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ON THE DOUBLE BONDAGE NUMBER OF GRAPHS PRODUCTS

ZEINAB KOUSHKI AND HAMIDREZA MAIMANI*

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Abstract. A set D of vertices of graph G is called *double dominating set* if for any vertex v, $|N[v] \cap D| \geq 2$. The minimum cardinality of *double domination* of G is denoted by $\gamma_d(G)$. The minimum number of edges E' such that $\gamma_d(G \setminus E) > \gamma_d(G)$ is called the double bondage number of G and is denoted by $b_d(G)$. This paper determines that $b_d(G \vee H)$ and exact values of $b(P_n \times P_2)$, and generalized corona product of graphs.

1. **Introduction**

Throughout this paper, all graphs are finite, undirected with neither loops nor multiple edges. We use [14] for any terminology and notation not defined here. Let G be a graph with and vertex set $V(G)$ edge set $E(G)$. Suppose that $x, y \in V(G)$. We recall that a *walk* between x and y is a sequence $x = v_0, e_1, v_1, e_2, \ldots, e_k, v_k = y$ of vertices and edges of G such that for every i with $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i . Also a *path* between x and y is a walk between x and y without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. A *cycle graph* is a graph that consists of a single cycle. We denote the cycle graph with n vertices by C_n. Also we write P_n for the path on n vertices. For a graph G and a nonempty subset $S \subseteq V(G)$, the *vertex-induced subgraph*, denoted $< S >$, is the subgraph of G with vertex-set S and edges incident to members of S. A graph G is called *connected* if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*. The maximal connected subgraphs of G are its *connected*

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∗Corresponding author.
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components. For a graph G and vertices x and y of G, the *distance* between x and y, denoted $d(x, y)$, is the number of edges in a shortest path between x and y. If there is no any path between x and y, then we write $d(x, y) = \infty$. Also we recall that the largest distance among all distances between pairs of the vertices of a graph G is called the *diameter* of G and is denoted by $diam(G)$. A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. A *clique* of a graph G is a complete subgraph of G

For a graph G, the *girth* of G is the length of a shortest cycle in G and is denoted by $girth(G)$. If G has no cycles, we define the girth of G to be infinite.

For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A *matching* of a graph G is a set M of edges of G such that each vertex of G is incident to at most one edge of M.

A vertex $v \in V(G)$ is said to *dominate* itself and all its neighbors. A set $D \subseteq V(G)$ is called a *dominating set* of G if every vertex $v \in V(G)$ is dominated by at least one vertex of D, and it is a *double dominating set*, abbreviated DDS, of G if every vertex $v \in V(G)$ is dominated by at least two vertices of D. The *domination number*, $\gamma(G)$, and *double domination number*, $\gamma_d(G)$, are equal to the minimum cardinalities of a dominating set and double dominating set of G, respectively. A dominating (double dominating) set of G minimum cardinality is called a $\gamma(G)-$ set ($\gamma_d(G) - set$). Note that an isolated vertex cannot be dominated by two vertices. Therefore, while considering double domination, we always assume that a graph has no isolated vertices. Double dominating sets were introduced by F. Harary and T.W. Haynes [5] and and studied further in [1, 2, 3, 6] and elsewhere.

In 1990, Fink et al. [4] formally introduced the bondage . number and then continued by others, for example see [7, 8, 9, 12, 13].

The *bondage number*, $b(G)$, of a nonempty graph G equals the minimum cardinality among all sets of edges X for which $\gamma(G \setminus X) > \gamma(G)$ holds.

In 2012, Krzywkowski [11] introduced the concept of double bondage number. The *double bondage number*, $b_d(G)$, of a nonempty graph G to be the minimum cardinality among all sets of edges X for which $\delta(G \setminus X) \geq 1$ and $\gamma_d(G \setminus X) > \gamma_d(G)$ holds. If for every $X \subseteq E$, either $\gamma_d(G \setminus X) = \gamma_d(G)$ or $\delta(G \setminus X) = 0$, then we define $b_d(G) = 0$.

The join of two graphs G and H, $G \vee H$, is the graph with vertex- set

$$V(G \vee H) = V(G) \cup V(H)$$

and edge- set

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

Let G be a graph of order n and $H_1, H_2, \ldots, H_n$ be n graphs. The generalized corona product, is the graph obtained by taking one copy of graphs $G, H_1, H_2, \ldots, H_n$ and joining the ith vertex of G to

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every vertex of H_i. This product is denoted by $G \circ \bigwedge_{i=1}^{n} H_i$. If each H_i is isomorphic to graph H, then generalized corona product is called the corona product of G and H and is denoted by $G \circ H$.

For two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *cartesian product*, $G_1 \Box G_2$, is the graph with vertex- set $V_1 \times V_2$ and $(x_1, y_1)(x_1, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 = y_1$ and $x_2 y_2 \in E_2$, or $x_2 = y_2$ and $x_1 y_1 \in E_1$. For further information about graph products see [10].

In section 2, we stat some results and bounds for double bondage number of graphs and stat a result about the double bondage number of graphs, G, with $\gamma_d(G) = n - 1$. In section 3, we study the double bondage number of graphs products and obtain the double bondage number of join, and generalized corona products of graphs.

2. Some bounds

In This section we stat some results and some bounds for bondage number of graphs.

Lemma 2.1. [5]
i) If v is a leaf of a graph G, then v is an element of every $\gamma_d(G)-$set.
ii) If v is a support vertex of a graph G, then v is an element of every $\gamma_d(G)-$set.
iii)If $G' = (V, E \setminus E')$ is a spanning subgraph of a graph $G = (V, E)$, then $\gamma_d(G') \geq \gamma_d(G)$.

Proposition 2.2. [5] *Let G be a graph with no isolated vertices and two vertex disjoint $\gamma-$ sets. Then $\gamma(G) + 1 \leq \gamma_d(G) \leq 2\gamma(G)$.*

Theorem 2.3. [5] *A graph G has V as its unique $\gamma_d(G)-$ set if and only if for each $v \in V$, there is a vertex with degree one in $N[v]$.*

Proposition 2.4. *Let G be a graph of order n. If G has $t \geq 3$ vertices of degree $n-1$, then $b_d(G) = \lfloor \frac{t}{2} \rfloor$.*

Proof. Clearly, $\gamma_d(G) = 2$. Set

$$W = \{x \in V(G) : deg(x) = n - 1\}.$$

Since W is a clique in G, then we can choose a matching, M, in $< W >$ of size $\lfloor \frac{t}{2} \rfloor$. By removing the edges of M of G, we obtain a subgraph G' which G' has at most one vertex of degree $n - 1$. Hence $\gamma_d(G') > 2$, and $b_d(G) \leq \lfloor \frac{t}{2} \rfloor$.
On the other hand if we remove l edges of G with $l < \lfloor \frac{t}{2} \rfloor$, then the remaining subgraph, H, has at least two vertices of degree $n - 1$, and $\gamma_d(H) = 2$. Hence $b_d(G) \geq \lfloor \frac{t}{2} \rfloor$. Therefore $b_d(G) = \lfloor \frac{t}{2} \rfloor$. □

Proposition 2.5. *Let H be a spanning subgraph obtained by removing k edges from a graph G. Then $b_d(G) \leq b_d(H) + k$.*

Proof. Let $E' = E(G) \backslash E(H)$ and B be a double bondage set of H. Then $\gamma_d((G \backslash E') \backslash B) = \gamma_d(H \backslash B) > \gamma_d(H) \geq \gamma_d(G)$ and so $b_d(G) \leq |B| + |E'| = b_d(H) + k$. □

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Theorem 2.6. *Let G be a graph. Suppose that $\{x, y, z\}$ is a clique of G and x, y are not support vertices of G. Then*

$$b_d(G) \leq d(x) + d(y) - 3.$$

Proof. Let

$$X = \{xv : v \in N(x)\} \cup \{yv : v \in N(y)\} \setminus \{xz, yz\},$$

and $H = G - X$. Since x, y are not support vertices of G, then H has not isolated vertex and therefore has a double domination set. Suppose that D is a γ_d-domination set of H. Since the degree of y and x in H is equal to 1 and z is the unique support vertex of y and x in H, we conclude that $x, y, z \in D$. It is not difficult to see that $D \setminus \{x\}$ is a double domination for G. Hence $\gamma_d(G) < \gamma_d(H)$ and we conclude that

$$b_d(G) \leq |X| = d(x) + d(y) - 3.$$

□

Theorem 2.7. *Let G be a graph. Suppose that $x - y - z - w - x$ is a cycle of order 4 in G and x, y are not support vertices of G. Then*

$$b_d(G) \leq d(x) + d(y) - 3.$$

Proof. The proof is similar to the proof of Theorem 2.6. □

Theorem 2.8. *Let G be a graph with $\gamma_d(G) = n - 1$ and S be the vertices of degree at least 2, which are not support vertices. Suppose that $H = \langle S \rangle$. Then*
i) $H \in \{P_1, P_2, P_3, C_3, C_4, C_5\}$,
ii)If $H \in \{P_2, C_4\}$, then $b_d(G) = 1$,
iii) If $H = C_5$, then $G = C_5$ and therefore $b_d(G) = 2$,
v) If $S = \{x\}$, then $b_d(G) = deg(x) - 1$.

Proof. i) Note that $\neq \varnothing$, since otherwise, we conclude that every vertex of G is leaf or support vertex and hence $\gamma_d(G) = n$, which is a contradiction. It is not difficult to see that $\gamma_d(G) \leq \gamma_d(H) + |V(G) \setminus S|$. Let H_1 and H_2 be two connected components of H and consider $x_i \in V(H_i)$ for $1 \leq i \leq 2$. Then $V \setminus \{x_1, x_2\}$ is a double dominating set for G and hence $\gamma_d(G) \leq n - 2$, which is a contradiction. Hence the graph H is a connected graph. We claim that $girth(G) \in \{3, 4, 5, \infty\}$. Suppose that H is a tree with $diam(H) \geq 3$. Consider two leaves x, y of H. Hence $V(G) \setminus \{x, y\}$ is a double domination of G, which is a contradiction. Hence $diam(H) \leq 2$ and therefore H is an star. Let x be the center of this star. If $deg_H(x) \geq 3$ and z, y are two adjacent vertices of x, then $V(G) \setminus \{z, y\}$ is double domination of G, which is a contradiction. Hence $deg_H(x) \leq 2$ and therefore $H \in \{P_1, P_2, P_3\}$.
Now suppose that H has a cycle, C_t. If $t \geq 6$, we can choose two vertices x and y of cycle C_t, with $d(x, y) = 3$. Then $V(G) \setminus \{x, y\}$ is a double domination of G, which is contradiction. Hence $girth(H) \in \{3, 4, 5\}$. At first suppose that $girth(H) = 3$ and $x - y - z - x$ is a cycle of length 3 in H. If $deg_H(x), deg_H(y), deg_H(z) \geq 3$, then $V(G) \setminus \{x, y\}$ is a double domination set for G, which is

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a contradiction. Suppose that $deg_H(y) = 2$. If $deg_H(x) \geq 3$, then there exists $w \in N(x) \bigcap S$ with $w \neq z, y$. Hence $V \setminus \{y, w\}$ is a double dominating set for G and this is a contradiction. Hence $deg_H(x) = deg_H(z) = 2$ and we conclude that $H = C_3$. By the same argument, we have $H = C_4$ or $H = C_5$ where $girth(H) = 4$ or $girth(H) = 5$, respectively.

ii) Suppose that $H = P_2$ and $S = \{x, y\}$. If $deg_G(x), deg_G(y) \geq 3$, then $V \setminus \{x, y\}$ is a double dominating set of G, which is a contradiction. Hence $deg_G(x) = 2$ or $deg_G(y) = 2$. Suppose that $deg_G(x) = 2$. Let z be the unique support vertex which is adjacent to x. Hence all vertices of $G \setminus \{xz\}$ are leaves or support vertices. Hence $\gamma_d(G \setminus \{xz\}) = n$ and therefore $b_d(G) = 1$. If $H = C_4$, then by the same argument, H has at least 3 vertices of degree 2 (in G). Suppose that x and y are two adjacent vertices of H of degree 2. Hence $\gamma_d(G \setminus \{xy\}) = n$ and therefore $b_d(G) = 1$.

iii)Let $H = C_5$. If x is a vertex of H of degree at least 3 in G, then consider two vertices y and z of H which are adjacent to x. Hence $V \setminus \{y, z\}$ is a double domination of G, which is a contradiction. Hence $G = C_5$ and $b_d(G) = 2$.

v)The result is obvious. □

3. Bondage number of graph products

In this section we study the double bondage number of some products of graphs.
In [3] , studied the double domination number of join of two graph. We summarized their results in the following proposition.

Proposition 3.1. *Let G and H be any non- trivial graphs. Then,*

$$\gamma_d(G \vee H) = \begin{cases} 4 & \gamma(G), \gamma(H) > 2 \\ 3 & \gamma(G) = 2, \gamma(H) \geq 2 \text{ or } \gamma(\mathrm{G}) = 1, \gamma_\mathrm{d}(\mathrm{G}) \geq 3, \gamma(\mathrm{H}) \geq 2, \\ 2 & \gamma(G) = \gamma(H) = 1 \text{ or } \gamma_\mathrm{d}(\mathrm{G}) = 2, \gamma(\mathrm{H}) \geq 2. \end{cases}$$

Remark 3.2. *Let G and H be two non- trivial graphs of orders n and m, respectively. Let $\gamma(G) = \gamma(H) = 1$. If G has r vertices of degree $n-1$ and H has s vertices of degree $m-1$, then $b_d(G \vee H) = \lfloor \frac{r+s}{2} \rfloor$.*

Proof. It follows from proposition 2.4. □

Lemma 3.3. *Let G and H be two non- trivial graphs of orders n and m, respectively. If $\gamma_d(G) = 2$ and $\gamma(H) > 1$, then $b_d(G \vee H) = b_d(G)$.*

Proof. By Proposition 3.1, we have $\gamma_d(G \vee H) = 2$. Since $\gamma_d(G) = 2$, we conclude that, G has at least two vertices of degree $n-1$. Suppose that G has t vertices $x_1, x_2, \ldots, x_t$, of degree $n-1$. Then $b_d(G) = \lfloor \frac{t}{2} \rfloor$. Let $E_0 \subseteq E(G \vee H)$ such that $\gamma_d(G \vee H \setminus E_0) < \gamma_d(G \vee H)$. If $E_0 \subseteq E(G)$, then $G \vee H \setminus E_0 = (G \setminus E_0) \vee H$. Hence $\gamma_d(G \setminus E_0) \geq 3$, which means that $b_d(G \vee H) \geq b_b(G)$. Suppose that $E_0 \nsubseteq E(G)$. Let $E_1 = E_0 \cap E(G)$. If $\gamma_d(G) < \gamma_d(G \setminus E_1)$, then $|E_0| \geq b_d(G)$ and hence $b_d(G \vee H) \geq b_d(G)$. Suppose that $|E_1| < b_d(G) = \lfloor \frac{t}{2} \rfloor$ and $l = t - |V(E_1)|$. Hence $l \geq 2$. Suppose that

$$\{x_1, x_2, \ldots, x_l\} = \{x_1, x_2, \ldots, x_t\} \setminus V(E_1).$$

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There are at least $l-1$ edge with an end vertex in H and other end vertex in $\{x_1, x_2, \ldots, x_l\}$, since otherwise

$$(G \vee H) \setminus E_0 \cong K_2 \vee L,$$

for some graph L and therefore $\gamma_d((G \vee H) \setminus E_0) = 2$, which is a contradiction. Hence

$$|E_0| \geq l - 1 + |E_1| = t - |V(E_1| - 1 + |E_1| \geq t - 1 - |E_1| > t - 1 - \lfloor \frac{t}{2} \rfloor$$

and hence $|E_0| \geq \lfloor \frac{t}{2} \rfloor$. Therefore$b_d(G \vee H) \geq b_d(G)$.
Now choose a maximum matching, M, of size $\lfloor \frac{t}{2} \rfloor$ in the set $\{x_1, x_2, \ldots, x_t\}$. Hence

$$\gamma_d(G \vee H \setminus M) = \gamma_d((G \setminus M) \vee H) \geq 3,$$

since $\gamma_d(G \setminus M) \geq 3$. Hence $b_d(G \vee H) \leq b_d(G)$. □

Lemma 3.4. *Let G and H be two non- trivial graphs of orders n and m, respectively. If $\gamma(G) = 1$, $\gamma_d(G) \geq 3$ and $\gamma(H) \geq 2$, then $b_d(G \vee H) \leq b_d(G) + b(H)$.*

Proof. Clearly, $\gamma_d(G \vee H) = 3$. Let $E_1 \subseteq E(G)$ and $E_2 \subseteq E(H)$ with $|E_1| = b_d(G)$ and $|E_2| = b_d(H)$. By removing the edges of E_1 from G and E_2 from H, we obtain subgraphs G' and H' which $\gamma_d(G') \geq 4$ and $\gamma(H) \geq 3$. Hence $\gamma_d(G' \vee H') = 4$ and we conclude that $b_d(G) \leq b_d(G) + b(H)$. □

Lemma 3.5. *Let G and H be two non- trivial graphs of orders n and m, respectively. Let $\gamma(G) = 2$,*
i) If $\gamma(H) = 2$, then $b_d(G \vee H) \leq b(G) + b(H)$.
ii) If $\gamma(H) > 2$, then $b_d(G \vee H) = b(G)$.

Proof. i) Clearly, $\gamma_d(G \vee H) = 3$. Let $E_1 \subseteq E(G)$ and $E_2 \subseteq E(H)$ with $|E_1| = b(G)$ and $|E_2| = b(H)$. By removing the edges of E_1 from G and E_2 from H, we obtain subgraphs G' and H' which $\gamma(G') \geq 3$ and $\gamma(H') \geq 3$. Hence $\gamma_d(G' \vee H') = 4$, and $b_d(G) \leq b(G) + b(H)$.

ii) Clearly, $\gamma_d(G \vee H) = 3$. Let $E_1 \subseteq E(G)$ with $|E_1| = b(G)$. By removing the edges E_1 from G, we obtain subgraphs G' which $\gamma(G') > 2$ and hence $\gamma_d(G' \vee H) = 4$, which implies that $b_d(G \vee H) \leq b(G)$.

On the other hand if we remove l edges of $G \vee H$ with $l < b(G)$, then we obtain subgraph, G', with $\gamma(G') = 2$ and $\gamma(H') \geq 3$ and $\gamma_d(G' \vee H') = 3$.
So $b_d(G \vee H) \geq b(G)$. Therefore $b_d(G \vee H) = b(G)$. □

Example 3.6. *Let $G = 2K_2$ and $H = 2K_2$, therefore $\gamma(G) = 2$ and $\gamma(H) = 2$. Then Theorem 3.5 implies that $b_d(G \vee H) \leq 2$ and it is easy to see that $b_d(G \vee H) = 2$.*

Theorem 3.7. *Let G be a non- trivial graph of order n.*

$$b_d(G \vee K_1) = \begin{cases} \lfloor \frac{t}{2} \rfloor & G \text{ has } t\text{- } 1 \text{ vertices of degree } n\text{- } 1, \\ \leq b(G) & \gamma(G) > 1, \end{cases}.$$

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Proof. a) Set $W_1 = \{x \in V(G) : deg_G(x) = n-1\}$ and $W = W_1 \cup \{u\}$ such that $\{u\} = V(K_1)$. Clearly, $\gamma_d(G \vee K_1) = 2$. Since W is an clique in $G \vee K_1$, then we can choose a matching, M, in $< W >$ from size $\lfloor \frac{t}{2} \rfloor$. By removing the edges of M from $G \vee K_1$, we obtain a subgraph K which K has at most one vertex of degree n. Hence $\gamma_d(K) > 2$ and so $b_d(G \vee K_1) \leq \lfloor \frac{t}{2} \rfloor$.
On the other hand if we remove l edges of $G \vee K_1$ with $l < \lfloor \frac{t}{2} \rfloor$, then the remaining subgraph, K, has at least two vertices of degree n and $\gamma_d(K) = 2$. Hence $b_d(G \vee K_1) \geq \lfloor \frac{t}{2} \rfloor$. Therefore $b_d(G \vee K_1) = \lfloor \frac{t}{2} \rfloor$.
b) Let $\gamma(G) > 1$, then $\gamma_d(G \vee K_1) = 1 + \gamma(G)$. Suppose that $E \subseteq E(G)$ with $|E| = b(G)$. By removing the edges of E from G, we obtain a subgraph G' of G, which $\gamma(G') > \gamma(G)$. Hence $\gamma_d(G' \vee K_1) > \gamma_d(G \vee K_1)$ and So $b_d(G \vee K_1) \leq b(G)$. □

Example 3.8. *Theorem 3.7 is sharp for arbitrarily many graphs. Let* $W_6 := C_5 \vee K_1$, *then* $b_d(W_6) = 2 = b(C_5)$.

For the cartesian product of graphs we have following lemma.

Proposition 3.9. *Let G and H be two graphs,*

$$\gamma_d(G \square H) \leq min\{|V(G)|\gamma_d(H), |V(H)|\gamma_d(G)\}.$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. Suppose that $\gamma_d(H) = \{u_1, u_2, \ldots, u_k\}$. It is not difficult to see that the set

$$S = \{(v_i, u_j) : 1 \leq i \leq n, 1 \leq j \leq k\}$$

is a double dominating set of $G \square H$. Hence $\gamma_d(G \square H) \leq |V(G)|\gamma_d(H)$. Similarly $\gamma_d(G \square H) \leq |V(H)|\gamma_d(G)$. □

Corollary 3.10. $\gamma_d(G \square K_2) \leq 2\gamma_d(G)$.

Proposition 3.11. *Let $n \geq 2$ be an integer. Then $\gamma_d(P_n \square P_2) = n + 1$.*

Proof. Let $G = P_n \square P_2$ and D be a minimum double dominating set for G. Let

$$T = \{(x, y) : x \in D, y \in N[x]\}.$$

By counting $|T|$ into two ways, we conclude that $|D| \geq \frac{2V(G)}{\Delta(G)+1}$. Clearly D contains at least one vertex of degree two. Since $\Delta(G) = 3$, we find that $|D| > \frac{2V(G)}{\Delta(G)+1}$. Hence $\gamma_d(P_n \square P_2) \geq n + 1$. On the other hand it is not difficult to see that $P_n \square P_2$ has a double dominating set of size $n + 1$. □

Proposition 3.12. *Let $n \geq 2$ be an integer. Then $b_d(P_n \square P_2) = 1$*

Proof. This is a simple results of Theorem 2.6. □

Now we study the generalized corona product of graphs. The proof of the following lemma is easy.

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Lemma 3.13. *Let G and $H_1, H_2, \ldots, H_n$ be graphs without isolated vertices. Then*

$$\gamma_d(G \circ \bigwedge_i^n H_i) = \sum_{i=1}^n (1 + \gamma(H_i)).$$

Theorem 3.14. *Let G and $H_1, H_2, \ldots, H_n$ be graphs without isolated vertices. Then $b_d(G \circ \bigwedge_i^n H_i) = min_{1 \le i \le n} b(H_i)$*

Proof. Suppose that $min_{1 \le i \le n} b(H_i) = b(H_j) = l$. Consider $E \subseteq E(H_j)$ of size l such that $\gamma(H_i \setminus E) > \gamma(H_i)$. Hence

$$\gamma_d(G \circ \bigwedge_{i=1}^n H_i \setminus E) = \gamma_d(G \circ \bigwedge_i^n H_i^{'}),$$

where $H_i^{'}$ is isomorphic to H_i for any $i \neq j$ and $H_j^{'}$ is isomorphic to $H_j \setminus E$. Hence $\gamma(H_j^{'}) > \gamma(H_j)$. Therefore

$$\gamma_d(G \circ \bigwedge_i^n H_i \setminus E) > \gamma_d(G \circ \bigwedge_i^n H_i).$$

Hence

$$b_d(G \circ \bigwedge_i^n H_i) \le min_{1 \le i \le n} b(H_i).$$

It is obvious that

$$b_d(G \circ \bigwedge_i^n H_i) \ge min_{1 \le i \le n} b(H_i)$$

and the result is obtained. □

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References

[1] M. Blidia, M. Chellali and T.W. Haynes, Characterizations of trees with equal paired and double domination numbers , *Discrete Math.*, **306** (2006) 1840–1845.

[2] M. Blidia, M. Chellali, T. W. Haynes and M. A. Henning, Independent and double domination in trees, *Utilita Mathematica*, **70** (2006) 159–173.

[3] A. M. Cuivillas and S. R. Canoy Jr., Double domination in graphs under some binary operations, *Applied Mathematical Science*, **8** (2014) 2015–2024.

[4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph, *Discrete Math.*, **86** (1990) 47–57.

[5] F. Harary and T. Haynes, Double domination in graphs, *Ars combin.*, **55** (2000) 201–213.

[6] J. Harant and M. Henning, On double domination in graphs, *Discussiones Math. Graph Theory*, **25** (2005) 29–34.

[7] B. L. Hartnell and D. F. Rall, A bound on the size of a graph with given order and bondage number, *Discrete Math.*, **197-198** (1999) 409-413.

[8] B. L. Hartnell and D. F. Rall, Bounds on the bondage number of a graph, *Discrete Math.*, **128**(1994) 173–177.

DOI: http://dx.doi.org/10.22108/toc.2018.114111.1605

[9] F. T. Hu and J. M. Xu, On the complexity of the bondage and reinforcement problems, *J. Complexity*, **28** (2012) 192–201.

[10] W. Imrich and S. Klavzar, *Product Graphs*, Wiley-Interscience, New-York, 2000.

[11] M. Krzywkowski, Double bondage number in graphs, *Utilita Mathematica*, 2012.

[12] U. Teschner, New results about the bondage number of a graph, *Discrete Math.*, **171** (1997) 249–259.

[13] Y. L. Wang, On the bondage number of a graph, *Discrete Math.*, **159** (1996) 291-294.

[14] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

Zeinab Koushki

Department of Mathematics, Department of Mathematics. Science and Research branch. Islamic Azad University(IAU), Tehran, Iran

Email: z.koushki@srbiau.ac.ir

Hamidreza Maimani

Department of Mathematics, Shahid Rajaee Teacher Training University, Tehran, Iran

Email: maimani@ipm.ir

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