OMEGAS OF AGEMOS IN POWERFUL GROUPS

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ABSTRACT. In this note we show that for any powerful p-group G, the subgroup $\Omega_i(G^{p^j})$ is powerfully nilpotent for all $i, j \geq 1$ when p is an odd prime, and $i \geq 1$, $j \geq 2$ when $p = 2$. We provide an example to show why this modification is needed in the case $p = 2$. Furthermore we obtain a bound on the powerful nilpotency class of $\Omega_i(G^{p^j})$.

1. Introduction

It is well known that for a powerful p-group G, the ith Agemo subgroup, $\mho_i(G) = G^{p^i}$, coincides with the set of p^ith powers, and that this subgroup is itself powerful [5, Corollary 1.2, Proposition 1.7]. In [8] we introduced the notion of powerful nilpotence, and showed that for a powerful p-group G the groups G^{p^i}, for $i \geq 1$, are powerfully nilpotent. In some sense dual to the Agemo subgroups are the Omega subgroups, $\Omega_i(G)$. For a powerful p-group G these Omega subgroups are studied in [2].

In [8] we observed how powerfully nilpotent groups often occur as characteristic subgroups of powerful groups. For example the proper terms of the derived and lower central series of a powerful group G are powerfully nilpotent. One aim of this paper is to further motivate the study of the relationship between powerful groups and the powerfully nilpotent groups within them, by showing another important class of characteristic subgroups of powerful groups to be powerfully nilpotent.

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Furthermore, in [8] it was proved that for a powerfully nilpotent group G of order p^n, rank r, exponent p^e and powerful nilpotency class c, we have that $e \leq n - c + 1$ and $r \leq n - c + 1$. We call the quantity $n - c$ the powerful coclass of G. It follows that there are only finitely many groups of any given powerful coclass, and as such a weak classification of powerfully nilpotent groups exists in terms of an "ancestry tree" [8]. In [6] the question is asked "which p-groups are subgroups of powerful p -groups?" and this was studied further in [7]. Thus in light of all of this it is interesting to note which subgroups must be powerfully nilpotent.

By [4, Theorem 1.1] we know that for a powerful p-group G, if $N \lhd G$ and $N \leq G^p$ then N is powerful in the case where p is an odd prime. Similarly in the even case if $N \lhd G$ and $N \leq G^4$ then N is powerful. It follows from this that $\Omega_i(G^p)$ is powerful for p an odd prime, and in the even case that $\Omega_i(G^4)$ is powerful. However in what follows we give an elementary proof of the fact that these Omega subgroups are powerful. In particular in this note we prove that for an odd prime p and a powerful p-group G, the Omega subgroups of any proper Agemo subgroup are powerfully nilpotent (and hence powerful), and moreover we can obtain a bound on the powerful nilpotency class.

Theorem 3.8. *Let G be a powerful p-group for an odd prime p. Then $\Omega_i(G^{p^j})$ is powerfully nilpotent for $i, j \geq 1$. The powerful nilpotency class of $\Omega_i(G^{p^j})$ is at most i.*

We obtain a similar result for $p = 2$ with a small modification.

Theorem 3.11. *Let G be a powerful 2 group, then $\Omega_i(G^{2^j})$ is powerfully nilpotent for all $i \geq 1$, $j \geq 2$. Furthermore for $i > 1$ the powerful nilpotency class of $\Omega_i(G^{2^j})$ is at most $i - 1$. For $i = 1$ the powerful nilpotency class is 1.*

2. **Preliminaries**

In this section we set up notation and terminology. For a group G, we denote the centre of G by $Z(G)$, the commutator subgroup of G by G', and G^n denotes the subgroup generated by all nth powers of elements of G. For a p-group G, the group G^{p^i} is sometimes denoted as $\mho_i(G)$ and known as the ith Agemo subgroup of G. The ith Omega subgroup of G, denoted $\Omega_i(G)$, is the subgroup generated by all elements of G whose order divides p^i. The exponent of G is denoted by $\exp G$.

Definition. A finite p-group G is termed **powerful** if $p > 2$ and $G' \leq G^p$, or $p = 2$ and $G' \leq G^4$.

Definition. A powerful p-group G is said to be **powerfully nilpotent** if it has a chain of subgroups $\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that $[H_i, G] \leq H_{i-1}^p$ for $i = 1, \ldots, n$. Such a chain is called a **powerfully central chain**. If G is powerfully nilpotent, then the **powerful nilpotency class** of G is the shortest length that a powerfully central chain of G can have.

Notice that the assumption that G is powerful is not needed if p is odd.

Definition. For any prime p, a finite p-group G is **strongly powerful** if $G' \leq G^{p^2}$.

In [8], it is shown that a strongly powerful group must be powerfully nilpotent, thus in particular any powerful 2-group is powerfully nilpotent. The theory of powerful p-groups is developed in [1, 5].

For the convenience of the reader we now repeat, without proof, some results which are used in this paper.

In [2, Theorem 1] the following theorem is proved. We make extensive use of this theorem in our paper. In keeping with [2], for a p-group G and $x \in G$, we define the meaning of the inequality $o(x) \leq p^i$ with $i < 0$ to be that $x = 1$. Similarly we define $\Omega_i(G) = \{1\}$ for $i < 0$.

Theorem 2.1 (Fernández-Alcober). *Let G be a powerful p-group. Then, for every $i \geq 0$:*

(i) If $x, y \in G$ and $o(y) \leq p^i$, then $o([x, y]) \leq p^i$.

(ii) If $x, y \in G$ are such that $o(x) \leq p^{i+1}$ and $o(y) \leq p^i$, then $o([x^{p^j}, y^{p^k}]) \leq p^{i-j-k}$ for all $j, k \geq 0$.

(iii) If p is odd, then $\exp \Omega_i(G) \leq p^i$.

(iv) If $p = 2$, then $\exp \Omega_i(T) \leq 2^i$ for any subgroup T of G which is cyclic over G^2. In particular, $\exp \Omega_i(G^2) \leq 2^i$.

In [8, Proposition 1.1] we prove the following proposition.

Proposition 2.2. *Let G be any finite p-group of exponent p^e where $e \geq 2$. If G/G^{p^2} is powerfully nilpotent, then G is powerfully nilpotent. Furthermore if G/G^{p^2} has powerful class m, then the powerful class of G is at most $(e-1)m$.*

Remark 2.3. *In the proof of Proposition 2.2, we show that if $G/G^{p^2} = \bar{H}_0 \geq \bar{H}_1 \geq \cdots \geq \bar{H}_{m-1} \geq \{\bar{1}\}$ is a powerfully central series, where $\bar{H}_i = H_i/G^{p^2}$, then the descending chain*

$$
\begin{array}{ccccccc}
G = & H_0 \geq & H_1 \geq & \cdots \geq & H_{m-1} \geq & H_m = & G^p \\
G^p = & H_0^p \geq & H_1^p \geq & \cdots \geq & H_{m-1}^p \geq & H_m^p = & G^{p^2} \\
& \vdots & \vdots & & & & \\
G^{p^{e-2}} = & H_0^{p^{e-2}} \geq & H_1^{p^{e-2}} \geq & \cdots \geq & H_{m-1}^{p^{e-2}} \geq & 1 &
\end{array}
$$

is powerfully central.

3. Omega Subgroups of Agemo Subgroups

The natural place to start when considering Omega subgroups of powerful p-groups is $\Omega_i(G)$. However it is not true in general that $\Omega_i(G)$ is powerful and such counter examples are easy to find. Consider the following example.

Example 3.1. Let p be an odd prime, the p-group

$$G = \langle a, b, c | a^p = b^p = c^{p^2} = [c, b] = [c, a] = 1, [b, a] = c^p \rangle$$

is powerful (in fact it is powerfully nilpotent), but $\Omega_1(G) = \langle a, b, c^p \rangle$ is not powerful.

Thus we turn our attention to $\Omega_i(G^p)$. First we shall use Theorem 2.1 to prove that for a powerful p-group G, elements in G^p of order p commute with each other and with elements in G^p of order p^2.

Lemma 3.2. *Let G be a powerful p-group. Let $g_1, g_2 \in G^p$ where $o(g_1) = p$ and $o(g_2) \leq p^2$. Then $[g_1, g_2] = 1$.*

Proof. As G is powerful, we know that elements of G^p are pth powers, and so we may assume $g_1 = a^p, g_2 = b^p$ for $a, b \in G$ where $o(a) = p^2$ and $o(b) \leq p^3$. Using Theorem 2.1(ii) and taking $x = b$, $y = a$ and $i = 2$ we see that $o([x^p, y^p]) \leq p^{2-1-1} = 1$, hence $[g_2, g_1] = 1$. It follows that the elements in G^p of order p commute with the elements in G^p of order at most p^2. $\qquad\square$

Notice that from this we obtain that $\Omega_1(G^p)$ is abelian. The next result is needed in the proof of Proposition 3.4, although it is also of independent interest in the context of better understanding the relationship between Agemo and Omega subgroups in powerful p-groups.

Proposition 3.3. *Let G be a powerful p-group. Then $\left(\Omega_i(G^{p^k})\right)^{p^j} \leq \Omega_{i-j}(G^{p^{k+j}})$ and $\exp(\Omega_i(G^{p^k})^{p^j}) \leq p^{i-j}$ for $i, j \geq 0$ and $k \geq 1$.*

Proof. Consider an element $x \in \left(\Omega_i(G^{p^k})\right)^{p^j}$. This element can be written in the form $g_1^{p^j} \cdots g_t^{p^j}$ where $g_l \in \Omega_i(G^{p^k})$ for each $l \in \{1, \ldots, t\}$. Note that $g_l \in G^{p^k}$ and so $g_l^{p^j} \in G^{p^{k+j}}$. Using Theorem 2.1(iii) if p is odd and Theorem 2.1(iv) if $p = 2$, it follows that the order of each g_l is at most p^i. Then the order of each $g_l^{p^j}$ is at most p^{i-j}. Thus each $g_l^{p^j} \in \Omega_{i-j}(G^{p^{k+j}})$. As $\Omega_{i-j}(G^{p^{k+j}})$ is a group, it is closed under taking products and so $x = g_1^{p^j} \cdots g_t^{p^j} \in \Omega_{i-j}(G^{p^{k+j}})$. Hence $\left(\Omega_i(G^{p^k})\right)^{p^j} \leq \Omega_{i-j}(G^{p^{k+j}})$. Then by Theorem 2.1(iii) if p is odd and Theorem 2.1(iv) if $p = 2$, we obtain that $\exp(\Omega_i(G^{p^k})^{p^j}) \leq p^{i-j}$. $\quad\square$

We now consider the case where p is an odd prime. We seek to show that $\Omega_i(G^p)$ is powerfully nilpotent for all $i \geq 1$. Recall by Proposition 2.2 that for any p-group G we have that G is powerfully nilpotent if and only if G/G^{p^2} is powerfully nilpotent. Thus in what follows we consider $H = \frac{\Omega_i(G^p)}{(\Omega_i(G^p))^{p^2}}$, for some powerful p-group G. Let $K = \frac{G}{(\Omega_i(G^p))^{p^2}}$. Notice that K and K^p are powerful and that $H \leq K^p$.

Proposition 3.4. *H is a powerful group.*

Proof. The exponent of H is at most p^2, and so it follows from Lemma 3.2 that all elements of order p are central. We thus only need to consider commutators between elements of order p^2. Since $H \leq K^p$, we can thus assume these commutators are of the form $[a^p, b^p]$ where $o(a) = p^3 = o(b)$. Applying Theorem 2.1(ii) with $x = a$, $y = b$ and $i = 3$ we see that $o([a^p, b^p]) \leq p$. Since K is powerful, we have that $[a^p, b^p] \in [K^p, K^p] = [K, K]^{p^2} \leq K^{p^3}$, and hence there exists some $g \in K$ such that $[a^p, b^p] = g^{p^3}$, where g has order at most p^4. Let $g = x\left(\Omega_i(G^p)^{p^2}\right)$. Then $x^{p^4} \in \Omega_i(G^p)^{p^2}$, which is of exponent at

most p^{i-2}, by Proposition 3.3. Hence $o(x) \leq p^{4+i-2} = p^{i+2}$. Then x^{p^2} has order at most p^i and so $x^{p^2} \in \Omega_i(G^{p^2})$. Then $g^{p^3} = x^{p^3} \left(\Omega_i(G^p)^{p^2} \right) \in \left(\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \right)^p \leq H^p$. Thus H is powerful. $\square$

Lemma 3.5. *H is powerfully nilpotent of powerful nilpotency class at most 2, in particular $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$ is a powerfully central chain.*

Proof. We will show that $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$ is a powerfully central chain. In the proof of Proposition 3.4 we saw that $[H, H] \leq \left(\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \right)^p$. We now show that $\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \leq Z(H)$, to do this we will show that $[\Omega_i(G^{p^2}), \Omega_i(G^p)] \leq \Omega_i(G^p)^{p^2}$. Consider $[g^{p^2}, h^p]$ for $g, h \in G$ with $o(g) \leq p^{i+2}$ and $o(h) \leq p^{i+1}$. Using Theorem 2.1(ii) we obtain that $o([g^{p^2}, h^p]) \leq p^{i-2}$. As $[g^{p^2}, h^p] \in G^{p^4}$ we may write $[g^{p^2}, h^p] = k^{p^4}$ for some $k \in G$. Then $o(k^{p^2}) \leq p^i$ and so $[g^{p^2}, h^p] = (k^{p^2})^{p^2} \in \Omega_i(G^p)^{p^2}$. Thus $\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \leq Z(H)$. Hence it follows that $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$ is a powerfully central chain. $\square$

Using Lemma 3.5 and Proposition 2.2 one can obtain a powerfully central chain for $\Omega_i(G^p)$ of length $2i - 1$. However a shorter chain is possible. The following Lemma will be used to reduce the length of the chain.

Lemma 3.6. *$[\Omega_i(G^{p^2})^{p^j}, \Omega_i(G^p)] \leq \Omega_i(G^{p^2})^{p^{j+2}}$ for $i \geq 1$ and $j \geq 0$.*

Proof. By Proposition 3.3 we know that $[\Omega_i(G^{p^2})^{p^j}, \Omega_i(G^p)] \leq [\Omega_{i-j}(G^{p^{2+j}}), \Omega_i(G^p)]$, hence it suffices to show that $[\Omega_{i-j}(G^{p^{2+j}}), \Omega_i(G^p)] \leq \Omega_i(G^{p^2})^{p^{j+2}}$. Consider $g, h \in G$ with $o(g) \leq p^{i+2}$ and $o(h) \leq p^{i+1}$ then $g^{p^{2+j}} \in \Omega_{i-j}(G^{p^{2+j}})$ and $h^p \in \Omega_i(G^p)$. Using Theorem 2.1(ii) we obtain that $o([g^{p^{2+j}}, h^p]) \leq p^{i-j-2}$. Also notice that $[g^{p^{2+j}}, h^p] \in G^{p^{4+j}}$, and hence we may write $[g^{p^{2+j}}, h^p] = k^{p^{4+j}}$ for some $k \in G$, where $o(k^{p^{4+j}}) \leq p^{i-j-2}$. It follows that $k^{p^2} \in \Omega_i(G^{p^2})$ and $[g^{p^{2+j}}, h^p] = k^{p^{4+j}} \in \Omega_i(G^{p^2})^{p^{j+2}}$. Hence $[\Omega_{i-j}(G^{p^{2+j}}), \Omega_i(G^p)] \leq \Omega_i(G^{p^2})^{p^{j+2}}$. $\square$

Note that if $j > i$ in the above, the inclusion still holds, with both sides of the inequality being equal to the trivial group.

Theorem 3.7. *If G is a powerful p-group where p is an odd prime, then $\Omega_i(G^p)$ is powerfully nilpotent for all $i \geq 1$ and the powerful nilpotency class of $\Omega_i(G^p)$ is at most i.*

Proof. As we observed above, for $i = 1$ the group is abelian, thus we may assume $i \geq 2$. Note that if $p^e = \exp(\Omega_i(G^p)) < p^2$ then by Lemma 3.2 it follows the group is abelian and so of powerful class 1 and so the claim holds in this case. If $\exp(\Omega_i(G^p)) = p^2$ then $H \cong \Omega_i(G^p)$ and so the claim follows by Lemma 3.5. Thus we may assume that $e > 2$ and $i \geq 2$. In Lemma 3.5 we saw that $H = \frac{\Omega_i(G^p)}{(\Omega_i(G^p))^{p^2}}$ has a powerfully central chain $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$. Then by Remark 2.3 we have the following powerfully central chain for $\Omega_i(G^p)$:

$$
\begin{array}{lll}
\Omega_i(G^p) & \geq \Omega_i(G^{p^2}) & \geq \Omega_i(G^p)^p \\
\Omega_i(G^p)^p & \geq \Omega_i(G^{p^2})^p & \geq \Omega_i(G^p)^{p^2} \\
& \vdots & \\
\Omega_i(G^p)^{p^{e-2}} & \geq \Omega_i(G^{p^2})^{p^{e-2}} & \geq 1
\end{array}
$$

Now using Lemma 3.6 we see that the terms $\Omega_i(G^p)^{p^j}$ for $j \in \{1, \ldots, e-2\}$ are redundant. Noting that by Theorem 2.1(iii) we have that $\exp \Omega_i(G^p) \leq p^i$, we obtain the following powerfully central chain for $\Omega_i(G^p)$ of length at most i.

$$
\Omega_i(G^p) \geq \Omega_i(G^{p^2}) \geq \Omega_i(G^{p^2})^p \geq \cdots \geq \Omega_i(G^{p^2})^{p^{i-2}} \geq 1.
$$

$\square$

Later we shall see an example where this bound is attained. Recall that for a powerful p-group G, we have that G^{p^j} is powerful for all $j \geq 0$. Given a powerful group G, applying Theorem 3.7 to G^{p^j} gives that $\Omega_i(G^{p^{j+1}})$ is powerfully nilpotent for all $i \geq 1$. Thus we have that for a powerful p-group G, where p is an odd prime, all Omega subgroups of the proper Agemo subgroups are powerfully nilpotent.

Theorem 3.8. *Let G be a powerful p-group for an odd prime p. Then $\Omega_i(G^{p^j})$ is powerfully nilpotent for $i, j \geq 1$. The powerful nilpotency class of $\Omega_i(G^{p^j})$ is at most i.*

We now turn to the case $p = 2$. Due to the modification in the definition of a powerful 2-group, that is the requirement that $G' \leq G^{2^2}$, the arguments used above would require us to show that the group H is abelian. However, this is not true in general. Below we exhibit an example of a powerful 2-group such that $\Omega_2(G^2)$ is not powerful, and so we see that Theorem 3.7 cannot hold in its current form for $p = 2$.

Example 3.9. Consider the $2-$group

$$
G = \langle a, b, c \, | \, a^{2^3} = 1, b^{2^3} = 1, c^{2^5} = 1, [a, c] = 1, [b, c] = 1, [a, b] = c^{2^2} \rangle.
$$

One can check either by hand or with GAP [3], that this is a consistent presentation defining a group of order 2^{11}. Clearly G is powerful and so $G^2 = \langle a^2, b^2, c^2 \rangle$. Consider $\Omega_2(G^2)$; this subgroup contains everything in G^2 of order less than or equal to 4. In particular it contains a^2, b^2 and c^{2^3}. Notice $[a^2, b^2] = c^{2^4}$. Hence $\Omega_2(G^2)$ is not abelian, but then it cannot be powerful for it has exponent at most 4 (Theorem 2.1(iv)) and any powerful group of exponent at most 4 is abelian.

Also note that in the example above, the prime $p = 2$ can be replaced with any odd prime p to give a consistent presentation for a powerfully nilpotent group of order p^{11}, where the property still holds

that $\Omega_2(G^p)$ is not abelian. Thus in particular $\Omega_2(G^p)$ is not strongly powerful, yet is still powerfully nilpotent. Thus for p odd we see that the subgroups $\Omega_i(G^p)$ are an example of characteristic subgroups of a powerful group G which are powerfully nilpotent but not necessarily strongly powerful. This is in contrast to the subgroups G^{p^i} for $i \geq 1$, and the proper terms of the derived and lower central series of G, which are all strongly powerful [8]. Furthermore observe that $\Omega_2(G^p)$ has powerful nilpotency class 2 and so the bound from Theorem 3.7 is attained.

For the case $p = 2$ we make the following modification - instead of looking at $\Omega_i(G^p)$ we look at $\Omega_i(G^{p^2})$.

Theorem 3.10. *If G is a powerful $2-$group, then $\Omega_i(G^4)$ is powerfully nilpotent for all $i \geq 1$ and furthermore for $i > 1$ the powerful nilpotency class of $\Omega_i(G^4)$ is at most $i - 1$, for $i = 1$ the powerful class is 1.*

Proof. Consider $\tilde{H} = \frac{\Omega_i(G^4)}{(\Omega_i(G^4))^4}$, we will show that $\tilde{H}$ is abelian. By Lemma 3.2 we only need to consider commutators between elements of order 4. Let $\tilde{K} = G/(\Omega_i(G^4))^4$, and notice that $\tilde{K}$ and $\tilde{K}^4$ are powerful and $\tilde{H} \leq \tilde{K}^4$. We only need to consider commutators of the form $[a^4, b^4]$ where $o(a) = 2^4$ and $o(b) = 2^4$. However then by Theorem 2.1(ii), setting $i = 4$ yields that $o([a^{2^2}, b^{2^2}]) \leq p^{4-2-2}$ and thus the commutator is trivial. It follows that $\tilde{H}$ is abelian. Suppose that $\exp(\Omega_i(G^4)) = p^e$. If $e = 1$ then $\Omega_i(G^4)$ is abelian and so of powerful nilpotency class 1, otherwise by Proposition 2.2 the powerful class of $\Omega_i(G^4)$ is at most $e - 1$. Since $\Omega_i(G^{p^2}) \leq \Omega_i(G^p)$, by Theorem 2.1(iv) we obtain that $e \leq i$ and so the result follows. $\square$

As in the odd case, we can apply the above result to G^{2^j} to obtain the following.

Theorem 3.11. *Let G be a powerful 2 group, then $\Omega_i(G^{2^j})$ is powerfully nilpotent for all $i \geq 1$, $j \geq 2$. Furthermore for $i > 1$ the powerful nilpotency class of $\Omega_i(G^{2^j})$ is at most $i - 1$. For $i = 1$ the powerful nilpotency class is 1.*

Acknowledgments

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