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THE ONE-PRIME POWER HYPOTHESIS FOR CONJUGACY CLASSES RESTRICTED TO NORMAL SUBGROUPS AND QUOTIENT GROUPS

JULIAN BROUGH

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Abstract. We say that a group G satisfies the one-prime power hypothesis for conjugacy classes if the greatest common divisor for all pairs of distinct conjugacy class sizes are prime powers. Insoluble groups which satisfy the one-prime power hypothesis have been classified. However it has remained an open question whether the one-prime power hypothesis is inherited by normal subgroups and quotients groups. In this note we construct examples to show the one-prime power hypothesis is not necessarily inherited by normal subgroups or quotient groups.

For G a finite group let $cs(G) := \{|x^G| \mid x \in G\}$ denote the set of conjugacy class sizes in G. A group G satisfies the one-prime power hypothesis for conjugacy classes if for every $m, n \in cs(G)$ with $m \neq n$, then the greatest common divisor of m and n is a prime power. The insoluble groups satisfying the one-prime power hypothesis have been classified in [4] and [1]. This problem is related to the one-prime power hypothesis for character degrees which has been studied for soluble groups in [2] and insoluble groups in [3].

It is well known that for $N \lhd G$, if $x \in N$ then $|x^N|$ divides $|x^G|$, while if $x \in G$ then $|(xN)^{G/N}|$ divides $|x^G|$. Therefore it is natural to ask whether the one-prime power hypothesis is inherited by normal subgroups and quotient groups. In [1] the authors prove the following lemma.

Lemma. [1, Lemma 3.1] *Suppose that G satisfies the one-prime power hypothesis and r is a prime dividing $|G|$. If N is a normal r-complement in G then N also satisfies the one-prime power hypothesis.*

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It is clear that the quotient analogue of [1, Lemma 3.1] holds as G/N being an r-group for some prime r, trivially implies that it G/N satisfies the one-prime power hypothesis. We first observe a minor generalisation of this result to normal subgroups and a weaker generalisation to quotients.

Lemma. *Let $N \lhd G$ such that $|G/N|$ and $|N|$ are coprime. If G satisfies the one-prime power hypothesis then N also satisfies the one-prime power hypothesis.*

Proof. This follows from the observation that if $x \in N$ then $|x^G| = |x^N|m$ for some m dividing $|G : N|$. □

Note that this proof does not work for quotient groups as it is not clear that $\frac{|x^G|}{|\overline{x}^{\overline{G}}|}$ will divide $|N|$. However when N is taken to be the centre then this does hold.

Lemma. *Let G be a finite group and $N \leq Z(G)$ such that $|G : N|$ and $|N|$ are coprime. If G satisfies the one-prime power hypothesis then so does $G = G/N$.*

Proof. Let $\overline{G} := G/N$. The result follows by observing that if $\overline{x} \in \overline{G}$ then $|x^G| = |\overline{x}^{\overline{G}}|m$ for m dividing $|N|$. □

It can be seen that these proofs are a special case, however the authors in [1] claim to not know of any example which shows that the one-prime power hypothesis is not inherited by normal subgroups. We now construct examples to show that in fact the one-prime power hypothesis is not inherited by normal subgroups and quotients Therefore to study soluble one-prime power groups alternative methods are required.

It is easy to construct a group G in which for any $x, y \in G$ the $\gcd(|x^G|, |y^G|)$ is a prime power but this does not hold for a normal subgroup. In particular, let $G = C_p \rtimes C_{p-1}$ such that 4 divides $p-1$, but $p-1$ is not a power of 2. Then $cs(G) = \{1, p-1, p\}$. However in the normal subgroup $N = C_p \rtimes C_{\frac{p-1}{2}}$ the class of size $p-1$ splits into two classes of size $\frac{p-1}{2}$. This is the idea we will try to manipulate, however we need to construct a group where we have at least two classes of the same size in G but in N or G/N at least one stays the same size and another splits into two classes.

A counter example for normal subgroups:
Let H be the permutation group generated by $g = (1, 2, 3, 4, 5, 6)$, $h_1 = (7, 8, 9, 10)$ and $h_2 = (8, 10)$ so that $H \cong C_6 \times D_8$. There exists an automorphism x of order 2 such that $g^x = g^{-1}h_1$, $h_1^x = h_1^{-1}$ and $h_2^x = h_1 h_2$. Let G be the group $H \rtimes \langle x \rangle$. As $cs(H) = \{1, 2\}$ the class sizes of elements from H in G are either $1, 2$ or 4. Thus it remains to consider those elements in $G \setminus H$. By using the relations given it follows that all the remaining classes have size 12. Thus $cs(G) = \{1, 2, 4, 12\}$ and G satisfies the one-prime power hypothesis.

It is clear that $N := \langle g^2, h_1, h_2 \rangle \rtimes \langle x \rangle \lhd G$. Moreover, the conjugacy classes of x and h_2x in G must either be the same in N or split into two classes of equal size. As $C_G(h_2x) = C_N(h_2x)$, it follows that the class $(h_2x)^G$ splits into two classes of size 6 in N. However $C_G(x) > C_N(x)$ and therefore $x^G = x^N$. Thus $cs(N) = \{1, 2, 4, 6, 12\}$ and N does not satisfy the one-prime power hypothesis.

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A counter example for quotient groups:
Let H be the permutation group generated by $g = (1,2,3,4,5,6), h_1 = (7,8,9,10)(11,12,13,14)$ and $h_2 = (7,11,9,13)(8,14,10,12)$ so that $H \cong C_6 \times Q_8$. Then there exists an automorphism x of order 4 such that $g^x = g^{-1}, h_1^x = g^3 h_1^{-1}$ and $h_2 = h_1 h_2$. Set $G = H \rtimes \langle x \rangle$. As before it is clear that all elements in H have class size in G contained in $\{1,2,4,8\}$. Thus we need to find $|(yx)^G|, |(yx^2)^G|$ and $|(yx^3)^G|$ for all $y \in H$. However as $(yx)^{-1} = x^3 y^{-1} = y' x^3$ for some $y' \in H$, the class size of $(yx^3)^G$ equals that of $(y'x)^G$. Also, $C_g(yx^2)$ contains g and h_1^2 and therefore $|(yx^2)^G| \in \{2,4,8\}$. Hence it is enough to compute $|(yx)^G|$ for all $y \in H$. It follows from the relations that $|(yx)^G| = 12$ for all $y \in H$ and thus $cs(G) = \{1,2,4,8,12\}$ and G satisfies the one-prime power hypothesis.

The action of x fixes g^2 and therefore $N := \langle g^2 \rangle$ forms a normal subgroup and the quotient $\overline{G} := G/N \cong (C_3 \times Q_8) \rtimes C_4$. Moreover, $\overline{x}^2$ now acts trivially and thus for $\overline{y} \in \overline{H}$ the class size $|(\overline{y}(\overline{x}^2))^{\overline{G}}| = |\overline{y}^{\overline{G}}| \in \{1,2,4\}$. In addition, as before it only remains to compute the class sizes $|(\overline{yx})^{\overline{G}})|$ with $\overline{y} \in \overline{H}$. As N is a central subgroup of order 2 it follows that $|x^G|/|\overline{x}^{\overline{G}}| = 1$ or 2. Moreover, it can be seen that $|\overline{x}^{\overline{G}}| = 12$ and $|(\overline{h_2}\overline{x})^{\overline{G}}| = 6$. Thus $cs(\overline{G}) = \{1,2,4,6,12\}$ and therefore $\overline{G}$ does not satisfy the one-prime power hypothesis.

Theorem. *The one-prime power hypothesis for conjugacy classes is not inherited by all normal subgroups and quotient groups.*

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Julian Brough
Fachgruppe Mathematik und Informatik, BU Wuppertal, 42119, Wuppertal, Germany

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