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# A LOWER BOUND ON THE $k$-CONVERSION NUMBER OF GRAPHS OF MAXIMUM DEGREE $k+1$

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Abstract. We derive a new sharp lower bound on the $k$-conversion number of graphs of maximum degree $k+1$. This generalizes a result of W. Staton [Induced forests in cubic graphs, *Discrete Math.*, **49** (1984) 175–178], which established a lower bound on the $k$-conversion number of $(k+1)$-regular graphs.

## 1. Introduction

An *irreversible k-threshold conversion process*, or *k-conversion process*, on a graph $G$ is a sequence of subsets $S_0, S_1, \dots$ of $V(G)$ such that for $t = 0, 1, \dots,$

$$S_{t+1} = S_t \cup \{v \in V - S_t : |N(v) \cap S_t| \geq k\}.$$

The set $S_0$ is called the *seed set* for the process, and if $S_t = V(G)$ for some finite $t$, we call the seed set a *k-conversion set* of $G$. The *k-conversion number* $c_k(G)$ of $G$ is the size of a minimum $k$-conversion set of $G$. It is common to view the vertices of $S_t$ and $V - S_t$, respectively, as being in different states, say black and white, at time $t$, and for a vertex to be converted from white to black at time $t+1$ if at

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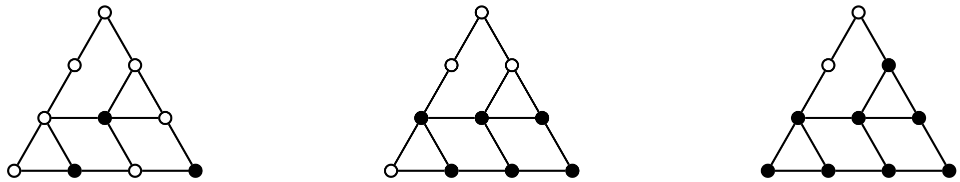
least $k$ of its neighbours are black at time $t$. Figure 1 illustrates an irreversible 2-conversion process on a graph $G$ with a seed set of size 3.



(A) $t = 0$ (seed set shown in black) (B) $t = 1$ (C) $t = 2$

FIGURE 1. A 2-conversion process at $t = 0, 1$ and 2.

Irreversible and reversible $k$-conversion processes[1] were introduced by Dreyer and Roberts [3, 4] as analogues of irreversible and reversible majority conversion processes, which had been studied earlier (see [5, 10], for example).

For $(k+1)$-regular graphs $G$, Dreyer and Roberts [4] have shown that a seed set $S$ is a $k$-conversion set of $G$ if and only if $G[V - S]$ is a forest. Given any graph $G$, a set $S$ satisfying this condition is called a *decycling set* or *feedback vertex set*, and the minimum size of such a set is the *decycling number* or *feedback vertex number* of $G$, denoted by $\phi(G)$. Clearly, finding a minimum decycling set of $G$ is equivalent to finding a maximum induced forest. Decycling sets and induced forests of regular graphs are studied in [1, 2, 8, 9, 12, 14, 15, 16, 17, 19], for example.

A result of Staton [17] yields the following sharp lower bound on the $k$-conversion number of a $(k+1)$-regular graph.

**Theorem 1.1.** [17, Proposition 2] *Let $k \geq 1$ and let $G$ be a $(k+1)$-regular graph of order $n$. Then $c_k(G) \geq \frac{n(k-1)+2}{2k}$.*

The main result of this paper generalizes Theorem 1.1, establishing a sharp lower bound on the $k$-conversion number of graphs of maximum degree $k+1$.

## 2. Definitions and preparation

The general strategy we use to extend the lower bound of Theorem 1.1 to graphs $G$ of maximum degree $k+1$ is to embed $G$ in a carefully chosen $(k+1)$-regular graph $G'$, and then use the bound on $c_k(G')$ given by Theorem 1.1 to deduce a bound on $c_k(G)$. In order to construct an appropriate $\Delta$-regular graph $G'$, we begin with the following definition.

[1] In a reversible $k$-conversion process, a vertex converts from white to black or from black to white at time $t+1$ if at least $k$ of its neighbours have the opposite colour at time $t$.

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**Definition 2.1.** *Let $G$ be a graph of maximum degree at most $\Delta$. For $v \in V(G)$, we define the* $\Delta$-deficiency *of $v$ by* $\operatorname{def}_\Delta(v) = \Delta - \deg(v)$, *and the* $\Delta$-deficiency *of $G$ by* $\operatorname{def}_\Delta(G) = \sum_{v \in V(G)} \operatorname{def}_\Delta(v)$.

We note that for a graph $G$ with $n$ vertices, $m$ edges and maximum degree $\Delta$,

$$\operatorname{def}_\Delta(G) = n\Delta - \sum_{v \in V(G)} \deg(v) = n\Delta - 2m. \tag{2.1}$$

Clearly, the deficiency of $G$ quantifies how close $G$ is to being regular. Specifically, the $\Delta$-deficiency of $G$ is the number of additional incidences required to make every vertex of $G$ have degree $\Delta$. If our goal was simply to obtain a $(k+1)$-regular graph from $G$, we would first try to do so by simply adding edges between existing vertices of $G$. However, we wish to obtain a $(k+1)$-regular graph whose $k$-conversion number will provide a good lower bound on the $k$-conversion number of $G$. Since adding edges to a graph may decrease its $k$-conversion number, we avoid adding new edges between vertices of $G$. Therefore, we add a graph $F$ to obtain a $(k+1)$-regular graph that contains $G$ as an induced subgraph. To ensure that the new (regular) graph $G'$ does not have a larger $k$-conversion number than $G$, we wish to construct $G'$ in such a way that any $k$-conversion set of $G$ is a $k$-conversion set of $G'$. The following conditions on the structure of $F$ are easily established and are stated without proof in Lemma 2.2 for referencing.

**Lemma 2.2.** *Let $G$ be a graph of order $n_G$ and maximum degree $k+1$. Suppose there exists a graph $F$ such that it is possible to obtain a simple $(k+1)$-regular graph $G \oplus F$ by adding edges between vertices of $G$ and vertices of $F$, with no new edges between vertices of $G$ or between vertices of $F$. Then*

*(a)* $\operatorname{def}_{k+1}(F) = \operatorname{def}_{k+1}(G)$, *and*

*(b) every $k$-conversion set of $G$ is a $k$-conversion set of $G \oplus F$ if and only if $F$ is a forest.*

Lemma 2.2 does not guarantee that it is possible to construct a simple $(k+1)$-regular graph from $G$ whose $k$-conversion number is at most the $k$-conversion number of $G$. However, it tells us that *if* it is possible, then the graph $F$ that we add to $G$ must be a forest. In fact, if such a forest exists, then it can be chosen to be a linear forest, since any two trees of the same order and maximum degree at most $\Delta$ have the same $\Delta$-deficiency. In Proposition 2.4 we derive a lower bound on the $k$-conversion number of a $(k+1)$-regular graph $G$ in the case where such a forest exists. From now on we use the notation $G \oplus H$ and $rG$ as follows.

**Definition 2.3.** *For graphs $G$ and $H$, we denote by $G \oplus H$ any graph constructed from $G$ and $H$ by adding edges between $G$ and $H$, with no new edges between vertices of $G$ or between vertices of $H$. We denote the disjoint union of $r$ copies of $G$ by $rG$.*

We note that the graph $G \oplus H$ is not necessarily unique for given $G$ and $H$.

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We can now use the bound of Theorem 1.1 to get a lower bound on the $k$-conversion number of graphs with maximum degree $k+1$ in the case where there exists an appropriate forest $F$.

**Proposition 2.4.** *Let $G$ be a graph of order $n_G$ and maximum degree $k+1$, and suppose there exists a forest $F$ such that it is possible to obtain a simple $(k+1)$-regular graph $G \oplus F$ by adding edges between vertices of $G$ and vertices of $F$ (with no new edges between vertices of $G$ or between vertices of $F$). Let $y \geq 1$ be the number of components of $F$. Then*

$$c_k(G) \geq \frac{n_G(k-1) + \operatorname{def}(G) - 2y + 2}{2k}.$$

*Proof.* Since $G \oplus F$ is $(k+1)$-regular, Theorem 1.1 gives

$$c_k(G \oplus F) \geq \frac{(n_G + n_F)(k-1) + 2}{2k}.$$

Since $F$ is a forest, Lemma 2.2 guarantees that a minimum $k$-conversion set of $G$ is a $k$-conversion set of $G \oplus F$, so $c_k(G) \geq c_k(G \oplus F)$. Therefore we have

$$c_k(G) \geq \frac{(n_F + n_G)(k-1) + 2}{2k}. \tag{2.2}$$

Let $m_F$ be the number of edges in $F$. By Lemma 2.2, $\operatorname{def}_{k+1}(G) = \operatorname{def}_{k+1}(F)$, where

$$\begin{aligned} \operatorname{def}_{k+1}(F) &= n_F(k+1) - 2m_F \\ &= n_F(k+1) - 2(n_F - y) \\ &= n_F(k-1) + 2y. \end{aligned}$$

Therefore $n_F = \frac{\operatorname{def}(G) - 2y}{k-1}$. Substituting this into (2.2) gives the result. □

The bound of Proposition 2.4 is optimized when $y = 1$, that is, when $F$ is a single tree (equivalently, path). In that case, we have the bound

$$c_k(G) \geq \frac{n_G(k-1) + \operatorname{def}(G)}{2k} \tag{2.3}$$

for the $k$-conversion number of a graph of maximum degree $\Delta = k+1$.

Unfortunately, not every graph of maximum degree $\Delta$ can be embedded in a $\Delta$-regular graph by adding a forest $F$, let alone a single path. Figure 2 displays such a graph $G$ for $\Delta = 4$. The vertex deficiencies are 0, 1, 1, 1 and 3, as shown in the figure, giving $G$ a 4-deficiency of 6. The vertex with deficiency 3 must be adjacent to three distinct vertices of $F$, but the only forest with 4-deficiency 6 is $K_2$.

Nonetheless, we show in Propositions 2.5 and 2.6, respectively, that the bound (2.3) still holds for $k = 2$ and $k = 3$. In Theorem 3.1 we modify the techniques used in these cases to show that (2.3) holds for all $k \geq 2$.

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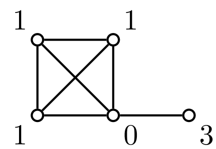


FIGURE 2. A graph of maximum degree 4, with vertex deficiencies indicated, from which it is impossible to obtain a simple 4-regular graph by adding a forest.

To prove Proposition 2.5 we first show that for graphs $G$ with sufficiently large 3-deficiency, there is a simple cubic graph $G \oplus P$, where $P$ is a single path (and hence $y = 1$ in the bound of Proposition 2.4). For the remaining cases we construct a cubic graph from multiple copies of $G$ and a path.

**Proposition 2.5.** *Let $G$ be a graph of order $n$ and maximum degree at most 3. Then*

$$c_2(G) \geq \frac{n + \operatorname{def}_3(G)}{4}.$$

*Proof.* We first claim that if $G$ has 3-deficiency at least 4 then there exists a simple cubic graph $G \oplus P_{\operatorname{def}_3(G)-2}$. The result then follows directly from Proposition 2.3.

To prove the claim, let $d \geq 4$ be the 3-deficiency of $G$. Clearly $P = P_{d-2}$ has 3-deficiency $d$ as well, so it is possible to obtain a cubic graph by adding $d$ edges between $G$ and $P$. We must show that this can be done without any parallel edges. If $G$ has no vertices of degree 1 (that is, no vertices with deficiency 2), then none of the added edges are incident with the same vertex of $G$, so there are no parallel edges. If $G$ has one vertex $x$ of degree 1, then add the edges $xu$ and $xv$, where $u$ and $v$ are the leaves of $P$. Clearly $G$ and $P$ still have the same deficiency (now $d-2$), and neither $G$ nor $P$ has any remaining vertices of deficiency 2. This means $G$ and $P$ both have $d-2$ vertices of deficiency 1; we obtain a simple cubic graph by adding a perfect matching between these vertices. If $G$ has two vertices $x$ and $y$ of deficiency 2, then add edges $xu$, $xv$, $yu$, $yv$. Again, $G$ and $P$ still have the same deficiency (now $d-4$), so it is possible to add $d-4$ more edges between $G$ and $P$ to get a cubic graph. Since $P$ has no remaining vertices of deficiency 2, none of these additional edges are incident with the same vertex of $P$, so no two are parallel.

It remains to show that the bound holds when $d \leq 3$. For these cases, we construct a graph $G'$ from some number $r$ of copies of $G$ and the path $P = P_{rd-2}$, which has the same deficiency as $r$ copies of $G$. We add $rd$ edges between $P$ and the copies of $G$ (and no other edges). For details of the construction, we refer the reader to [18, Corollary 5.7]. By Lemma 2.2 (b), $P$ converts once all vertices in all copies of $G$ have converted, so $rc_2(G) \geq c_2(G')$. Then by Theorem 1.1, $c_2(G) \geq \frac{rn+(rd-2)+2}{4r} = \frac{n+d}{4}$. □

For graphs of maximum degree $\Delta = 3$ and 3-deficiency $d \geq 3$, there is always a path with 3-deficiency $d$, namely $P_{d-2}$. In general, the path with $s$ vertices has $\Delta$-deficiency

$$\operatorname{def}_\Delta(P_s) = s(\Delta - 2) + 2. \tag{2.4}$$

Given a graph $G$ with maximum degree $\Delta$ and $\Delta$-deficiency $d$, we wish to find a path with the same $\Delta$-deficiency. However, for a given $\Delta$ and $d$, (2.4) gives $s = \frac{d-2}{\Delta-2}$, which is not necessarily an integer. Therefore, in many cases, there is no path $P$ such that $G \oplus P$ is simple and $\Delta$-regular. The proof of Theorem 2.5 suggests that we may overcome this obstacle by constructing a regular graph with $r$ disjoint copies of $G$ and an appropriate path $P_s$. The disjoint union of $r$ copies of $G$ has deficiency $rd$, so we wish to find integers $r$ and $s$ such that

$$rd = (\Delta - 2)s + 2. \tag{2.5}$$

In the case $\Delta = 4$ we use the above strategy to prove the bound (2.3) for $k = 3$. We state this result in Proposition 2.6 and refer the reader to [18, Proposition 5.9] for the proof.

**Proposition 2.6.** *Let $G$ be a graph of order $n$ and maximum degree 4. Then*

*(a) there are integers $r$ and $s$ such that it is possible to construct a simple 4-regular graph $G' = rG \oplus P_s$, or it is possible to construct a simple 4-regular graph by adding edges between two copies of $G$, and*

*(b) $c_3(G) \geq \frac{2n+\mathrm{def}_4(G)}{6}$.*

For $\Delta \geq 5$, Equation (2.5) reveals that it is not always possible to find suitable integers $r$ and $s$ to construct a simple $\Delta$-regular graph $G' = rG \oplus P_s$. To see this, suppose $d$ is divisible by $\Delta - 2$, and consider Equation (2.5) modulo $\Delta - 2$: we get $0 \equiv 2 \pmod{\Delta - 2}$, which is only possible when $\Delta = 3$ or 4. For larger values of $\Delta$ there is always a graph of maximum degree $\Delta$ for which it is impossible to construct a $\Delta$-regular graph from disjoint copies of $G$ and a path $P$ by adding edges between $P$ and the copies of $G$. For example, let $G$ be a graph whose vertices all have degree $\Delta$ or $\Delta - 2$, with at least one vertex of each type. For this graph, as observed above, there are no integers $r$ and $s$ satisfying (2.5). Nonetheless, as we will show in Section 3, the bound (2.3) still holds for $\Delta \geq 5$. We just need to modify our construction.

In our previous strategy, we aimed to find a suitable tree $T$ from which to construct a simple $(k+1)$-regular graph $G' = G \oplus T$ or $rG \oplus T$. In that case, $|V(G')|$ depends on $r$ and $\mathrm{def}_\Delta(G)$ and

$$c_k(G) \geq \frac{c_k(G')}{r}.$$

Applying Theorem 1.1 to $G'$ then conveniently yields the bound (2.3) for $G$. However, as we have seen, there may not exist a tree $T$ and an integer $r$ such that $T$ has the same deficiency as $rG$. Another irritation, since trees are not regular, is that their vertices have non-uniform deficiency, which makes it difficult to define a general construction for adding edges between $rG$ and $T$ such that $rG \oplus T$ is simple and $(k+1)$-regular. We can overcome both of these obstacles by adding a cycle $C$ instead of a tree to form a simple $(k+1)$-regular graph $G' = rG \oplus C$. The tradeoff is that a conversion set of $G$ (copied $r$ times) no longer converts $G'$. However, adding one seed vertex on $C$ fixes this problem, so we have $c_k(G') \leq rc_k(G) + 1$.

As before, $|V(G')|$ depends on $r$ and $\mathrm{def}_\Delta(G)$, but this time we have

$$c_k(G) \geq \frac{c_k(G')}{r} - \frac{1}{r}.$$

In other words, including a vertex of $C$ in the $k$-conversion set of $G'$ introduces an error term in the bound. Plainly, the error term is minimized by increasing $r$, the number of copies of $G$ (and, correspondingly, the size of the cycle $C$). This is the strategy we use in Section 3 to prove (2.3) for all $k \geq 2$ and all deficiencies.

The relationship between the number $r$ of copies of $G$ and the size of the cycle $C = C_s$ is governed by the requirement that $C_s$ have the same $\Delta$-deficiency as $rG$. That is,

$$(\Delta - 2)s = rd,$$

where $d = \mathrm{def}_\Delta(G)$. Since $s$ and $r$ must both be integers, we take $r = N(\Delta - 2)$ and $s = Nd$, with $N \in \mathbb{N}$.

## 3. The lower bound

**Theorem 3.1.** *Let $G$ be a graph of order $n$ and maximum degree $\Delta = k + 1$, with $k \geq 2$. Then*

$$c_k(G) \geq \frac{n(k-1) + \mathrm{def}_\Delta(G)}{2k}. \tag{3.1}$$

*Proof.* Let $\mathrm{def}_\Delta(G) = d$ and let $N$ be any integer satisfying $Nd \geq 3$. We construct a $\Delta$-regular graph $G'$ from $N(\Delta-2)$ disjoint copies of $G$ and a cycle $C_{Nd}$. The graphs $N(\Delta-2)G$ and $C_{Nd}$ both have $\Delta$-deficiency $N(\Delta-2)d$. We construct the graph $G' = N(\Delta-2)G \oplus C_{Nd}$ by adding edges between the copies of $G$ and the cycle $C_{Nd}$ as follows. (The construction is illustrated in Figure 3.)

Partition the vertices of $C_{Nd}$ into $N$ intervals $P^1, \ldots, P^N$ of $d$ vertices each, and arrange the $N(\Delta-2)$ copies of $G$ into $N$ groups $S_1, \ldots, S_N$ of $\Delta - 2$. For each $i = 1, \ldots, N$, let $v$ be a vertex in $P^i$ (so $\mathrm{def}_\Delta(v) = \Delta - 2$). Choose a deficient vertex $u$ of $G$, and join $v$ to the copy of $u$ in each of the $\Delta - 2$ copies of $G$ in $S_i$. The deficiency of $v$ becomes 0 and the deficiency of each copy of $u$ in $S_i$ decreases by one. Repeat this process until there are no more deficient vertices in $P^i$ (and, automatically, there will be no more deficient vertices in $S_i$ either). Since no vertex of $P_i$ gets joined to two vertices in the same copy of $G$, there are no parallel edges.

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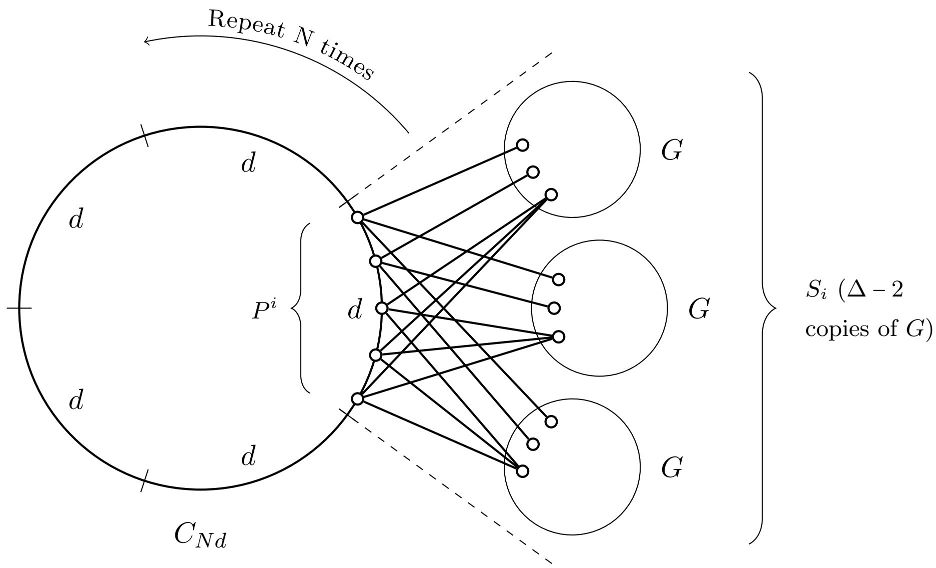


FIGURE 3. The construction of the $\Delta$-regular graph $G' = N(\Delta-2)G \oplus C_{Nd}$ in the proof of Theorem 3.1.

Let $S$ be a minimum $k$-conversion set of $G$. Then the $N(\Delta-2)$ copies of $S$ (one for each copy of $G$ in $G'$), together with one vertex $v$ from the cycle $C_{Nd}$, form a $k$-conversion set of $G'$, so $c_k(G') \le N(\Delta-2)c_k(G)+1$. This gives

$$c_k(G) \ge \frac{c_k(G')-1}{N(\Delta-2)}. \tag{3.2}$$

Since $G'$ is a regular graph, the bound of Theorem 1.1 applies, with $k+1=\Delta$. That is,

$$c_k(G') \ge \frac{n_{G'}(k-1)+2}{2k}. \tag{3.3}$$

Substituting (3.3) into (3.2) (and dropping a constant term) gives

$$c_k(G) \ge \frac{n_{G'}(k-1)-2k}{2kN(\Delta-2)}, \tag{3.4}$$

and substituting $n_{G'} = N(\Delta-2)n_G + Nd$ and $\Delta-2 = k-1$ into (3.4) then gives

$$\begin{aligned} c_k(G) &\ge \frac{(N(k-1)n_G+Nd)(k-1)-2k}{2k(k-1)N} \\ &= \frac{n_G(k-1)+d}{2k} - \frac{1}{(k-1)N}. \end{aligned} \tag{3.5}$$

The bound 3.5 holds for any $N \in \mathbb{N}$ sufficiently large such that $Nd \ge 3$. Taking the limit as $N$ approaches infinity gives the desired bound. □

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The bound (3.1) of Theorem 3.1 can be rewritten in terms of the number of edges in $G$, since

$$\operatorname{def}_\Delta(G) = \sum_{v\in V(G)} (\Delta - \deg(v)) = n\Delta - 2|E(G)| = n(k+1) - 2|E(G)|.$$

This gives the bound

$$c_k(G) \geq n - \frac{|E(G)|}{k}$$

for $k \geq 2$.

In Section 4 we prove that the bound of Theorem 3.1 is sharp by presenting an infinite family of graphs of maximum degree $k+1$ for which $c_k(G) = \frac{n(k-1)+\operatorname{def}_\Delta(G)}{2k}$. Then, in Section 5 we present infinite families of graphs whose true $k$-conversion numbers are much larger than the lower bound.

## 4. Graphs for which the lower bound is sharp

A *perfect $k$-ary tree* with $\ell+1$ levels (labelled 0 to $\ell$) is a rooted tree in which every internal vertex has exactly $k$ children and all leaves are on level $\ell$. For $k \geq 2$, every perfect $k$-ary tree with at least three levels meets the lower bound of Theorem 3.1 for graphs of maximum degree $k+1$. Such a tree $T$ has $\frac{k^{\ell+1}-1}{k-1}$ vertices, of which $k^\ell$ are leaves. The leaves form a $k$-conversion set, so $c_k(T) \leq k^\ell$. On the other hand, the $(k+1)$-deficiency is $k(k^\ell)+1$, so the lower bound from Theorem 3.1 is

$$c_k(T) \geq \frac{\frac{k^{\ell+1}-1}{k-1}(k+1) + k(k^\ell) + 1}{2k} = k^\ell.$$

## 5. Graphs of maximum degree $k+1$ with large $k$-conversion number

For $k = 2$ there is an infinite family of subcubic graphs for which $c_2(G)$ is far from the bound $\frac{n+\operatorname{def}_3(G)}{4}$. We call a tree *cubic* if all of its internal vertices have degree 3, and we define $\mathcal{G}$ to be the set of cubic graphs obtained from a cubic tree by replacing each internal vertex with a copy of $K_3$ and each leaf with a copy of the graph $H$ obtained by subdividing an edge of $K_4$. Let $G \in \mathcal{G}$ and let $G'$ be the subcubic graph obtained from $G$ by deleting from one of the copies of $H$ an edge incident with two vertices whose degree in $H$ is 3. (An example of such a graph is depicted in Figure 4, with a minimum 2-conversion set shown in black.) The 3-deficiency of $G'$ is 2, so by Proposition 2.5, the lower bound on the 2-conversion number of $G'$ is $c_2(G') \geq \frac{n+2}{4}$. However, the true 2-conversion number of $G'$, as given by [2, Theorem 4], is approximately 3/2 of this number: $c_2(G') = c_2(G) = \frac{3n+2}{8}$.

Below we present a construction that yields infinite families of graphs for which the true $k$-conversion number is much larger than the lower bound given by Theorem 3.1, for $k \geq 3$.

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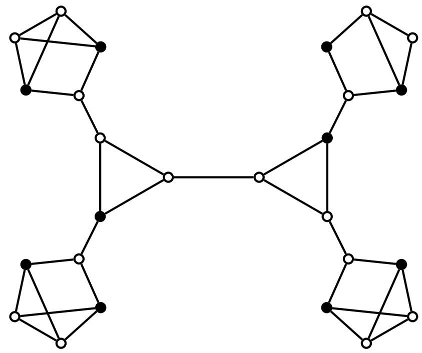

FIGURE 4. A subcubic graph for which $c_2(G)$ exceeds the lower bound.

**Proposition 5.1.** *Let $T$ be a tree with $r$ internal vertices and $\ell$ leaves, and suppose that every internal vertex of $T$ has degree $k+1 \geq 4$. Let $H$ be $K_{k+2}$ with an edge subdivided. Let $G$ be the graph obtained by replacing each internal vertex of $T$ with a copy of $K_{k+1}$ and each leaf $v$ of $T$ with a copy of $H$, identifying $v$ with the degree 2 vertex of $H$. Then*

*(a) $G$ has maximum degree $k+1$ and $(k+1)$-deficiency $(k-2)\ell$,*

*(b) $c_k(G) = (k-1)r + k\ell$, and*

*(c) the difference between $c_k(G)$ and the lower bound given by Theorem 3.1 is $\frac{k^2-1}{2k}(\ell + r) + \frac{3}{2k}\ell$.*

*Proof.* For a given copy of $H$, let $uv$ be the edge that is subdivided, and let $w$ be the vertex of degree 2 in $H$. Every vertex of $G$ has degree $k+1$ except for the vertices $w$, which have degree 3, and hence $(k+1)$-deficiency $k-2$. This proves (a).

For (b), let $S$ be a minimum $k$-conversion set of $G$. We claim that, for each copy of $H$, $|V(H) \cap \overline{S}| \leq 3$. To see this, let $X = V(H) - \{u, v, w\}$ and note that any three vertices of $X$ form a triangle, so $\overline{S}$ contains at most two vertices of $X$. Any two vertices of $X$ together with either $u$ or $v$ form a cycle, and any one vertex of $X$ together with $\{u, v, w\}$ forms a cycle. This proves the claim, and therefore $V(H) \cap S \geq k$. In $K_{k+1}$ any three vertices form a cycle, so $|S \cap V(K_{k+1})| \geq k-1$. Finally, we note that the set containing every copy of $X$ and any $k-1$ vertices from each copy of $K_{k+1}$ is a $k$-conversion set of $G$, so $c_k(G) = (k-1)r + k\ell$, as desired.

Finally, for (c),

$$\begin{aligned} c_k(G) - \frac{n(k-1) + \operatorname{def}_{k+1}(G)}{2k} &= (k-1)r + k\ell - \frac{(k-1)((k-1)r + (k+3)\ell) + (k-2)\ell}{2k} \\ &= \frac{k^2-1}{2k}(\ell + r) + \frac{3}{2k}\ell, \end{aligned}$$

as claimed. □

For the graphs $G$ described in Proposition 5.1, the difference between $c_k(G)$ and the lower bound given by Theorem 3.1 is positive for all $k \geq 2$ and increases without bound as the order of $T$ (the tree

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from which $G$ is contructed) increases. (We note that $r$ and $\ell$ are related by the equation

$$(k+1)r + \ell = 2(r + \ell - 1),$$

which is given by the degree sum formula.)

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