|H| = rn$. Then*

- (i) *if $*_i = *_0$, then its derivable graph is isomorphic to complete graph $K_{n/r}$;*
- (ii) *if $*_i = *_n$, then its derivable graph is isomorphic to complete graph $\overline{K}_{n/r}$.*

Proof. Since H' is a discrete complete hypergraph, where is a $2 \leq r$-hyperedge, then for all $\eta(x), \eta(y) \in H/\eta, |\eta(x)| = |\eta(y)| = r$.

(i) If $*_i = *_0$, then for every $\eta(x), \eta(y) \in H/\eta, \eta(x) *_0 \eta(y) = \widehat{\eta(x), \eta(y)}$. Since $|H| = n$ we get that $|H/\eta| = n/r$.

(ii) If for every $n \in \mathbb{N}$, $*_i = *_n$, then for every $\eta(x), \eta(y) \in H/\eta, \eta(x) *_n \eta(y) = \widehat{\emptyset}$. Hence H/η is a null graph and $|H| = n$ implies that $|H/\eta| = n/r$. □

Theorem 8.4. *Let $G = (V, E)$ be a derivable graph by $*_i = *_0$ and $|V| = n$. Then*

- (i) *if $G = (V, E)$ is a connected graph, then there exists $n \in \mathbb{N}^*$ so that $G \cong K_n$,*
- (ii) *if $G = (V, E)$ is a non–connected graph, then all connected components of G as G' are isomorphic to K_k or $\overline{K}_k$ whence, $1 \leq k \leq n \in \mathbb{N}^*$.*

Proof. Since $G = (V, E)$ is a derivable graph, then we have a hypergraph $H = (\overline{V}, \{\overline{E_i}\}_{i=1}^m)$ as associated graph to G, where $m \geq n$.

(i) Since $G = (V, E)$ is a connected derivable graph, $|V| = n$ and $*_i = *_0$, we get that there exists $a_1, a_2, \ldots, a_n \in H$ so that for every $1 \leq j, j' \leq n, \eta(a_j) *_i \eta(a_{j'}) = \widehat{\eta(a_j), \eta(a_{j'})}$.

(ii) Since $G = (V, E)$ is a non–connected derivable graph, then there exist $a_1, \ldots, a_n \in H$ such that for every $1 \leq j, j' \leq n, |\eta(a_j)| \neq |\eta(a_{j'})|$ or $|\eta(a_j)| = |\eta(a_{j'})|$. Let $1 \leq l \leq m$ and $\mathcal{P}_l = \{\eta(a_j) \mid |\eta(a_j)| = l\}$, then for every $1 \leq l \leq m$, it is easy to see that $\sum_{l=1}^{m}(l.|\mathcal{P}_l|) = m$. If G' be a connected components of G, then there exists $1 \leq l \leq m$ so that $G' \cong K_{|\mathcal{P}_l|}$.

□

Corollary 8.5. *Let $G, G' \in \mathcal{D}_c(H)$ and $G/\eta^* \cong K_k$. Then*

- (i) *$G \sim_i G'$ if and only if $i = 0$;*
- (ii) *$G \sim_i G'$ if and only if $k \mid \big||G'| - |G|\big|$;*
- (iii) $|[G]| = |\mathbb{N}|$.

Theorem 8.6. *Let $m, n \in \mathbb{N}$. Then for every $1 \leq i$ there exists a partitioned hypergraph H such that $H/\eta \cong K_{m,n}$, where $K_{m,n}$ is a complete bigraph.*

Proof. Let $1 \leq i$. Then we consider $t \in \mathbb{N}$ and a partitioned hypergraph $H' = (H, \{E_j\}_j)$, where has hyperedges $E_1, E_2, \ldots, E_m$ that for every $1 \leq j \leq m, |E_j| = t$ and hyperedges $E_{m+1}, E_{m+2}, \ldots, E_{n+m}$ that for every $m+1 \leq j \leq n, |E_j| = t+i$. Let for every $1 \leq j \leq m, |E_j| = \{a_j^1, a_j^2, \ldots, a_j^t\}$ and for every $m+1 \leq j \leq n+m, |E_j| = \{a_j^1, a_j^2, \ldots, a_j^{t+i}\}$. A simple calculation shows that for every $1 \leq j \leq m$, for every $1 \leq j' \leq t, \eta(a_j^{j'}) = \{a_j^1, a_j^2, \ldots, a_j^t\}$ and for every $m+1 \leq l \leq n+m$, for every $1 \leq l' \leq t+i, \eta(a_l^{l'}) = \{a_l^1, a_l^2, \ldots, a_l^{t+i}\}$. So $H/\eta = \{\eta(a_r^1), \eta(a_s^1) \mid 1 \leq r \leq m$ and $m+1 \leq s \leq n+m\}$, whence for every $1 \leq r \leq m, |\eta(a_r^1)| = t$ and for every $m+1 \leq s \leq n+m, |\eta(a_s^1)| = t+i$. Thus

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for every $1 \leq r \leq m$, and for every $m+1 \leq s \leq n+m$ we have $\eta(a_r^1) *_i \eta(a_s^1) = \eta(\widehat{a_r^1), \eta(}a_s^1)$, for every $1 \leq r \leq m, \eta(a_r^1) *_i \eta(a_r^1) = \widehat{\emptyset}$ and for every $m+1 \leq s \leq n+m, \eta(a_s^1) *_i \eta(a_s^1) = \widehat{\emptyset}$. Therefore $H/\eta \cong K_{m,n}$. □

Corollary 8.7. *Let $G, G' \in \mathcal{P}_h(H)$ and $G/\eta^* \cong K_{m,n}$. Then*

(i) *$G \sim_i G'$ if and only if $i \neq 0$;*

(ii) *$G \sim_i G'$ if and only if $(m+n) \mid \big||G'| - |G|\big|$;*

(iii) $|[G]| = |\mathbb{N}|$.

9. Conclusion

The current paper defined the concept of partitioned hypergraphs and enumerated the number of these hypergraphs. We obtained a relation ,η, on hypergraphs and showed that:

(i) On partitioned hypergraphs and discrete complete hypergraphs we have $\eta^* = \eta$, where η^* is as transitive closure of η.

(ii) Using the relation η^*, is constructed G/η^* and via this quotient hypergraph the concept of (self) derivable graph and extended derivable graphs are introduced.

(iii) It is shown that G is a self derivable graph if and only if G is a complete graph.

(iv) Cycle graph C_n is a derivable graph, if and only if $n = 3$ and $n = 4$.

(v) C_{2k} is an extended derivable graph.

(vi) It is obtained an equivalent condition that a graph is a non–derivable graph.

We hope that these results are helpful for further studies in graph theory. In our future studies, we hope to obtain more results regarding graphs, hypergraphs and their applications.

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