www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 2 No. 1 (2013), pp. 45-47.

www.ui.ac.ir

FACTORIZING PROFINITE GROUPS INTO TWO ABELIAN SUBGROUPS

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Communicated by Patrizia Longobardi

In memory of Narain Gupta

Abstract. We prove that the class of profinite groups G that have a factorization $G = AB$ with A and B abelian closed subgroups, is closed under taking inverse limits of surjective inverse systems. This is a generalization of a recent result by K.H. Hofmann and F.G. Russo. As an application we reprove their generalization of Iwasawa's structure theorem for quasihamiltonian pro-p groups.

1. **Introduction**

In a forthcoming paper, [3], Hofmann and Russo are concerned with pro-p quasihamiltonian groups. By definition, in such a group G every pair X, Y of closed subgroups commutes as sets, i.e. $XY = YX$. When G is finite then such a group satisfies Iwasawa's structure theorem – namely, $G = A\langle b\rangle$ with A abelian and $\langle b\rangle$ cyclic, and so that $b^{-1}ab = a^{1+p^s}$ holds for some $s \geq 1$ (and $s \geq 2$ if $p = 2$) and all $a \in A$ – see e.g. [1, Theorem 1.4.3]. Hofmann and Russo term a group *nearabelian* if it satisfies Iwasawa's structure theorem without the restriction on s for $p = 2$. One of their main results is the fact that nearabelian pro-p groups form a category that is closed under taking strict inverse limits. Here it is meant that the inverse limit over an inverse system $(G_i, I, \leq)$ is *strict* provided that all the bonding maps $\psi_{ij} : G_i \to G_j$ for $i \geq j$ are epimorphisms. We are going to reprove this inverse limit result in a slightly more general context and want to use a well-known result from topology.

For a boolean space X let $\mathcal{C}(X)$ denote the subset of all nonempty closed subsets. The latter set can be equipped with the *Vietoris topology*, namely, when $X = \varprojlim_i X_i$ is the inverse limit of finite discrete spaces, then $\mathcal{C}(X) = \varprojlim_i \mathcal{C}(X_i)$ and we consider the initial topology with respect to canonical epimorphisms $\mathcal{C}(X) \to \mathcal{C}(X_i)$. See e.g. [2].

MSC(2010): Primary: 20E18; Secondary: 20F20.

Keywords: group factorization, pro-p groups, limits.

Received: 1 November 2012, Accepted: 28 December 2012.

Lemma 1.1. *Let (X_j, ϕ_{jk}, I) be an inverse system of compact spaces with bonding maps ϕ_{jk}. Suppose there are non-empty closed subsets $F_j \subseteq \mathcal{C}(X_j)$ none of them containing the empty set such that $(F_j, C(\phi_{jk}), I)$ is an inverse system of closed subsets of $\mathcal{C}(X_j)$ then*

(1) *$F := \varprojlim_j F_j \in \mathcal{C}(X)$ is not empty; and*
(2) *Every $A \in F$ is the inverse limit $A = \varprojlim_j \phi_j(A)$ and $\phi_j(A) \in F_j$.*

Proof. (1) is a general property of inverse limits.

(2). Note first that $C(\phi_j)(F) \subseteq F_j$. Therefore, for all $j \in I$, $A_j := \phi_j(A) \in F_j$. For $j \leq k$ we have $A_j = \phi_j(A) = \phi_{jk}\phi_k(A) = \phi_{jk}(A_k)$. Now [4, Corollary 1.1.8] shows that $A = \varprojlim_j \phi_j(A_j)$. □

2. **The Main Result**

Theorem 2.1. *Let $G = \varprojlim_i G_i$ be a strict inverse limit of profinite groups G_i that allow a factorization $G_i = A_iB_i$ with A_i and B_i closed abelian subgroups. Then $G = AB$ for suitable abelian closed subgroups A and B of G.*

Proof. We want to employ Lemma 1.1 and set $X_i := G_i \times G_i$. The inverse system (G_i, ψ_{ij}, I) gives rise to an inverse system (X_i, ϕ_{ij}, I) with bonding maps defined by $\phi_{ij}(g, h) := (\psi_{ij}(g), \psi_{ij}(h))$ for all $(g, h) \in G_i \times G_i$. As $G = \varprojlim_i G_i$ is strict so is $X := \varprojlim_i G_i \times G_i$. Define F_i to be the set of all cartesian products $A \times B$ of closed abelian subgroups A and B in G_i with $G_i = AB$ and note that F_i is not the empty set by assumption. Moreover, if $A \times B \in F_i$ then certainly $C(\phi_{ij})(A \times B) = \psi_{ij}(A) \times \psi_{ij}(B) \in F_j$ since $G_i = AB$ implies $\psi_{ij}(G_i) = \psi_{ij}(AB) = \psi_{ij}(A)\psi_{ij}(B)$.

Having thus established the premises of the Lemma we find that $\varprojlim_i F_i$ is not empty. Hence there are closed sets A and B with $\phi_i(A \times B) = \psi_i(A) \times \psi_i(B) \in F_i$, i.e. $G_i = \psi_i(A)\psi_i(B)$, for every $i \in I$. By (2) of the Lemma the sets A and B must be closed subgroups of G and, since all $\psi_i(A)$ and $\psi_i(B)$ are abelian, so are A and B.

For showing that $G = AB$ pick $x \in G$ arbitrary. Then there are $(a_i, b_i) \in A \times B$ with $\psi_i(x) = \psi_i(a_i)\psi_i(b_i)$, i.e. $a_ib_ix^{-1} \in \ker \psi_i$. Fix any open normal subgroup N of G and pass to a cofinal subset of I so that a_i and b_i converge respectively to elements $a \in A$ and $b \in B$. Then $a_i \in aN$ and $b_i \in bN$ holds for a cofinal subset of I and, for the same subset we have $abx^{-1} \in N \ker \psi_i$. Since $\bigcap_i N \ker \psi_i = N$ by [4, Lemma 1.1.16] we can conclude that $abx^{-1} \in N$. As N was an arbitrary open normal subgroup and $\bigcap_N N = 1$ we arrive at $x = ab$ as desired. □

Remark that G in the theorem is metabelian since, by Iwasawa's result, every finite factorizable group is metabelian. As a consequence we can reprove [3, Theorem 7.2] in a more direct way.

Corollary 2.2. *Let G be a pro-p group in which any two closed subgroups commute as sets. Then there is a closed normal abelian subgroup A of G, an element $b \in G$ and $s \geq 1$ ($s \geq 2$ if $p = 2$) such that $G = A\overline{\langle b \rangle}$ and, for every $a \in A$, $b^{-1}ab = a^{1+p^s}$.*

Proof. For any clopen normal subgroup N of G the quotient group G/N is a finite quasihamiltonian p-group. Therefore we can present $G = \varprojlim_i G_i$ as the strict inverse limit of finite quasihamiltonian

p-groups. Then, by Iwasawa's theorem for finite groups, [1, Theorem 1.4.3], $G_i = A_i\langle b_i\rangle$ with A_i normal in G_i and abelian in G_i and $a_i^{b_i} = a_i^{1+p^{s_i}}$ where $s_i \geq 2$ for $p = 2$. By Theorem 2.1 there are abelian subgroups A and B such that $G = AB$. Restricting in the proof of the main theorem the groups A_i to be normal and B_i to be cyclic this proof also yields that A is normal and B is procyclic – the inverse limit of cyclic finite p-groups.

Finally observe that $\psi_i(a^b) = \psi_i(a)^{\psi_i(b)} = \psi_i(a)^{1+p^{s_i}}$ holds for all $a \in A$ and the topological generator b of B. We claim that for a cofinal subset of indices i we must have $s_i = s$ for a fixed number $s \in \mathbb{N}$. Indeed

$$\psi_i(a)^{1+p^{s_j}} = \psi_i(a^{1+p^{s_j}}) = \psi_{ij}(\psi_j(a^{1+p^{s_j}})) = \psi_{ij}(\psi_j(a^b)) = \psi_i(a)^{\psi_i(b)} = \psi_i(a)^{1+p^{s_i}}$$

hence $\psi_i(a)^{p^{s_i}-p^{s_j}} = 1$. So, if for a cofinal subset of I we have $s_i \neq s_j$, the latter equation implies that $a = 1$. If $s_i \geq 2$ for a cofinal subset of I then certainly $s \geq 2$. □

The same Lemma from topology can be used to derive a inverse limit result on profinite Frobenius groups. Recall from [4, page 142] that the semidirect product $G = F \rtimes H$ of profinite groups H and F so that for every open normal subgroup N of G the orders $|HN/N|$ and $|FN/N|$ are coprime and $C_G(f) \leq F$ holds for all $1 \neq f \in F$ is termed *profinite Frobenius group.*

Corollary 2.3. *The strict inverse limit $G = \varprojlim_i G_i$ of profinite Frobenius groups $G_i = F_i \rtimes H_i$ is a profinite Frobenius group.*

Proof. Using the Lemma as before one can find F and H so that $G = FH$. Note that $G_i = \psi_i(F) \rtimes \psi_i(H)$ and F becomes normal in G since all F_i are normal in G_i. When N is any open normal subgroup of G then $|\psi_i(FN)/\psi_i(N)|$ and $|\psi_i(HN)/\psi_i(N)|$ are coprime and therefore so are $|FN/N|$ and $|HN/N|$. Suppose next that $f^g = f \neq 1$ for some $g \in G$. Then $\psi_i(f)^{\psi_i(g)} = \psi_i(f) \neq 1$ holds for a cofinal subset of indices $i \in I$. Hence $\psi_i(g) \in \psi_i(F)$ for these indices i and so $g \in F$. Thus $G = F \rtimes H$ is Frobenius group. □

Acknowledgments

The author wishes to thank K.H. Hofmann and F.G. Russo for lively email discussions on the subject.

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