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Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 8 No. 2 (2019), pp. 1-12.

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BOUNDS FOR THE SKEW LAPLACIAN (SKEW ADJACENCY) SPECTRAL RADIUS OF A DIGRAPH

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Communicated by Peter Csikvari

ABSTRACT. For a simple connected graph G with n vertices and m edges, let $\overrightarrow{G}$ be a digraph obtained by giving an arbitrary direction to the edges of G. In this paper, we consider the skew Laplacian/skew adjacency matrix of the digraph $\overrightarrow{G}$. We obtain upper bounds for the skew Laplacian/skew adjacency spectral radius, in terms of various parameters (like oriented degree, average oriented degree) associated with the structure of the digraph $\overrightarrow{G}$. We also obtain upper and lower bounds for the skew Laplacian/skew adjacency spectral radius, in terms of skew Laplacian/skew adjacency rank of the digraph $\overrightarrow{G}$.

1. Introduction

Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges. Let $\overrightarrow{G}$ be a digraph obtained by assigning arbitrarily a direction to each of the edges of G. The digraph $\overrightarrow{G}$ is called an orientation of G or oriented graph corresponding to G and the graph G as the underlying graph of $\overrightarrow{G}$. Let $d_i^+ = d^+(v_i)$, $d_i^- = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \ldots, n$ be the out-degree, in-degree and degree of the vertices of $\overrightarrow{G}$, respectively. The out-adjacency matrix $A^+ = A^+(\overrightarrow{G}) = (a_{ij})$ of a digraph $\overrightarrow{G}$ is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in-adjacency matrix $A^- = A^-(\overrightarrow{G}) = (a_{ij})$ of a digraph $\overrightarrow{G}$ is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_j, v_i) is an arc and $a_{ij} = 0$, otherwise. It is clear that $A^- = (A^+)^t$.

MSC(2010): 05C20, 05C50, 15B36.

Keywords: Digraph, skew Laplacian matrix, skew Laplacian spectrum, skew Laplacian spectral radius.

Received: 27 August 2018, Accepted: 06 March 2019.

DOI: http://dx.doi.org/10.22108/toc.2019.112589.1582

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The skew adjacency matrix $S = S(\overrightarrow{G}) = (s_{ij})$ of a digraph $\overrightarrow{G}$ is the $n \times n$ matrix, where

$$s_{ij} = \begin{cases} 1, & \text{if there is an arc from } v_i \text{ to } v_j, \\ -1, & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $S(\overrightarrow{G})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The energy of the matrix $S(\overrightarrow{G})$ was considered in [1], and is defined as

$$E_s(\overrightarrow{G}) = \sum_{i=1}^{n} |\xi_i|,$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the eigenvalues of $S(\overrightarrow{G})$. This energy of a digraph $\overrightarrow{G}$ is called the skew energy by Adiga et al. [1]. For recent developments in the theory of skew spectrum and skew energy see the survey [14].

Let $D^+ = D^+(\overrightarrow{G}) = diag(d_1^+, d_2^+, \ldots, d_n^+)$, $D^- = D^-(\overrightarrow{G}) = diag(d_1^-, d_2^-, \ldots, d_n^-)$ and $D(\overrightarrow{G}) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex out-degrees, vertex in-degrees and vertex degrees of $\overrightarrow{G}$, respectively. Let A^+ and A^- be respectively, the out-adjacency and in-adjacency matrix of a digraph $\overrightarrow{G}$. If $S(\overrightarrow{G})$ is the skew adjacency matrix of $\overrightarrow{G}$ and $A(G)$ is the adjacency matrix of the underlying graph G of the digraph $\overrightarrow{G}$, then it is clear that $A(G) = A^+ + A^-$ and $S(\overrightarrow{G}) = A^+ - A^-$. The matrix

$$\widetilde{SL}(\overrightarrow{G}) = \widetilde{D}(\overrightarrow{G}) - S(\overrightarrow{G}),$$

where $\widetilde{D}(\overrightarrow{G}) = D^+(\overrightarrow{G}) - D^-(\overrightarrow{G})$, is called the skew Laplacian matrix of the digraph $\overrightarrow{G}$. Clearly the matrix $\widetilde{SL}(\overrightarrow{G})$ is not symmetric and so its eigenvalues need not be real. The characteristic polynomial

$$P_{sl}(\overrightarrow{G}, x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,$$

of the matrix $\widetilde{SL}(\overrightarrow{G})$ is called the *skew Laplacian characteristic polynomial* of the digraph $\overrightarrow{G}$. The zeros of the polynomial $P_{sl}(\overrightarrow{G}, x)$, that is, the eigenvalues of the matrix $\widetilde{SL}(\overrightarrow{G})$ are the skew Laplacian eigenvalues of the digraph $\overrightarrow{G}$ and are denoted by $\nu_1, \nu_2, \ldots, \nu_n$. The skew Laplacian spectrum of the digraph $\overrightarrow{G}$ is denoted by $\text{Spect}_{sl}(\overrightarrow{G})$. The sign of the even cycle $C_k = u_1 u_2 \ldots u_k u_1$, denoted by $sgn(C_k)$, is defined as $sgn(C_k) = s_{12} s_{23} \ldots s_{k-1k} s_{k1}$, where s_{ij} is the entry of the skew matrix $S(\overrightarrow{G})$ indexed by u_i row and u_j column. An even oriented cycle C_k is called evenly-oriented (oddly-oriented) if its sign is positive (negative). If every even cycle in $\overrightarrow{G}$ is evenly-oriented, $\overrightarrow{G}$ is called evenly-oriented. An even oriented cycle C_{2k} is said to be uniformly oriented if $sgn(C_{2k}) = (-1)^k$.

The following observations are immediate from the definition of $\widetilde{SL}$.

Theorem 1.1. [4]

(i) *If $\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of $\widetilde{SL}(\overrightarrow{G})$, then $\sum_{i=1}^{n} \nu_i = 0$.*

(ii) 0 *is an eigenvalue of* $\widetilde{SL}(\overrightarrow{G})$ *with multiplicity at least* p*, where* p *is the number of components of* $\overrightarrow{G}$ *with all ones vector* $(1, 1, \ldots, 1)$ *as the corresponding eigenvector.*

(iii) *If* $P_{sl}(\overrightarrow{G}, x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$ *is the skew Laplacian characteristic polynomial of the digraph* $\overrightarrow{G}$*, then* $a_1 = 0,\ a_2 = m + \sum_{i<j}(d_i^+ - d_i^-)(d_j^+ - d_j^-),\ a_n = 0.$

Evidently much research has been done on spectral theory of skew matrices of oriented graphs, see [14], but the research on the skew Laplacian spectrum of a digraph $\overrightarrow{G}$ has recently started and it will be of interest to develop the theory in this direction. For some recent papers on the spectral theory of graphs and digraphs we refer to [2, 7, 8, 9, 10, 11, 15, 19] and the references therein.

As usual, we denote the complete graph on n vertices by K_n, the complete bipartite graph on $s+t$ vertices by $K_{s,t}$, the cycle on n vertices by C_n. For other undefined notations and terminology from graphs and spectral graph theory, the readers are referred to [3, 16].

The rest of the paper is organised as follows. In Section 2, we obtain the upper bounds for the skew Laplacian/skew adjacency spectral radius of an oriented graph, in terms of various parameter (like oriented degree, average oriented 2-degree) associated with the structure of the digraph $\overrightarrow{G}$ and the underlying graph G. These bounds improve some known upper bounds for some families of graphs. In Section 3, we obtain the upper and lower bounds for the skew Laplacian/skew adjacency spectral radius, in terms of the skew Laplacian/skew adjacency rank of the digraph $\overrightarrow{G}$.

2. Bounds for skew Laplacian/skew adjacency spectral radius

In this section, we obtain the upper bounds for the skew Laplacian/skew adjacency spectral radius of $\overrightarrow{G}$ in terms of various parameters associated with the structure of the digraph $\overrightarrow{G}$ and the underlying graph G.

The skew Laplacian spectral radius of the digraph $\overrightarrow{G}$ is denoted by $\rho_{sl}(\overrightarrow{G})$ and is defined as

$$\rho_{sl}(\overrightarrow{G}) = \max_i\{|\nu_i| :\ i = 1, 2, \ldots, n\}.$$

The singular values of a square matrix X of order n are defined as the positive square roots of the eigenvalues of the matrix X^*X. If $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the singular values and $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ are the absolute values of the eigenvalues of X, then it is well known that $|\lambda_1| \leq \sigma_1$ [13], with equality if and only if X is a normal matrix (a matrix X is said to be normal if $XX^* = X^*X$). This observation implies that any upper bound for the largest singular value σ_1 gives an upper bound for the spectral radius.

The following result can be found in [13].

Lemma 2.1. *The spectral radius* $\rho(X)$ *of an* $n \times n$ *matrix* $X = (x_{ij})$ *always satisfies*

$$\rho(X) \leq \min_i\{R_i,\ C_i\},$$

where $R_i = \max_i\{\sum_{k=1}^{n} |x_{ik}| : 1 \leq i \leq n\}$ *and* $C_i = \max_i\{\sum_{k=1}^{n} |x_{ki}| : 1 \leq i \leq n\}$*. Moreover, for a non-negative irreducible matrix, equality occurs if and only if all the row sums are equal.*

http://dx.doi.org/10.22108/toc.2019.112589.1582

A digraph $\overrightarrow{G}$ is said to be Eulerian if $d_i^+ = d_i^-$ for all $v_i \in V(\overrightarrow{G})$ and non-Eulerian otherwise.

The following upper bound for $\rho_{sl}(\overrightarrow{G})$, in terms of the adjacency spectral radius and signless Laplacian spectral radius of the underlying graph G was obtained in [2].

Theorem 2.2. *Let $\overrightarrow{G}$ be an orientation of a connected graph G of order n. Let λ_1 and q_1 be respectively, the largest adjacency eigenvalue and the largest signless Laplacian eigenvalue of the graph G. Then*

$$\rho_{sl}(\overrightarrow{G}) \le \begin{cases} \lambda_1, & \text{if } \overrightarrow{G} \text{ is Eulerian} \\ q_1, & \text{if } \overrightarrow{G} \text{ is non-Eulerian.} \end{cases}$$

If $\overrightarrow{G}$ is Eulerian, equality occurs if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

For each vertex $v_i \in V(\overrightarrow{G})$, let $\alpha_i = d_i^+ - d_i^-$ be its oriented degree. Let t_i be the sum of the absolute values of the oriented degrees of the vertices which are adjacent to v_i, that is, $t_i = \sum_{v_j, v_iv_j \in E(G)} |\alpha_j|$. We call t_i the 2-oriented degree and $m(v_i) = \frac{t_i}{|\alpha_i|}$ as the average oriented degree of the vertex v_i. Let $m^+(v_i) = \frac{t_i^+}{d_i^+}$, where $t_i^+ = \sum_{v_j, v_iv_j \in E(G)} d_j^+$, be the average positive degree of the vertex v_i.

The following gives an upper bound for $\rho_{sl}(\overrightarrow{G})$, in terms of oriented degrees α_i and the average oriented degrees $m(v_i)$ of the digraph $\overrightarrow{G}$.

Theorem 2.3. *Let $\overrightarrow{G}$ be an orientation of a connected graph G. If α_i is the oriented degree and $m(v_i)$ is the average oriented degree of the vertex v_i, then*

$$\rho_{sl}(\overrightarrow{G}) \le \begin{cases} \max_i\{|\alpha_i| + \sum\limits_{v_j,\ v_iv_j \in E(G)} |\alpha_i^{-1}\alpha_j| \sqrt{\frac{m_j}{m_i}}\}, & \text{if } \alpha_i \ne 0, \text{ for all } i \\ \max_i\{\overline{\delta}_i^k\}, & \text{if } \alpha_i = 0, \text{ for } 1 \le i \le k \\ \max_i\{\overline{\delta}_i\}, & \text{if } \alpha_i = 0, \text{ for } 1 \le i \le n, \end{cases}$$

where for $1 \le i \le k$, we have $\overline{\delta}_i^k = \sum\limits_{\substack{v_j,\ v_iv_j \in E(G) \\ j \le k}} \beta_i^{-1}\beta_j \sqrt{\frac{\overline{m}_j}{\overline{m}_i}} + \sum\limits_{\substack{v_j,\ v_iv_j \in E(G) \\ j \ge k+1}} \beta_i^{-1}|\alpha_j| \sqrt{\frac{m_j}{\overline{m}_i}}$, and for $k+1 \le i \le n$, we have $\overline{\delta}_i^k = \sum\limits_{\substack{v_j,\ v_iv_j \in E(G) \\ j \le k}} \beta_j|\alpha_i^{-1}| \sqrt{\frac{\overline{m}_j}{m_i}} + |\alpha_i| + \sum\limits_{\substack{v_j,\ v_iv_j \in E(G) \\ j \ge k+1}} |\alpha_i^{-1}\alpha_j| \sqrt{\frac{m_j}{m_i}}$; and $\overline{\delta}_i = \sum\limits_{v_j,\ v_iv_j \in E(G)} \beta_i^{-1}\beta_j \sqrt{\frac{\overline{m}_j}{\overline{m}_i}}$, $\beta_i = \min_i\{d_i^+, d_i^-\}$. If $\overrightarrow{G}$ is an Eulerian digraph, then equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

Proof. Let G be a connected graph of order n having m edges and let $\overrightarrow{G}$ be an orientation of G. In $\overrightarrow{G}$, suppose that the oriented degree α_i of each of the vertex v_i is non-zero. Let $\tilde{D} = diag(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $K = diag(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_n})$, where $m_i = m(v_i)$ is the average oriented degree. Then the matrices $\tilde{D}^{-1} = diag(\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1})$ and $K^{-1} = diag(m_1^{\frac{-1}{2}}, m_2^{\frac{-1}{2}}, \ldots, m_n^{\frac{-1}{2}})$ exist. Consider the matrix $M = K^{-1}(\tilde{D}^{-1}\widetilde{SL}\tilde{D})K$. It is easy to see that the $(i,i)^{th}$ entry of this matrix is α_i and the $(i,j)^{th}$ entry is $\alpha_i^{-1}\alpha_j\sqrt{\frac{m_j}{m_i}}s_{ij}$, where $s_{ij} = -1$ or 0 or 1. Clearly the matrices $\widetilde{SL}$ and M are similar, therefore $\rho_{sl}(\overrightarrow{G}) = \rho(M)$. Taking $X = M$, and using Lemma 2.1 the result follows in this case.

http://dx.doi.org/10.22108/toc.2019.112589.1582

Suppose that some k, $1 \leq k \leq n-1$ vertices of $\overrightarrow{G}$ has oriented degree zero. With out loss of generality, suppose these vertices are $v_1, v_2, \ldots, v_k$. Label the vertices of $\overrightarrow{G}$ in such a way that the first k rows and columns of the matrix $\widetilde{SL}$ correspond to the vertices $v_1, v_2, \ldots, v_k$. Let $D_1 = diag(\beta_1, \beta_2, \ldots, \beta_k, \alpha_{k+1}, \ldots, \alpha_n)$, where $\beta_i = \min_i\{d_i^+, d_i^-\}$ and $K_1 = diag(\sqrt{\overline{m}_1}, \sqrt{\overline{m}_2},\ dots, \sqrt{\overline{m}_k}, \sqrt{m_{k+1}}, \ldots, \sqrt{m_n})$, where $\overline{m}_i = \frac{\overline{t}_i}{\beta_i}$, with $\overline{t}_i = \sum_{v_j, v_iv_j \in E(G)} \beta_j$ and $\beta_i = \min_i\{d_i^+, d_i^-\}$, if $d_i^+ d_i^- \neq 0$ and $\beta_i = \max_i\{d_i^+, d_i^-\}$, if $d_i^+ d_i^- = 0$. It is easy to see that for the matrix $K_1^{-1}(D_1^{-1}\widetilde{SL}D_1)K_1$ the $(i,i)^{th}$ entry is equal to 0 for $1 \leq i \leq k$ and equal to α_i, for $k+1 \leq i \leq n$. Also for $1 \leq i \leq k$ the $(i,j)^{th}$ entry of this matrix is equal to $\beta_i^{-1}\beta_j\sqrt{\frac{\overline{m}_j}{\overline{m}_i}}s_{ij}$, for all $1 \leq j \leq k$ and equal to $\beta_i^{-1}\alpha_j\sqrt{\frac{m_j}{\overline{m}_i}}s_{ij}$, for all $k+1 \leq j \leq n$; and for $k+1 \leq i \leq n$ its $(i,j)^{th}$ entry is equal to $\beta_i\alpha_j^{-1}\sqrt{\frac{\overline{m}_j}{m_i}}s_{ij}$, for all $1 \leq j \leq k$ and equal to $\alpha_i^{-1}\alpha_j\sqrt{\frac{m_j}{m_i}}s_{ij}$, for $k+1 \leq j \leq n$, where $s_{ij} = -1$ or 0 or 1. Proceeding similarly as above the result follows in this case.

Lastly, suppose that all vertices of $\overrightarrow{G}$ have oriented degree zero. Let $D_2 = diag(\beta_1, \beta_2, \ldots, \beta_n)$ and $K_2 = diag(\sqrt{\overline{m}_1}, \sqrt{\overline{m}_2}, \ldots, \sqrt{\overline{m}_n})$. Now, consider the matrix $K_2^{-1}(D_2^{-1}\widetilde{SL}D_2)K_2$ and proceeding similarly as above, it can be seen that the result follows in this as well.

If $\overrightarrow{G}$ is an Eulerian digraph, then as shown in Theorem 2.2 that $\rho_{sl}(\overrightarrow{G}) \leq \lambda_1(G)$, with equality if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$. Let λ_1 be the adjacency spectral radius of G, then $\lambda_1(G) = \lambda_1(|\widetilde{SL}(\overrightarrow{G}|) = \lambda_1(|K_2^{-1}(D_2^{-1}\widetilde{SL}D_2)K_2|)$. Using Lemma 2.1, it follows that

$$\lambda_1(G) \leq \max_i\{\sum_{v_j,\ v_iv_j \in E(G)} \beta_i^{-1}\beta_j\sqrt{\frac{\overline{m}_j}{\overline{m}_i}}\},$$

with equality if and only if all the row sums of $|K_2^{-1}(D_2^{-1}\widetilde{SL}D_2)K_2|$ are equal. It can be seen that all the row sums of $|K_2^{-1}(D_2^{-1}\widetilde{SL}D_2)K_2|$ are equal if and only if G is a regular graph. Thus combining these observations it follows that for an Eulerian digraph equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$. This completes the proof. $\square$

Proceeding similarly as in Theorem 2.3, we obtain the following upper bound for skew adjacency spectral radius $\rho_s(\overrightarrow{G})$ of the digraph $\overrightarrow{G}$.

Theorem 2.4. *Let $\overrightarrow{G}$ be an orientation of a connected graph G of order n. Then*

$$\rho_s(\overrightarrow{G}) \leq \max_i\{\sum_{v_j, v_iv_j \in E(G)} \beta_i^{-1}\beta_j\sqrt{\frac{\overline{m}_j}{\overline{m}_i}}\},$$

where $\overline{m}_i = \frac{\overline{t}_i}{\beta_i}$, with $\overline{t}_i = \sum_{v_j, v_iv_j \in E(G)} \beta_j$ and $\beta_i = \min_i\{d_i^+, d_i^-\}$, if $d_i^+ d_i^- \neq 0$ and $\beta_i = \max_i\{d_i^+, d_i^-\}$, if $d_i^+ d_i^- = 0$. Equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

Remark 2.5. *It is clear that unlike the upper bound given by Theorem 2.2, the upper bound given by Theorem 2.3 for the skew Laplacian spectral radius is orientation dependent. Therefore, there can be*

http://dx.doi.org/10.22108/toc.2019.112589.1582

orientations of G for which the upper bound given by Theorem 2.3 will be better than the upper bound given by Theorem 2.2.

Let G be an r-regular graph with r odd and let $\overrightarrow{G}$ be an orientation of G, obtained when among the r edges incident at a vertex v_i, $\frac{r-1}{2}$ edges are directed away and $\frac{r-1}{2}$ edges are directed towards the vertex v_i, the remaining edges are directed randomly. For the digraph $\overrightarrow{G}$, we have $d_i^+ = \frac{r-1}{2}$ or $\frac{r-1}{2}+1$; $d_i^- = \frac{r-1}{2}+1$ or $\frac{r-1}{2}$, for all i and so $\beta_i = \frac{r-1}{2}$ for all i. Also $|\alpha_i| = 1$ and $m_i = r$ for all i. Since $q_1(G) = 2r$ and the digraph $\overrightarrow{G}$ is non-Eulerian, from Theorem 2.2, it follows that $\rho_{sl}(\overrightarrow{G}) \leq 2r$. Theorems 2.3 implies that,

$$\begin{aligned}\rho_{sl}(\overrightarrow{G}) &\leq \max_i\{|\alpha_i| + \sum_{v_j,\ v_iv_j\in E(G)} |\alpha_i^{-1}\alpha_j|\sqrt{\frac{m_j}{m_i}}\}\\ &= 1 + \frac{1}{\sqrt{r}}\left(\sqrt{r}+\sqrt{r}+\cdots+\sqrt{r}\right) = r+1,\end{aligned}$$

which is clearly an improvement to the upper bound given by Theorem 2.2.

Let $N_i^+ = N^+(v_i) = \{v_j : v_iv_j \in E(\overrightarrow{G})\}$ and $N_i^- = N^-(v_i) = \{v_j : v_jv_i \in E(\overrightarrow{G})\}$, be respectively, the set of out-neighbours and in-neighbours of the vertex v_i in $\overrightarrow{G}$. Clearly, $N_i^+ \cup N_i^- = N_i$, the neighbourhood set of the vertex v_i and $N_i^+ \cap N_i^- = \phi$. The following result gives an upper bound for the skew spectral radius $\rho_s(\overrightarrow{G})$, in terms of out-neighbours N_i^+, in-neighbours N_i^- and the neighbours N_i of any vertex v_i of the digraph $\overrightarrow{G}$.

Theorem 2.6. *Let $\overrightarrow{G}$ be an orientation of a connected graph G of order n. Then*

$$\rho_s(\overrightarrow{G}) \leq \max_{1\leq i\leq n}\sqrt{d_i+\gamma_{ij}},$$

where $\gamma_{ij} = \sum_{j=1, i\neq j}\left(\left|n(N_i^+\cap N_j^-) + n(N_i^-\cap N_j^+) - n(N_i^+\cap N_j^+) - n(N_i^-\cap N_j^-)\right|\right)$ *and by $n(U)$, we mean the number of elements in the set U. Equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.*

Proof. Let $S(G) = (s_{ij})$ be the skew adjacency matrix of the digraph $\overrightarrow{G}$. Since $|z^2| = |z|^2$, for any complex number $z = a + ib$, it follows that

$$\rho_s(\overrightarrow{G}) = \rho_s(S(\overrightarrow{G})) = \sqrt{\rho_s(S^2(\overrightarrow{G}))}. \tag{2.1}$$

Let us consider the matrix $S^2(\overrightarrow{G}) = S(\overrightarrow{G})S(\overrightarrow{G})$. It is easy to see that the $(i,i)^{th}$ entry of the matrix $S^2(\overrightarrow{G})$ is $s_{i1}s_{1i} + s_{i2}s_{2i} + \cdots + s_{in}s_{ni} = -n(N_i) = -d_i$, as if $s_{ij} = 1$, then $s_{ji} = -1$. Also, it is easy to see that the $(i,j)^{th}$ entry of the matrix $S^2(\overrightarrow{G})$ is $\sum_{k=1}^n s_{ik}s_{kj} = s_{i1}s_{1j} + s_{i2}s_{2j} + \cdots + s_{ik}s_{kj} + \cdots + s_{in}s_{nj}$. Since s_{ij} can be 0 or 1 or -1, it follows that $s_{ik}s_{kj}$ can be 0 or 1 or -1. If $s_{ik}s_{kj} = 1$, then $s_{ik} = 1$ and $s_{kj} = 1$ or $s_{ik} = -1$ and $s_{kj} = -1$. This gives that the number of $1^{'s}$ in the sum $\sum_{k=1}^n s_{ik}s_{kj}$ is the sum of the number of out-neighbours of v_i which are the in-neighbours of v_j and the number of in-neighbours of v_i which are the out-neighbours of v_j . Similarly, if $s_{ik}s_{kj} = -1$, then $s_{ik} = 1$ and $s_{kj} = -1$ or $s_{ik} = -1$ and $s_{kj} = 1$ giving that the number of $-1^{'s}$ in the sum $\sum_{k=1}^n s_{ik}s_{kj}$ is the sum of the number of out-neighbours of v_i which are the out-neighbours of v_j and the number of in-neighbours of v_i which are the in-neighbours of v_j. Thus, it follows that

$$\sum_{k=1}^{n} s_{ik}s_{kj} = n(N_i^+ \cap N_j^-) + n(N_i^- \cap N_j^+) - n(N_i^+ \cap N_j^+) - n(N_i^- \cap N_j^-)$$

Applying Lemma 2.1 to the matrix $S^2(\overrightarrow{G})$ and using equation (2.1), the result follows. Equality can be discussed similarly as in Theorem 2.3. This completes the proof. □

Since for an Eulerian digraph $\rho_{sl}(\overrightarrow{G}) = \rho_s(\overrightarrow{G})$. Therefore, we have the following observation, which is immediate from Theorem 2.6.

Theorem 2.7. *Let $\overrightarrow{G}$ be an orientation of a connected graph G of order n. If $\overrightarrow{G}$ is an Eulerian digraph, then*

$$\rho_{sl}(\overrightarrow{G}) \le \max_{1\le i\le n} \sqrt{d_i + \gamma_{ij}},$$

where $\gamma_{ij} = \sum_{j=1, i\neq j} \left(\left| n(N_i^+ \cap N_j^-) + n(N_i^- \cap N_j^+) - n(N_i^+ \cap N_j^+) - n(N_i^- \cap N_j^-) \right| \right)$ *and by $n(U)$, we mean the number of elements in the set U. Equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.*

Remark 2.8. *For the graph $S_n^* = K_{1,n-1} + e$, $n \ge 4$ consider the orientation H_1 as shown in Figure 1. For the digraph H_1, we have $d_1^+ = n-2, d_2^+ = d_3^+ = \cdots = d_{n-2}^+ = 0, d_{n-1}^+ = d_n^+ = 1; d_1^- = 1, d_2^- = d_3^- = \cdots = d_n^- = 1$ or $d_1^+ = 1, d_2^+ = d_3^+ = \cdots = d_n^+ = 1; d_1^- = n-2, d_2^- = d_3^- = \cdots = d_{n-2}^- = 0, d_{n-1}^- = d_n^- = 1$. In each of the cases, we obtain $\beta_1 = \beta_2 = \beta_3 = \cdots = \beta_{n-1} = \beta_n = 1$. Also, $|\alpha_1| = n-3, |\alpha_2| = |\alpha_3| = \cdots = |\alpha_{n-2}| = 1, |\alpha_{n-1}| = |\alpha_n| = 0$, $m_1 = 1, m_2 = m_3 = \cdots = m_{n-2} = n-3, \overline{m}_{n-1} = \overline{m}_n = 2$ and $\overline{m}_1 = n+1, \overline{m}_2 = \cdots = \overline{m}_{n-2} = 1, \overline{m}_{n-1} = \overline{m}_n = 2$. Since $q_1(S_n^*) = n$, it follows from Theorem 2.2, that $\rho_{sl}(H_1) \le n$. But, it can be seen that Theorems 2.3, tells us that $\rho_{sl}(H_1) \le n - 2 + \frac{2}{n-3}$, which is clearly an improvement to the upper bound given by Theorem 2.2.*

For the skew spectral radius, each of the Theorems 2.4 and 2.6, gives that $\rho_s(H_1) \le \sqrt{n+1}$.

3. Bounds for skew Laplacian/skew adjacency spectral radius in terms of skew Laplacian/skew adjacency rank

Let $S(\overrightarrow{G})$ be the skew matrix of the digraph $\overrightarrow{G}$. The rank of the skew matrix $S(\overrightarrow{G})$ denoted by $r_s = r_s(\overrightarrow{G})$ is called the skew rank of the graph G [14]. Like wise, we denote the rank of the matrix $\widetilde{SL}(\overrightarrow{G})$ by $r_{sl} = r_{sl}(\overrightarrow{G})$ and call it the skew Laplacian rank of the graph G. It is clear that for Eulerian digraphs $\overrightarrow{G}$, we have $r_{sl} = r_s$ (as for Eulerian digraphs $\widetilde{SL}(\overrightarrow{G}) = S(\overrightarrow{G})$). For recent works on skew rank, we refer to [18] and the references therein.

In this section, we obtain the lower and upper bounds for skew Laplacian spectral radius $\rho_{sl}(\overrightarrow{G})$ and skew adjacency spectral radius $\rho_s(\overrightarrow{G})$, in terms of the skew Laplacian rank r_{sl} and the skew adjacency rank r_s, respectively of the graph G.

The following observation can be found in [17].

http://dx.doi.org/10.22108/toc.2019.112589.1582

Lemma 3.1. *Let X be a complex $n \times n$ matrix with rank$(X) \leq r \leq n$. Then*

$$\rho(X) \leq \frac{|tr(X)|}{r} + \Big(\frac{r-1}{2r}\Big)^{\frac{1}{2}}\Big(c(X) - \frac{|tr(X)|^2}{r} + \Big|tr(X^2) - \frac{(tr(X))^2}{r}\Big|\Big)^{\frac{1}{2}},$$

$$\text{where } c(X) = \frac{|tr(X)|^2}{r} + \Big[\Big(\|A\|^2 - \frac{|tr(X)|^2}{r}\Big)^2 - \frac{1}{2}\|A^*A - AA^*\|^2\Big]^{\frac{1}{2}},$$

where $\|A\|$ denoted the Euclidean norm of the matrix A.

The following gives an upper bound for the skew Laplacian spectral radius $\rho_{sl}(\overrightarrow{G})$, in terms of the oriented degrees α_i of the digraph $\overrightarrow{G}$, the skew Laplacian rank r_{sl} and the number of edges m of the underlying graph G.

Theorem 3.2. *Let G be a connected graph of order $n \geq 3$ having m edges. Let $\overrightarrow{G}$ be an orientation of G having skew Laplacian rank $r_{sl} \geq 1$, then*

$$\rho_{sl}(\overrightarrow{G}) \leq \Big(\frac{r_{sl}-1}{2r_{sl}}\Big)^{\frac{1}{2}}\Big(\Big[\Big(2m + \sum_{i=1}^{n}\alpha_i^2\Big)^2 - 2\sum_{i,j=1}^{n}\Big(\alpha_j - \alpha_i\Big)^2\Big]^{\frac{1}{2}} + |\sum_{i=1}^{n}\alpha_i^2 - 2m|\Big)^{\frac{1}{2}},$$

where $\alpha_i = d_i^+ - d_i^-$.

Proof. Let G be a connected graph and let $\overrightarrow{G}$ be an orientation of G having skew Laplacian rank $r_{sl} \geq 1$. Let $\widetilde{SL}(\overrightarrow{G}) = (l_{ij})$ be the skew Laplacian matrix of the digraph $\overrightarrow{G}$, then $\widetilde{SL}(\overrightarrow{G}) = \widetilde{D}(\overrightarrow{G}) - S(\overrightarrow{G})$, where $S(\overrightarrow{G}) = (s_{ij})$ is the skew adjacency matrix of $\overrightarrow{G}$. From Theorem 1.1, it is well known that $tr(\widetilde{SL}) = 0$. Also

$$\begin{aligned} tr(\widetilde{SL}^2) &= tr((\widetilde{D} - S(\overrightarrow{G}))^2)\\ &= tr(\widetilde{D}^2) + tr(S(\overrightarrow{G})^2) - tr(\widetilde{D}S(\overrightarrow{G})) - tr(S(\overrightarrow{G})\widetilde{D}). \end{aligned}$$

Since $tr(S(\overrightarrow{G})) = 0$, it follows that $tr(S(\overrightarrow{G})\widetilde{D}) = tr(\widetilde{D}S(\overrightarrow{G})) = 0$. Using this in above, we obtain

$$tr(\widetilde{SL}^2) = -2m + \sum_{i=1}^{n}(d_i^+ - d_i^-)^2 = \sum_{i=1}^{n}\alpha_i^2 - 2m,$$

where $\alpha_i = d_i^+ - d_i^-$.
Further, since $\widetilde{SL}$ is a real matrix, it follows that $\widetilde{SL}^* = \widetilde{SL}^t = \widetilde{D} + S(\overrightarrow{G})$ and so

$$\widetilde{SL}^*\widetilde{SL} = \widetilde{SL}^t\widetilde{SL} = (\widetilde{D} + S(\overrightarrow{G}))(\widetilde{D} - S(\overrightarrow{G})) = \widetilde{D}^2 - \widetilde{D}S(\overrightarrow{G}) + S(\overrightarrow{G})\widetilde{D} - S(\overrightarrow{G})^2.$$

Similarly, we have $\widetilde{SL}\widetilde{SL}^* = \widetilde{D}^2 + \widetilde{D}S(\overrightarrow{G}) - S(\overrightarrow{G})\widetilde{D} - S(\overrightarrow{G})^2$. Using these we get

$$\widetilde{SL}^*\widetilde{SL} - \widetilde{SL}\widetilde{SL}^* = 2S(\overrightarrow{G})\widetilde{D} - 2\widetilde{D}S(\overrightarrow{G}).$$

From this we obtain that the

$$\begin{aligned} &(i,j)^{th} \text{ entry of the matrix } \widetilde{SL}^*\widetilde{SL} - \widetilde{SL}\widetilde{SL}^*\\ &= (i,j)^{th} \text{ entry of the matrix } 2S(\overrightarrow{G})\widetilde{D} - 2\widetilde{D}S(\overrightarrow{G})\\ &= 2s_{ij}(d_j^+ - d_j^-) - 2(d_i^+ - d_i^-)s_{ij} = 2s_{ij}\Big[(d_j^+ - d_j^-) - (d_i^+ - d_i^-)\Big]. \end{aligned}$$

So we have

$$\|\widetilde{SL}^*\widetilde{SL}-\widetilde{SL}\widetilde{SL}^*\|^2=\sum_{i,j=1}^{n}4\Big[(d_j^+-d_j^-)-(d_i^+-d_i^-)\Big]^2.$$

Also, $\|\widetilde{SL}\|^2=\sum\limits_{i,j=1}^{n}l_{ij}^2=\sum\limits_{i=1}^{n}\Big[(d_i^++d_i^-)+(d_i^+-d_i^-)^2\Big]=2m+\sum\limits_{i=1}^{n}\alpha_i^2.$

Now, taking $X=\widetilde{SL}(\overrightarrow{G})$, in Lemma 3.1, and using the above discussion, we obtain

$$\rho_{sl}(\overrightarrow{G})\leq\Big(\frac{r_{sl}-1}{2r_{sl}}\Big)^{\frac{1}{2}}\Big(\Big[\Big(2m+\sum_{i=1}^{n}\alpha_i^2\Big)^2-2\sum_{i,j=1}^{n}\Big(\alpha_j-\alpha_i\Big)^2\Big]^{\frac{1}{2}}+|\sum_{i=1}^{n}\alpha_i^2-2m|\Big)^{\frac{1}{2}},$$

where $\alpha_i=d_i^+-d_i^-$. This completes the proof. □

Since the skew adjacency matrix of a digraph $\overrightarrow{G}$ is normal with zero diagonals, it follows that $S(\overrightarrow{G})S(\overrightarrow{G})^*=S(\overrightarrow{G})^*S(\overrightarrow{G})$. Therefore, proceeding similarly as in Theorem 3.2, we obtain the following upper bound for $\rho_s(\overrightarrow{G})$, in terms of skew adjacency rank r_s and the number of edges m of the graph G.

Theorem 3.3. *Let G be a connected graph of order $n\geq 3$ having m edges. Let $\overrightarrow{G}$ be an orientation of G having skew adjacency rank $r_s\geq 1$. Then*

$$\rho_s(\overrightarrow{G})\leq\sqrt{2m\Big(1-\frac{1}{r_s}\Big)}.$$

Since 0 is a skew eigenvalue of an Eulerian digraph $\overrightarrow{G}$, it follows that $r_s\leq n-1$, and so we obtain

$$\rho_s(\overrightarrow{G})\leq\sqrt{2m\Big(1-\frac{1}{n-1}\Big)},$$

which is similar to Wilf's upper bound to λ_1, see [3].

If the orientation $\overrightarrow{G}$ is Eulerian with rank $r_s=2$, then we obtain

$$\rho_s(\overrightarrow{G})\leq\sqrt{m}. \tag{3.1}$$

If the underlying graph G is a complete bipartite graph, then it can be seen that the equality occurs in (3.1).

The following observation can be found in [17].

Lemma 3.4. *Let X be a complex $n\times n$ matrix with rank$(X)\leq r\leq n$. Then for $r\geq 2$,*

$$(i).\ \rho(X)\geq\sqrt{\frac{|tr(X)^2-tr(X^2)|}{r(r-1)}},\quad (ii).\ \rho(X)\geq\Big(\frac{|tr(X)tr(X^2)-tr(X^3)|}{r(r-1)}\Big)^{\frac{1}{3}}$$

The following gives lower bounds for the skew Laplacian spectral radius $\rho_{sl}(\overrightarrow{G})$, in terms of the oriented degrees α_i of the digraph $\overrightarrow{G}$, the skew Laplacian rank r_{sl} and the number of edges m of the underlying graph G.

Theorem 3.5. *Let G be a connected graph of order $n \geq 3$ having m edges. If $\overrightarrow{G}$ is an orientation of G having skew Laplacian rank $r_{sl} \geq 2$, then*

$$\rho_{sl}(\overrightarrow{G}) \geq \Big(\frac{|\sum_{i=1}^{n}(d_i^+ - d_i^-)^2 - 2m|}{r_{sl}(r_{sl}-1)}\Big)^{\frac{1}{2}} \quad and \quad \rho_{sl}(\overrightarrow{G}) \geq \Big(\frac{|\theta|}{r_{sl}(r_{sl}-1)}\Big)^{\frac{1}{3}},$$

where $\theta = \sum_{i=1}^{n}(d_i^+ - d_i^-)^3 + 3M_1^-(\overrightarrow{G}) - 3M_1^+(\overrightarrow{G}) - 6\Big(t^+(\overrightarrow{G}) - t^-(\overrightarrow{G})\Big)$.

Proof. Let $\overrightarrow{G}$ be an orientation of G having skew Laplacian rank $r_{sl} \geq 2$. It is clear that $tr(\widetilde{SL}) = 0$ and $tr(\widetilde{SL}^2) = \sum_{i=1}^{n}(d_i^+ - d_i^-)^2 - 2m$. If C_3 is a triangle in $\overrightarrow{G}$ with vertices v_i, v_j and v_k, then we can write $C_3 = v_iv_jv_kv_i$ and so its sign is given by $sgn(C_3) = s_{ij}s_{jk}s_{ki}$. Let $t^+(\overrightarrow{G})$ be the number of triangles in $\overrightarrow{G}$ which have positive sign and let $t^-(\overrightarrow{G})$ be the number of triangles in $\overrightarrow{G}$ which have negative sign. We have

$$tr(\widetilde{SL}^3) = tr([\tilde{D} - S]^3) = tr(\tilde{D}^3) - 3tr(\tilde{D}^2S) + 3tr(\tilde{D}S^2) - tr(S^3). \tag{3.2}$$

Since $tr(S) = 0$, it follows that $tr(\tilde{D}^2S) = 0$. Also, using the fact that the $(i,i)^{th}$ entry of $\tilde{D}S^2$ is $-d_i(d_i^+ - d_i^-)$, we obtain

$$tr(\tilde{D}S^2) = \sum_{i=1}^{n} -d_i(d_i^+ - d_i^-) = -\sum_{i=1}^{n}\Big((d_i^+)^2 - (d_i^-)^2\Big) = M_1^-(\overrightarrow{G}) - M_1^+(\overrightarrow{G}),$$

where $M_1^+(\overrightarrow{G})$ and $M_1^-(\overrightarrow{G})$ are respectively the positive and the negative Zagreb indices of the digraph $\overrightarrow{G}$. Using these observation in (3.2), we obtain

$$\begin{aligned} tr(\widetilde{SL}^3) &= \sum_{i=1}^{n}(d_i^+ - d_i^-)^3 + 3\Big(M_1^-(\overrightarrow{G}) - M_1^+(\overrightarrow{G})\Big) - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} s_{ij}s_{jk}s_{ki} \\ &= \sum_{i=1}^{n}(d_i^+ - d_i^-)^3 + 3M_1^-(\overrightarrow{G}) - 3M_1^+(\overrightarrow{G}) - 6\sum_{C_3 \in \overrightarrow{G}} sgn(C_3) \\ &= \sum_{i=1}^{n}(d_i^+ - d_i^-)^3 + 3M_1^-(\overrightarrow{G}) - 3M_1^+(\overrightarrow{G}) - 6\Big(t^+(\overrightarrow{G}) - t^-(\overrightarrow{G})\Big). \end{aligned}$$

Now, taking $X = \widetilde{SL}$ in Lemma 3.4, and using above discussion we obtain the desired result. □

Proceeding similarly as in Theorem 3.5, we obtain the following lower bound for the skew adjacency spectral radius $\rho_s(\overrightarrow{G})$ of a digraph, in terms of skew adjacency rank r_s of the graph G.

Theorem 3.6. *Let G be a connected graph of order $n \geq 3$ having m edges. If $\overrightarrow{G}$ is an orientation of G having skew adjacency rank $r_s \geq 2$, then*

$$\rho_s(\overrightarrow{G}) \geq \Big(\frac{2m}{r_s(r_s-1)}\Big)^{\frac{1}{2}} \quad and \quad \rho_s(\overrightarrow{G}) \geq \Big(\frac{6|t^+(\overrightarrow{G}) - t^-(\overrightarrow{G})|}{r_s(r_s-1)}\Big)^{\frac{1}{3}}.$$

http://dx.doi.org/10.22108/toc.2019.112589.1582

X. Chen in [5] *obtained the following lower bound for* $\rho_s(\overrightarrow{G})$*, in terms of maximum degree* Δ *of the graph* G*:*

$$\rho_s(\overrightarrow{G}) \geq \sqrt{\Delta}. \tag{3.3}$$

Remark 3.7. *For a connected graph* G *of order* $n \geq 3$*. It can be easily seen that the lower bound for* $\rho_s(\overrightarrow{G})$ *given by the first half of the Theorem 3.6 is better than the lower bound (3.3) if* $2m \geq \Delta r_s(r_s - 1)$*. In particular, if* $r_s = 2$*, then for a connected graph* G *of order* $n \geq 3$*, the lower bound for* $\rho_s(\overrightarrow{G})$ *given by the first half of the Theorem 3.6 is always better than the lower bound (3.3). X. Li and G. Yu* [14] *have completely characterised the graphs* G *having skew rank* $r_s(\overrightarrow{G}) = 2$*. In fact, they have shown that for* $n \geq 5$*, the skew rank* $r_s(\overrightarrow{G}) = 2$ *if and only if* G *is a complete bipartite graph or a tripartite graph with all the 4-vertex cycles evenly-oriented in* $\overrightarrow{G}$*. From this it follows that for the digraphs* $\overrightarrow{G}$ *of order* $n \geq 5$ *having underlying graph a complete bipartite graph or a tripartite graph with all the 4-vertex cycles evenly-oriented in* $\overrightarrow{G}$*, the lower bound for* $\rho_s(\overrightarrow{G})$ *given by the first half of the Theorem 3.6 is always better than the lower bound (3.3).*

The above remark shows that for graphs with small skew rank, the lower for $\rho_s(\overrightarrow{G})$ given by the first half of the Theorem 3.6 can be better than the lower bound (3.3).

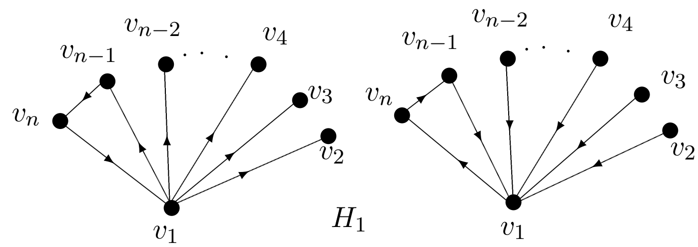

FIGURE 1. showing the orientation H_1 for the graph S_n^*

Acknowledgments

The author would like to express their sincere gratitude to the anonymous referee for his comments and suggestions, which improved the presentation of this paper.

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http://dx.doi.org/10.22108/toc.2019.112589.1582