LIMITS OF GENERALIZED QUATERNION GROUPS

REZA HOBBI AND MOHAMMAD SHAHRYARI*

Communicated by Evgeny Vdovin

ABSTRACT. In the space of marked groups, we determine the structure of groups which are limit points of the set of all generalized quaternion groups.

In the space of marked groups, consider the situation, a sequence of generalized quaternion groups in which converges to a marked group (G, S). We will prove that there exists a finitely generated abelian group $A = \mathbb{Z}^l \oplus \mathbb{Z}_{2^k}$, such that

$$G \cong \frac{\mathbb{Z}_4 \ltimes A}{\langle (2, 2^{k-1}) \rangle}.$$

Here, the cyclic group $\mathbb{Z}_4$ acts on A by $x.a = (-1)^x a$. This gives a partial answer to a question of Champetier and Guirardel on the limits of finite groups, [1]. Already, Guyot in [4] studied the same problem for the class of dihedral groups. As any dihedral group is a semidirect product of two cyclic groups, determining their limit points is more straightforward than the case of generalized quaternion groups. The generalized quaternion group Q_{2^n} has the standard presentation

$$\langle x, y | x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{-1}, x^{2^{n-2}} = y^2 \rangle,$$

and in the same time it can be defined as the quotient

$$Q_{2^n} = \frac{\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{n-1}}}{\langle (2, 2^{n-2}) \rangle},$$

.

where the action of $\mathbb{Z}_4$ on $\mathbb{Z}_{2^{n-1}}$ is given by $x \cdot a = (-1)^x a$. In this article, the word *quaternion* will be used instead of *generalized quaternion*. Four facts about quaternion groups will be used in our arguments:

1- they are 2-groups;

2- they have unique involution;

3- any subgroup of a quaternion group is cyclic or quaternion;

4- the order two subgroup $\langle (2, 2^{n-2}) \rangle$ is central in $\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{n-1}}$.

1. Basic notions

The idea of Gromov-Grigorchuk metric on the space of finitely generated groups is proposed by M. Gromov in his celebrated solution to the Milnor's conjecture on the groups with polynomial growth (see [3]). It is extensively studied by Grigorchuk in [2]. For a detailed discussion of this metric, the reader can consult [1]. Here, we give some necessary basic definitions. A marked group (G, S) consists of a group G and an m-tuple of its elements $S = (s_1, \ldots, s_m)$ such that G is generated by S. Two marked groups (G, S) and (G', S') are *the same*, if there exists an isomorphism $G \to G'$ sending any s_i to s_i'. The set of all such marked groups is denoted by $\mathcal{G}_m$. This set can be identified by the set of all normal subgroup of the free group $\mathbb{F}_m$. Since the later is a closed subset of the compact topological space $2^{\mathbb{F}_m}$ (with the product topology), so it is also a compact space. It is easy to see that, if (N_i) is a convergent sequence in this space, then

$$\lim N_i = \liminf_i N_i = \limsup_i N_i,$$

where by definition

$$\liminf_i N_i = \bigcup_{j=1}^{\infty} \bigcap_{i \geq j} N_i, \quad \limsup_i N_i = \bigcap_{j=1}^{\infty} \bigcup_{i \geq j} N_i.$$

This space is in fact metrizable: let B_λ be the closed ball of radius λ in $\mathbb{F}_m$ (having the identity as the center) with respect to its word metric. For any two normal subgroups N and N', we say that they are in distance at most $e^{-\lambda}$, if $B_\lambda \cap N = B_\lambda \cap N'$. So, if Λ is the largest of such numbers, then we can define

$$d(N, N') = e^{-\Lambda}.$$

This induces a corresponding metric on $\mathcal{G}_m$. To see what is this metric exactly, let (G, S) be a marked group. For any non-negative integer λ, consider the set of *relations* of G with length at most λ, i.e.

$$\mathrm{Rel}_\lambda(G, S) = \{w \in \mathbb{F}_m : \|w\| \leq \lambda, w(S) = 1\}.$$

Then $d((G, S), (G', S')) = e^{-\Lambda}$, where Λ is the largest number such that $\mathrm{Rel}_\Lambda(G, S) = \mathrm{Rel}_\Lambda(G', S')$. This metric on $\mathcal{G}_m$ is the so called Gromov-Grigorchuk metric. Equivalently, two marked groups (G, S) and (G', S') are close, if large enough balls (around identity) in the corresponding marked Cayley graphs of (G, S) and (G', S') are isomorphic.

Many topological properties of the space $\mathcal{G}_m$ are discussed in [1]. In this article, we will need some basic results from [1]. The first result, describes the limits of convergent marked quotient groups.

Theorem 1.1. *Suppose* $\lim(G_i, S_i) = (G, S)$ *and for any* i, *assume that* K_i *is a normal subgroup of* G_i. *Assume that* $\overline{S}_i$ *is the canonical image of* S_i *in* G_i/K_i. *If we have* $\lim(G_i/K_i, \overline{S}_i) = (H, T)$, *then* $H = G/K$ *for some normal subgroup* K *and* T *is the canonical image of* S *in* G/K.

In our main argument, we will give an explicit description of this normal subgroup K in terms of the normal subgroups K_i. The second result, concerns the notion of fully residualness with respect to some classes of groups. Let $\mathfrak{X}$ be a class of groups. We say that a group G is fully residually $\mathfrak{X}$, if for any finite subset $E \subseteq G$, there exists a group $H \in \mathfrak{X}$ and a homomorphism $\alpha : G \to H$ such that the restriction of α to E is injective.

Theorem 1.2. *Any finitely generated fully residually* $\mathfrak{X}$*-group is a limit of a sequence of marked groups from* $\mathfrak{X}$. *Conversely, any finitely presented limit of such marked groups is fully residually* $\mathfrak{X}$.

To explain the next result from [1], we need some logical concepts. Let $\mathcal{L} = (1, ^{-1},)$ be the first order language of groups. For a group G, we denote by $\mathrm{Th}(G)$, the first order theory of G, i.e. the set of all first order sentences in the language $\mathcal{L}$ which are true in G. The universal theory of G is denoted by $\mathrm{Th}_\forall(G)$ and consists of all elements of $\mathrm{Th}(G)$ which have just universal quantifiers in their normal form.

Theorem 1.3. *Suppose a sequence* (G_i, S_i) *of marked groups converges to* (G, S). *Then we have* $\limsup_i \mathrm{Th}_\forall(G_i) \subseteq \mathrm{Th}_\forall(G)$. *Conversely, if* $\bigcap_i \mathrm{Th}_\forall(G_i) \subseteq \mathrm{Th}_\forall(G)$, *then for any marking* (G, S), *there exits a sequence of integers* (n_i) *and subgroups* $H_i \leq G_{n_i}$ *such that a sequence of suitable markings of* $H_i s$ *converges to* (G, S).

There is also a logical connection between convergence in the space of marked groups and ultra-products. Let (G_i) be a sequence of groups and $\mathcal{U}$ be an ultra-filter on $\mathbb{N}$. Define a congruence over $\prod_i G_i$ by

$$(x_i) \sim (y_i) \Leftrightarrow \{i : \ x_i = y_i\} \in \mathcal{U}.$$

The quotient group $\prod_i G_i/ \sim$ is called the ultra-product of the groups G_i with respect to $\mathcal{U}$. We denote this new group by $\prod_i G_i/\mathcal{U}$. A special case of the well-known theorem of Łos says that

$$\mathrm{Th}_\forall(\prod_i G_i/\mathcal{U}) = \lim_{\mathcal{U}} \mathrm{Th}_\forall(G_i),$$

where, $\lim_\mathcal{U}$ of any sequence of sets (A_i) is the set of all elements which belong to $\mathcal{U}$-almost all number of A_is.

Theorem 1.4. *Let* $\lim(G_i, S_i) = (G, S)$. *Then* G *can be embedded in an ultra-product* $\prod_i G_i/\mathcal{U}$, *for some ultra-filter* $\mathcal{U}$. *Conversely, let* G *be any finitely generated subgroup of some ultra-product* $\prod_i G_i/\mathcal{U}$. *Then for any marking* (G, S), *there exits a sequence of integers* (n_i) *and subgroups* $H_i \leq G_{n_i}$ *such that a sequence of suitable markings of* $H_i s$ *converges to* (G, S).

2. **Main result**

We work within the space of marked groups $\mathcal{G}_m$. In [4], Guyot determined the structure of limits of dihedral groups. The main result of [4] is the following.

Theorem 2.1. *Let G be a non-abelian finitely generated group. Then the following conditions are equivalent:*

1- G is a limit of dihedral groups.

2- G is fully residually dihedral.

3- G is isomorphic to a semidirect product $\mathbb{Z}_2 \ltimes A$, where A is a finitely generated abelian group with a cyclic torsion part, such that $\mathbb{Z}_2$ acts by multiplication by -1.

4- $\bigcap_{n \geq 3} \mathrm{Th}_\forall(D_{2n}) \subseteq \mathrm{Th}_\forall(G)$.

5- G can be embedded in some ultra-product of dihedral groups.

Our aim is to give the same characterization for the case of quaternion groups. Recall that by a quaternion group, we mean in fact a generalized quaternion group.

Theorem 2.2. *Let G be a non-abelain finitely generated group. Then the following conditions are equivalent:*

1- G is a limit of quaternion groups.

2- G is fully residually quaternion.

3- G is isomorphic to a group of the form

$$\frac{\mathbb{Z}_4 \ltimes (\mathbb{Z}^l \oplus \mathbb{Z}_{2^k})}{\langle (2, 2^{k-1}) \rangle},$$

for some integers l and k, such that the action of $\mathbb{Z}_4$ on $\mathbb{Z}^l \oplus \mathbb{Z}_{2^k}$ is given by $x \cdot a = (-1)^x a$.

4- $\bigcap_{n \geq 3} \mathrm{Th}_\forall(Q_{2^n}) \subseteq \mathrm{Th}_\forall(G)$.

5- G can be embedded in some ultra-product of quaternion groups.

Proof. Our pattern for the proof is the following:

$$1 \Rightarrow 5 \Rightarrow 4 \Rightarrow 1, \quad 1 \Leftrightarrow 3, \quad 1 \Leftrightarrow 2$$

$(1 \Rightarrow 5)$. Let G be a limit of quaternion groups. Then by Theorem 1.4, there exists an ultra-filter $\mathcal{U}$ such that G embeds in $\prod_{n \geq 3} Q_{2^n}/\mathcal{U}$.

$(5 \Rightarrow 4)$. Suppose for some ultra-filter $\mathcal{U}$, we have $G \leq \prod_{n \geq 3} Q_{2^n}/\mathcal{U}$. Then,

$$\mathrm{Th}_\forall(\prod_{n \geq 3} Q_{2^n}/\mathcal{U}) \subseteq \mathrm{Th}_\forall(G).$$

By the theorem of Łos, we have

$$\mathrm{Th}_\forall(\prod_{n\geq 3} Q_{2^n}/\mathcal{U}) \quad = \quad \lim_{\mathcal{U}}(\mathrm{Th}_\forall(Q_{2^n}))$$

$$\supseteq \quad \bigcap_{n\geq 3} \mathrm{Th}_\forall(Q_{2^n}),$$

and so 4 follows.

$(4 \Rightarrow 1)$. By Theorem 1.3, there exists a sequence (n_i) of integers and subgroups $H_i \leq Q_{2^{n_i}}$ such that for suitable markings, we have $\lim(H_i, T_i) = (G, S)$. Every H_i is cyclic or quaternion. If almost all H_i are cyclic then G is abelian, which is not the case. Because of convergence, almost all H_i are quaternion and 1 follows.

$(1 \Rightarrow 3)$. Suppose that G is a limit of quaternion groups. Then for suitable markings, we have

$$\lim(Q_{2^i}, T_i) = (G, T).$$

Recall that $Q_{2^i} = (\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{i-1}})/K_i$, where $K_i = \langle (2, 2^{i-2}) \rangle$. Let

$$T_i = (a_{i1}K, \ldots, a_{im}K), \quad t_i = (2, 2^{i-2}).$$

Then $S_i = (a_{i1}, \ldots, a_{im}, t_i)$ is a generating set for $\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{i-1}}$. We have $(\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{i-1}}, S_i) \in \mathcal{G}_{m+1}$ and since $\mathcal{G}_{m+1}$ is compact, so a subsequence of this later sequence is convergent, i.e. there exists a sequence (n_i) and a marked group $(H, S) \in \mathcal{G}_{m+1}$, such that

$$\lim(\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{n_i-1}}, S_{n_i}) = (H, S).$$

On the other hand, in $\mathcal{G}_{m+1}$, we have $\lim(Q_{2^{n_i}}, T_{n_i}+1) = (G, T+1)$, where $T+1$ denotes T extended by one extra 1 from right (and similarly $T_{n_i}+1$). By Theorem 1.1, we see that $G = H/K$, for some normal subgroup K. Before computing H, we show that $K \subseteq Z(H)$. For simplicity, we put $H_i = \mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{n_i-1}}$ and $S_i = S_{n_i}$. Suppose that

$$N_i = \{w \in \mathbb{F}_{m+1} : w(S_i) = 1\}, \quad N = \{w \in \mathbb{F}_{m+1} : w(S) = 1\}.$$

Then we have

1- $H_i \cong \mathbb{F}_{m+1}/N_i$ and $H \cong \mathbb{F}_{m+1}/N$.
2- $N = \liminf_i N_i$.

Similarly, we know that the marked group $(H_i/K_i, T_i + 1)$ is corresponding to a normal subgroup M_i in $\mathbb{F}_{m+1}$. We have

$$M_i \quad = \quad \{w \in \mathbb{F}_{m+1} : w(T_i + 1) = 1\}$$

$$= \quad \{w \in \mathbb{F}_{m+1} : w(S_i) \in K_i\}.$$

By a similar argument, $(H/K, T+1)$ corresponds to

$$M = \{w \in \mathbb{F}_{m+1} : w(S) \in K\}.$$

Therefore, we have

$$\{w \in \mathbb{F}_{m+1} : w(S) \in K\} = \liminf_i \{w \in \mathbb{F}_{m+1} : w(S_i) \in K_i\}.$$

Note that this description of K is generally true for all cases of Theorem 1.1. Recall that $K_i \subseteq Z(H_i)$. We now can show that $K \subseteq Z(H)$. Let $a \in K$ and $b \in H$. There are words w and v such that $a = w(S)$ and $b = v(S)$. Moreover $w \in M$. So, there is a j_0 such that for all $i \geq j_0$, $w(S_i) \in K_i$. So, for $i \geq j_0$, we have $[w(S_i), H_i] = 1$. As a special case $[w(S_i), v(S_i)] = 1$. Let R be the length of the commutator word $[w, v]$. Since $(H_i, S_i) \to (H, S)$, so there is j_1, such that for all $i \geq j_1$, two closed balls $B_R(H_i, S_i)$ and $B_R(H, S)$ are marked isomorphic. Let $j = \max\{j_0, j_1\}$. Then for $i \geq j$,

$$B_R(H_i, S_i) \cong B_R(H, S), \quad [w(S_i), v(S_i)] = 1.$$

Hence, we have also $[w(S), v(S)] = 1$ and this shows that $a \in Z(H)$.

It remains to determine the structure of H. But this is completely similar to the process of finding limits of dihedral groups in [4]. Hence, we know that $H = \mathbb{Z}_4 \ltimes A$, where A is a finitely generated abelian group with a cyclic torsion part. As

$$\bigcap_{n \geq 2} \mathrm{Th}_\forall(\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^n}) \subseteq \mathrm{Th}_\forall(\mathbb{Z}_4 \ltimes A),$$

we see that for all odd prime p, the universal sentence

$$\forall x(x^p = 1 \to x = 1)$$

which is true in all groups $\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^n}$, is already true in $\mathbb{Z}_4 \ltimes A$. This shows that in the later group, the torsion part is a cyclic 2-group. Hence for some integers l and k, we have $A = \mathbb{Z}^l \oplus \mathbb{Z}_{2^k}$. By a simple computation in semidirect product, we see that

$$Z(H) = \{(0, 0), (2, 0), (0, 2^{k-1}), (2, 2^{k-1})\}.$$

Therefore we have five alternatives for K:

$$
\begin{aligned}
K &= \{(0,0)\}, \\
K &= \{(0,0), (2,0)\}, \\
K &= \{(0,0), (0,2^{k-1})\}, \\
K &= \{(0,0), (2,2^{k-1})\}, \\
K &= Z(H).
\end{aligned}
$$

We will prove that the only acceptable case is $K = \{(0,0), (2, 2^{k-1})\}$. Recall that all quaternion groups have a unique involution. This fact can be translated into a universal sentence as

$$\forall x, y(x^2 = y^2 = 1 \to (x = 1 \lor y = 1 \lor x = y)).$$

Since G is a limit of quaternion groups, so the above sentence is also true in G, i.e. G has a unique involution. Computation in semidirect product, reveals the following facts:

i- in the case one, there are at least three involutions

$$(2,0)K, (0, 2^{k-1})K, (2, 2^{k-1})K.$$

ii- in the second case, there are at least two involutions

$$(1,0)K, (0, 2^{k-1})K.$$

iii- in the third case, there is also at least two involutions

$$(2,0)K, (0, 2^{k-2})K.$$

iv- in the case five, there are at least two involutions

$$(1,0)K, (0, 2^{k-2})K.$$

It remains only the case four where actually the resulting group has a unique involution. Summarizing, we conclude that G is isomorphic to a group of the form

$$\frac{\mathbb{Z}_4 \ltimes (\mathbb{Z}^l \oplus \mathbb{Z}_{2^k})}{\langle (2, 2^{k-1}) \rangle},$$

for some integers l and k, such that the action of $\mathbb{Z}_4$ on $\mathbb{Z}^l \oplus \mathbb{Z}_{2^k}$ is given by $x \cdot a = (-1)^x a$.

$(3 \Rightarrow 1)$. We know that the abelian group $\mathbb{Z}^l \oplus \mathbb{Z}_{2^k}$ is a limit of cyclic 2-groups, and hence $\mathbb{Z}_4 \ltimes (\mathbb{Z}^l \oplus \mathbb{Z}_{2^k})$ is a limit of groups of the form $\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{n-1}}$. By Theorem 1.1, we have

$$\frac{\mathbb{Z}_4 \ltimes \mathbb{Z}_{2^{n-1}}}{K_n} \to \frac{\mathbb{Z}_4 \ltimes (\mathbb{Z}^l \oplus \mathbb{Z}_{2^k})}{L},$$

for some normal subgroup L. Again checking the number of involutions, we conclude that $L = K$.

$(2 \Rightarrow 1)$. Let G be fully residually quaternion. Suppose S is an arbitrary generating set for G. For any $R > 0$, the closed ball $B_R(G, S)$ is finite. Hence, there is a $n \geq 3$ and a homomorphism $\alpha : G \to Q_{2^n}$, such that its restriction to $B_R(G, S)$ is injective. Let $T = \alpha(S)$. Then clearly we have

$$d((G, S), (Q_{2^n}, T)) \leq e^{-R}.$$

This shows that G is a limit of quaternion groups.

$(1 \Rightarrow 2)$. By 3, the group G is finitely presented and hence by Theorem 1.2, it is fully residually quaternion. $\qquad \square$

As a final note, it must be said that the same argument gives us the limits of *dicyclic* groups: let $r \geq 2$ be a natural number and consider the cyclic group $\mathbb{Z}_{4r}$. The group $\mathbb{Z}_4$ acts on $\mathbb{Z}_{4r}$ by $x.a = (-1)^x a$. It is easy to see that the subgroup $\langle (2, r) \rangle$ has order 2 and contained in the center of $\mathbb{Z}_4 \ltimes \mathbb{Z}_{4r}$. The group

$$Dic(r) = \frac{\mathbb{Z}_4 \ltimes \mathbb{Z}_{4r}}{\langle (2, r) \rangle}$$

is called the dicyclic group of order $4r$. It has a presentation of the form

$$Dic(r) = \langle x, y \mid x^{2r} = 1, y^2 = x^r, yxy^{-1} = x^{-1} \rangle.$$

By a similar argument as in the case of quaternion groups one can prove the following theorem.

Theorem 2.3. *Let G be a non-abelian finitely generated group. Then the following conditions are equivalent:*

1- G is a limit of dicyclic groups.

2- G is fully residually dicyclic.

3- G is isomorphic to a group of the form

$$\frac{\mathbb{Z}_4 \ltimes (\mathbb{Z}^l \oplus \mathbb{Z}_{2k})}{\langle (2, k) \rangle},$$

for some integers l and k, such that the action of $\mathbb{Z}_4$ on $\mathbb{Z}^l \oplus \mathbb{Z}_{2k}$ is given by $x \cdot a = (-1)^x a$.

4- $\bigcap_{r \geq 2} \mathrm{Th}_\forall(Dic(r)) \subseteq \mathrm{Th}_\forall(G)$.

5- G can be embedded in some ultra-product of dicyclic groups.

References

[1] C. Champtier and V. Guirardel, Limit groups as limits of free groups, *Israel J. Math.*, **146** (2004) 1–75.

[2] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, *Izv. Akad. Nauk SSSR Ser. Mat.*, **48** (1984) 939–985.

[3] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Etudes Sci. Publ. Math.*, **53** (1981) 53–73.

[4] L. Guyot, Limits of dihedral groups, *Geom. Dedicata*, **147** (2010) 159–171.

Reza Hobbi

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Email: rhobbi@tabrizu.ac.ir

Mohammad Shahryari

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Email: mshahryari@tabrizu.ac.ir