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ON THE HILBERT SERIES OF BINOMIAL EDGE IDEALS OF GENERALIZED TREES

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ABSTRACT. In this paper we introduce the concept of generalized trees and compute the Hilbert series of their binomial edge ideals.

1. Introduction

Let G be a simple graph on the vertex set $[n] = \{1, \ldots, n\}$ and K be a field. Definition of binomial edge ideal first appears independently in [3, 5]. The binomial edge ideal associated to G by definition is the ideal J_G, generated by $\{f_e : e = \{i, j\} \in E(G) \text{ and } i < j\}$ in $R = R_G = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where $f_e = x_i y_j - x_j y_i$. In[1, 10, 5, 3], some algebraic properties of J_G were studied and proved that J_G is a radical ideal and determined when J_G is a prime ideal. Recently Zafar has given a characterization of approximately Cohen-Macaulay binomial edge ideals for trees and proved that the binomial edge ideal of any cycle is approximately Cohen-Macaulay; in addition he computed the Hilbert series of the corresponding ideals[10]. The notion of closed graphs was introduced in [3] and Cohen-Macaulay closed graphs are completely classified in [1] and the Hilbert series of their binomial edge ideal is computed in[1]. Mohammadi and Sharifan used closed graphs with Cohen-Macaulay binomial edge ideals to compute the depth and the Hilbert function of further graphs. They computed the Hilbert function of a quasi cycle and gave a combinatorial description for the quotient ideal $J_G : f_e$ and showed that $J_G : f_e$ is a binomial edge ideal of another known graph in some cases[4]. In this paper we introduce the concept of generalized trees and generalized sun-graphs and compute the Hilbert series of binomial edge ideal of these classes of graphs.

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2. Preliminaries

Let $G_1, G_2, \cdots, G_c$ be connected components of G and $R_i = K[\{x_j, y_j\}|j \in V(G_i)]$. Note that $R/J_G = R_1/J_{G_1} \otimes \cdots \otimes R_c/J_{G_c}$ and the Hilbert series of R/J_G is $H(R/J_G, t) = \prod_{i=1}^{c} H(R_i/J_{G_i}, t)$.

Following the notation of [4], let $i, j \in V(G)$ and $e = \{i, j\} \notin E(G)$ and let $N(i)$ be the neighbor set of the vertex i in G, define G_e as the graph on $[n]$ with $E(G_e) = E(G) \cup \{\{k, l\} : k, l \in N(i) \text{ or } k, l \in N(j)\}$. If $e = \{i, j\} \notin E(G)$ is a bridge in $G \cup \{e\}$, then $J_G : f_e = J_{G_e}$.[4, Theorem 3.4]

Adding a whisker at a vertex i to G, is adding a new vertex k to $V(G)$ and $\{i, k\}$ to $E(G)$. The obtained graph will be denoted by $G \cup W(i)$ and the graph obtained by adding r_i whiskers at i will be denoted by $G \cup W_{r_i}(i)$.

A **m-complete rooted graph** is the graph obtained from a complete graph, K_m, by adding r_i whiskers at its ith vertex, i.e. $K_m \cup (\bigcup_{i=1}^{m} W_{r_i}(i))$. It will be denoted by $V(m, r_1, \ldots, r_m)$. We can also use the notaition $V(m, r_1, \ldots, r_k)$ instead of $V(m, r_1, \ldots, r_k, 0, \ldots, 0)$.

A **T-sun graph** is the graph of the form $K_m \cup (\bigcup_{s \in T} W(s))$ where $T \subseteq V(G)$. The set of all T-sun graphs with $|T| = p$, $\overline{S(m, p)}$, is called (m, p)- class of T-sun graphs.

The glued graphs are mathematically defined befor[8, 9]. Some combinatorial properties of glued graphs can be found in[6, 7].

Definition 2.1. *Let $G_i, 1 \leq i \leq l$ be arbitrary graphs and H_i be a nontrivial connected subgraph of G_i such that $H_1 \cong \cdots \cong H_l$, which named H, up to an isomorphism $f_i : H_i \longrightarrow H_{i+1}$, $1 \leq i \leq l$. The glued graph of $G_1, \ldots, G_l$ at H with respect to f_i, denoted by $G_1 \bigtriangledown G_2 \bigtriangledown \cdots \bigtriangledown G_l$, is the graph that results from combining $G_1, \ldots, G_l$ by identifying $H_1, \ldots, H_l$, with respect to the isomorphism f_i.*

Without loss of generality we choose a labeling on $G_1 \bigtriangledown G_2 \bigtriangledown \cdots \bigtriangledown G_l$ which is $\{1, \ldots, n\}$ on H. To avoid multiple edge, if $\{m, p\} \notin E(H)$, for any $m, p \in [n]$, we suppose that there is at most one k, $1 \leq k \leq l$ which $\{m, p\} \in E(G_k)$.

A **generalized tree** is a graph obtained by gluing some complete rooted graphs on their whiskers such that no new cycle arise. Similarly, a **generalized sun-graph** is a graph obtained by gluing some sun graphs on their whiskers such that no new cycle arise.

It is clear that a tree is a generalized tree by considering all complete rooted graphs in the definition as $V(1, r)$.

3. Hilbert series of binomial edge ideal of complete rooted graphs

In this section we are going to compute the Hilbert series of binomial edge ideals of complete rooted graphs.

Lemma 3.1. *Let $G = S(m, p) \in \overline{S(m, p)}$, then*

$$H(R_G/J_G, t) = (1 - t^2)^p H(R_G/J_{K_m}, t).$$

Proof. For the case $p = 1$, let $x_1, \cdots, x_m$ be the vertices of the complete subgraph of G, K_m, and y is its extra vertex. Suppose that g is the whisker connecting y to x_i in G. We have the following short exact sequence

$$0 \longrightarrow R_G/J_{G-g} : f_g(-2) \longrightarrow R_G/J_{G-g} \longrightarrow R_G/J_G \longrightarrow 0$$

and by [4, Theorem 3.4], $J_{G-g} : f_g = J_{G-g}$.

$$H(R_G/J_G, t) = (1-t^2)H(R_G/J_{G-g}, t) = (1-t^2)H(R_G/J_{K_m}, t)$$

Now by induction on p, suppose that

$$H(R_G/J_{S(m,p-1)}, t) = (1-t^2)^{p-1}H(R_G/J_{K_m}, t)$$

and let $y_1, \cdots, y_p$ be the extra vertices of whiskers and g be the whisker connecting y_p to x_j, so we have $J_{S(m,p-1)} : f_g = J_{S(m,p-1)}$. Considering the following short exact sequence

$$0 \longrightarrow R_G/J_{S(m,p-1)} : f_g(-2) \longrightarrow R_G/J_{S(m,p-1)} \longrightarrow R_G/J_{S(m,p)} \longrightarrow 0$$

we have $H(R_G/J_{S(m,p)}, t) = (1-t^2)H(R_G/J_{S(m,p-1}, t)$ and by induction,

$\mathrm{H}(\mathrm{R}_G/J_{S(m,p)}, t) = (1-t^2)(1-t^2)^{p-1}H(R_G/J_{K_m}, t) = (1-t^2)^p H(R_G/J_{K_m}, t)$

□

Proposition 3.2. *Let $G = V(m, r, a_1, \ldots, a_{m-1})$ such that $r \neq 0$, $a_i \in \{0, 1\}$ and $\sum_{i=1}^{m-1} a_i = p$, then*

$$H(R_G/J_G, t) = (1-t^2)^{p+1}H(R_G/J_{K_m}, t) - t^2(1-t^2)^p \sum_{i=1}^{r-1} H(R_G/J_{K_{i+m}}, t).$$

Proof. The proof is by induction on r. For $r = 1$ Lemma 3.1 implies the proposition.

Let $x_1, \ldots, x_m$ be the vertices of the complete subgraph K_m of G and y_1 be the vertex of one of the r whiskers, noted by g, connecting to a vertex x_i in G. By induction, the Hilbert series of $R_1/J_{V(m,r-1,a_1,\ldots,a_{m-1})}$ is

$$(1-t^2)^{p+1}H(R_1/J_{K_m}, t) - t^2(1-t^2)^p \sum_{i=1}^{r-2} H(R_1/J_{K_{i+m}}, t)$$

where R_1 is the ring associated to $V(m, r-1, a_1, \ldots, a_{m-1})$. Considering the following short exact sequence

$0 \longrightarrow R_G/J_{G-g} : f_g(-2) \longrightarrow R_G/J_{G-g} \longrightarrow R_G/J_G \longrightarrow 0$

we have

$H(R_G/J_G, t) = (1-t)^{-2}(H(R_1/J_M, t) - t^2H(R_1/J_N, t)) = H(R_G/J_M, t) - t^2H(R_G/J_N, t)$

with $M = V(m, r-1, a_1, \ldots, a_{m-1})$ and $N = V(m+r-1, 0, a_1, \ldots, a_{m-1})$. So by induction,

$H(R_G/J_G, t) = (1-t^2)^{p+1}H(R_G/J_{K_m}, t) - t^2(1-t^2)^p \sum_{i=1}^{r-2} H(R_G/J_{K_{i+m}}, t) - t^2(1-t^2)^p H(R_G/J_{K_{m+r-1}}, t)$

$= (1-t^2)^{p+1}H(R_G/J_{K_m}, t) - t^2(1-t^2)^p \sum_{i=1}^{r-1} H(R_G/J_{K_{i+m}}, t).$ □

For $G = V(m, r_1, \ldots, r_k, 0, \ldots, 0)$, we use the following notations:

$T_m = H(R_G/J_{K_m}, t)$

$T_{m,r_1} = (1-t^2)T_m - t^2 \sum_{i=1}^{r_1-1} T_{m+i} = (1-t^2)H(R_G/J_{K_m}, t) - t^2 \sum_{i=1}^{r_1-1} H(R_G/J_{K_{m+i}}, t)$

$T_{m,r_1,\ldots,r_k} = (1-t^2)T_{m,r_1,\ldots,r_{k-1}} - t^2 \sum_{i=1}^{r_k-1} T_{m+i,r_1,\ldots,r_{k-1}}$

Theorem 3.3. *Let $G = V(m, r_1, \dots, r_k, 0, \dots, 0)$ such that $r_i \neq 0$ for $i = 1, \dots, k$, then*

$$H(R_G/J_G, t) = T_{m,r_1,\dots,r_k}$$

Proof. For $k = 1$ proposition 3.2 implies the theorem.

Set $N = V(m, r_1, \dots, r_{k-1}, 0 \dots, 0)$ and $G' = V(m, r_1, \dots, r_{k-1}, 1, 0 \dots, 0)$ so $N \cup W_{r_k}(k) = G$ and $N \cup W(k) = G'$. Let $x_1, \dots, x_m$ be the vertices of the complete subgraph K_m of G and y is one of the extra vertices. Suppose that g be the whisker connecting y to x_k in G'. Considering the following exact sequence,

$$0 \longrightarrow R_{G'}/J_{G'-g} : f_g(-2) \longrightarrow R_{G'}/J_{G'-g} \longrightarrow R_{G'}/J_{G'} \longrightarrow 0$$

the Hilbert series of $R_{G'}/J_{G'}$ will be
$H(R_{G'}/J_{G'}, t) = H(R_{G'}/J_{G'-g}, t) - t^2 H(R_{G'}/J_{G'-g} : f_g, t) = H(R_{G'}/J_N, t) - t^2 H(R_{G'}/J_N, t)$
$= (1 - t^2)H(R_{G'}/J_N, t)$.
So $H(R_G/J_{G'}, t) = (1 - t^2)H(R_G/J_N, t) = (1 - t^2)T_{m,r_1,\dots,r_{k-1}} = T_{m,r_1,\dots,r_{k-1},1}$.

Set $G'' = V(m, r_1, \dots, r_{k-1}, l, 0, \dots, 0)$ and $G''' = V(m, r_1, \dots, r_{k-1}, l-1, 0, \dots, 0)$ so $G''' = G' \cup W_{l-2}(k)$ and $G'' = G''' \cup W(k)$. Induction on l shows that $H(R_G/J_{G''}, t) = T_{m,r_1,\dots,r_{k-1},l}$ and it completes the proof.

Let y' be the vertex of the lth whisker, noted by g, connecting to x_k in G''.
Considering the following short exact sequence,

$$0 \longrightarrow R_{G''}/J_{G''-g} : f_g(-2) \longrightarrow R_{G''}/J_{G''-g} \longrightarrow R_{G''}/J_{G''} \longrightarrow 0$$

the Hilbert series of $R_{G''}/J_{G''}$ will be,
$H(R_{G''}/J_{G''-g}, t) - t^2 H(R_{G''}/J_{G''-g} : f_g, t) = H(R_{G''}/J_{G'''}, t) - t^2 H(R_{G''}/J_M, t)$
So $H(R_G/J_{G''}, t) = H(R_G/J_{G'''}, t) - t^2 H(R_G/J_M, t)$
with $M = V(m + l - 1, r_1, \dots, r_{k-1})$ and by induction,
$H(R_G/J_{G''}, t) = (1 - t^2)T_{m,r_1,\dots,r_{k-1}} - t^2 \sum_{i=1}^{l-2} T_{m+i,r_1,\dots,r_{k-1}} - t^2 T_{m+l-1,r_1,\dots,r_{k-1}}$
$= (1 - t^2)T_{m,r_1,\dots,r_{k-1}} - t^2 \sum_{i=1}^{l-1} T_{m+i,r_1,\dots,r_{k-1}} = T_{m,r_1,\dots,r_{k-1},l}$. □

Corollary 3.4. *Let $G = V(m, r_1, \dots, r_m)$ such that $r_i \neq 0$ for $i = 1, \dots, m$, then*

$$H(R_G/J_G, t) = T_{m,r_1,\dots,r_m}$$

Theorem 3.5. *Let $G = V(m, r_1, \dots, r_m)$ and $s_1, \dots, s_k$ be the nonzero elements of $r_1, \dots, r_m$. Then,*

$$H(R_G/J_G, t) = T_{m,s_1,\dots,s_k}$$

Proof. As we know, $V(m, r_1, \dots, r_m) \cong V(m, s_1, \dots, s_k)$. So $H(R_G/J_G, t) = T_{m,s_1,\dots,s_k}$ by theorem 3.3. □

Theorem 3.6. *Let $G = V(m, r_1, \dots, r_m)$ and σ be a permutation on $\{1, \dots, m\}$. Then*

$$H(R_G/J_{V(m,r_1,\dots,r_m)}, t) = H(R_G/J_{V(m,r_{\sigma(1)},\dots,r_{\sigma(m)})}, t)$$

Proof. One can check that $R_G/J_{V(m,r_1,\dots,r_m)} \cong R_G/J_{V(m,r_{\sigma(1)},\dots,r_{\sigma(m)})}$ as graded algebras, so they have the same Hilbert series. □

4. Hilbert series of binomial edge ideal of generalized trees

A k-generalized tree, denoted by W_k, is the graph obtained by gluing k complete rooted graphs on $k-1$ bridge whiskers such that no new cycle arise.

The graph W_k can be shown by a $(k-1)\times 2$ matrix, each rows of this matrix is one of $k-1$ bridge whiskers. Suppose that g is a bridge whisker which connects $(V(m, r_1, \ldots, r_m), r_i)$ and $(V(m', r'_1, \ldots, r'_{m'}), r'_j)$. It means that one of r_i whiskers of x_i is glued to one of r_j whiskers of x_j. We may describe W_2 by a matrix as follows: (see figure 1)
$W_2 = \Big((V(m, r_1, \ldots, r_m), r_i) \quad (V(m', r'_1, \ldots, r'_{m'}), r'_j)\Big)$.

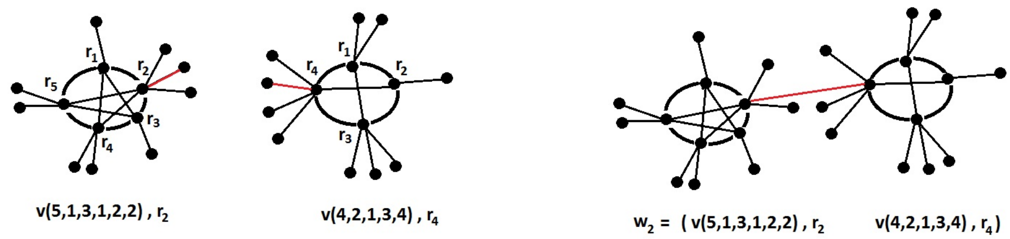

Figure 1.

If there are some isomorphic complete rooted graphs in W_k, then we separate them by different indices such as $V_1(m, r_1, \ldots, r_m), \ldots, V_l(m, r_1, \ldots, r_m)$ and $V(1,1)$ is not placed in matrix.

For example, the matrix of the generalized tree in figure 2, is

$$W_5 = \begin{pmatrix} (V(3,2,1,0), r_1) & (V(4,3,2,2,0), r_1) \\ (V(4,3,2,2,0), r_2) & (V(3,1,2,0), r_1) \\ (V(3,1,2,0), r_2) & (V_1(2,1,1), r_1) \\ (V_1(2,1,1), r_2) & (V_2(2,1,1), r_1) \end{pmatrix}$$

We consider the case that each whisker connects $(V(m, r_1, \ldots, r_m), r_i)$ to at most one $(V(q, s_1, \ldots, s_q), s_j)$ with $q \geq 3$. This class contains all of trees. In the following, we compute the Hilbert series of binomial edge ideals of **this class of generalized trees.**

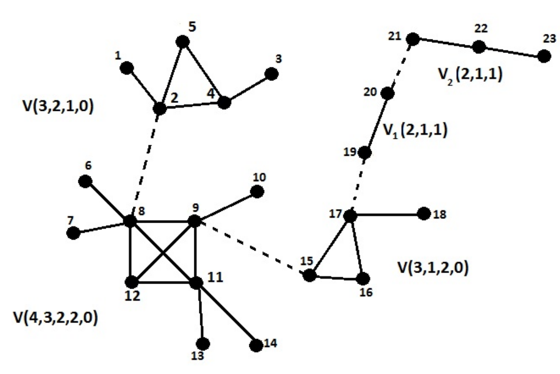

Figure 2.

Note that if $[(V(m, r_1, \ldots, r_m), r_i)\ (V(p, s_1, \ldots, s_p), s_j)]$ and $[(V(m, r_1, \ldots, r_m), r_i)\ (V(q, u_1, \ldots, u_q), u_l)]$ are two rows of associated matrix, then in this class, by the above definition at most one of p and q can be more than 2.

The Hilbert series of binomial edge ideal of W_k will be computed by induction on k. If $k = 1$, then W_1 is a complete rooted graph and the Hilbert series of binomial edge ideal of which is computed in section 3.

Now let the Hilbert series of binomial edge ideal of W_l be computed for all $l \leq k$, it is not difficult to see that a (k+1)-generalized tree may be constructed by gluing a complete rooted graph $V(n, s_1, \ldots, s_n)$, and a k-generalized tree W_k, on a bridge whisker.

Let $V(m, r_1, \ldots, r_m)$ be one of k complete rooted graphs of W_k. The graph W_{k+1}, may be considered as a graph obtained by gluing $V(m, r_1, \ldots, r_m)$ of W_k, and $V(n, s_1, \ldots, s_n)$, on a bridge whisker g and g connects ith vertex of K_m of $V(m, r_1, \ldots, r_m)$ of W_k and jth vertex of K_n of $V(n, s_1, \ldots, s_n)$. Using the following short exact sequence,

$$(1)\ 0 \longrightarrow R_{W_{k+1}}/J_{W_{k+1}-g} : f_g(-2) \longrightarrow R_{W_{k+1}}/J_{W_{k+1}-g} \longrightarrow R_{W_{k+1}}/J_{W_{k+1}} \longrightarrow 0$$

the Hilbert series of binomial edge ideal of W_{k+1} can be computed by the Hilbert series of $R_{W_{k+1}}/J_{W_{k+1}-g}$ and the Hilbert series of $R_{W_{k+1}}/J_{W_{k+1}-g} : f_g$.

Let $V(p_1, t_{11}, \ldots, t_{1p_1}), V(p_2, t_{21}, \ldots, t_{2p_2}), \ldots, V(p_l, t_{l1}, \ldots, t_{lp_l})$ be the complete rooted graphs, which are connected to $V(m, r_1, \ldots, r_m)$ in W_k.

So the graph can be represented by the following matrix:

$W_k = (v_{ij})_{(k-1)\times 2}$ where $v_{i1} = (V(m, r_1, \ldots, r_m), r_{a_i})$ for $i = 1, \ldots, l$ and $1 \leq a_i \leq m$. $v_{i2} = (V(p_i, t_{i1}, \ldots, t_{ip_i}), t_{ib_i})$ for $i = 1, \ldots, l$ and $1 \leq b_i \leq p_i$. $v_{ij} = (V(u_{ij}, c_1, \ldots, c_{u_{ij}}), c_{q_i z_j})$ for $j = 1, 2$ and $i = l + 1, \ldots, k - 1$, $1 \leq q_i z_j \leq u_{ij}$.

The above condition implies that there are two cases:

I) The graph $V(n, s_1, \ldots, s_n)$ is glued to $V(m, r_1, \ldots, r_m)$ of W_k on a bridge whisker g, which meets the ith vertex of K_m, and there is no complete rooted graph of W_k, connected to ith vertex of K_m of $V(m, r_1, \ldots, r_m)$ by a bridge whisker.

II) The graph $V(n, s_1, \ldots, s_n)$ is glued to $V(m, r_1, \ldots, r_m)$ of W_k on a bridge whisker g, which meets the ith vertex of K_m, and there are some 1-complete rooted graphs $V_1(1, a_1), \ldots, V_l(1, a_l)$, and some 2-complete rooted graphs $V_1(2, b_1, c_1), \ldots, V_f(2, b_f, c_f)$, connected to ith vertex of K_m of $V(m, r_1, \ldots, r_m)$ by some bridge whiskers, and $l + f \leq r_i - 1$. (one of b_i whiskers of $V_i(2, b_i, c_i)$ is glued to $V(m, r_1, \ldots, r_m)$)

Now for the first case (I), we introduce the operators L and L' as follows:

For all $V(m, r_1, \ldots, r_m)$ of each W_k and all $1 \leq i \leq m$, $L_{V(m,r_1,\ldots,r_m),i}(W_k)$ by definition is the matrix obtained by changing r_i of $V(m, r_1, \ldots, r_m)$ to $r_i - 1$ in each entries. So, $L_{V(m,r_1,\ldots,r_m),i}(W_k)$ is a k-generalized tree.

For all $V(m, r_1, \ldots, r_m)$ of each W_k and all $1 \leq i \leq m$, $L'_{V(m,r_1,\ldots,r_m),i}(W_k)$ by definition is the matrix obtained by changing r_i and m of $V(m, r_1, \ldots, r_m)$, to 0 and $m + r_i - 1$ in each entries. So $L'_{V(m,r_1,\ldots,r_m),i}(W_k)$ is also a k-generalized tree.

Set $N = V(n, s_1, \ldots, s_{j-1}, s_j - 1, s_{j+1}, \ldots, s_n)$, $N' = V(n + s_j - 1, s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_n)$ and $M = V(m, r_1, \ldots, r_m)$. The Hilbert series of binomial edge ideal of $W_{k+1} - g$, using the above notations, will be the product of $(1-t)^{2|V(W_{k+1})|}$, $H(R_{W_{k+1}}/J_N, t)$ and $H(R_{W_{k+1}}/J_{L_{M,i}(W_k)}, t)$. In the other hand the Hilbert series of $R_{W_{k+1}}/J_{W_{k+1}-g} : f_g$, by [4, theorem 3.4], is the product of $(1-t)^{2|V(W_{k+1})|}$, $H(R_{W_{k+1}}/J_{N'}, t)$ and $H(R_{W_{k+1}}/J_{L'_{M,i}(W_k)}, t)$.

So using the short exact sequence (1), the Hilbert series of binomial edge ideal of W_{k+1} will be:

$H(R_{W_{k+1}}/J_{W_{k+1}}, t) = H(R_{W_{k+1}}/J_{W_{k+1}-g}, t) - t^2 H(R_{W_{k+1}}/J_{W_{k+1}-g} : f_g, t) =$
$H(R_{W_k}/J_{L_{M,i}(W_k)}, t)H(R_N/J_N, t) - t^2 H(R_{W_k}/J_{L'_{M,i}(W_k)}, t)H(R_N/J_{N'}, t) =$
$(1-t)^{2|V(W_{k+1})|}[H(R_{W_{k+1}}/J_{L_{M,i}(W_k)}, t)H(R_{W_{k+1}}/J_N, t) - t^2 H(R_{W_{k+1}}/J_{L'_{M,i}(W_k)}, t)H(R_{W_{k+1}}/J_{N'}, t)]$.

For the second case (II), we introduce the operator L'' as follows:

For all $V(m, r_1, \ldots, r_m)$ of each W_k and all $1 \leq i \leq m$, $L''_{V(m,r_1,\ldots,r_m),i}(W_k)$ by definition is the matrix obtained by following changes:

- Omitting each rows which contains of $V(m, r_1, \ldots, r_m)$ and $V_i(1, a_i)$, for $i = 1, \ldots, l$, and changing $V_i(1, a_i)$, for $i = 1, \ldots, l$ to $V(m, r_1, \ldots, r_m)$ in other rows.

-Changing r_i and m of $V(m, r_1, \ldots, r_m)$, to $\sum_{i=1}^{l} a_i + \sum_{i=1}^{f} b_i - l - f$ and $m + r_i - 1$, in each entries.

-Changing $V(2, b_i, c_i)$ to $V(1, c_i)$, for $i = 1, \ldots, f$ in each rows of matrix.

So by definition, $L''_{V(m,r_1,\ldots,r_m),i}(W_k)$ is a (k-l)-generalized tree.

Again, the Hilbert series of $R_{W_{k+1}}/J_{W_{k+1}-g} : f_g$, using the same theorem, will be the product of $(1-t)^{2|V(W_{k+1})|}$, $H(R_{W_{k+1}}/J_{N'}, t)$ and $H(R_{W_{k+1}}/J_{L''_{M,i}(W_k)}, t)$

and using the short exact sequence (1), the Hilbert series of binomial edge ideal of W_{k+1} will be computed as follows:

$H(R_{W_{k+1}}/J_{W_{k+1}}, t) = H(R_{W_{k+1}}/J_{W_{k+1}-g}, t) - t^2 H(R_{W_{k+1}}/J_{W_{k+1}-g} : f_g, t) =$
$[H(R_{W_{k+1}}/J_{L_{M,i}(W_k)}, t)H(R_{W_{k+1}}/J_N, t) - t^2 H(R_{W_{k+1}}/J_{L''_{M,i}(W_k)}, t)H(R_{W_{k+1}}/J_{N'}, t)](1-t)^{2|V(W_{k+1})|}$.

Theorem 4.1. *Let G be a generalized sun-graph constructed by gluing of some sun-graphs ($S(n_i, p_i) \in \overline{S(n_i, p_i)}$, $i = 1, \ldots, k$). Then*

$$H(R_G/J_G, t) = (1-t)^{2(k-1)|V(G)|}(1-t^2)^{\alpha} \prod_{i \in \Gamma} H(R_G/J_{K_i}, t)$$

where, $\alpha = \sum_{i=1}^{k} p_i - k + 1, \Gamma = \{n_1, \ldots, n_k\}$.

Proof. The proof is by induction on k. For $k = 1$ Lemma 3.1 implies the theorem.

Suppose that a $(k-1)-$generalized sun-graph, W_{k-1}, is constructed by gluing of $S(n_i, p_i) \in \overline{S(n_i, p_i)}$, $i = 2, \cdots, k$ on $k-2$ bridge whiskers, and G is obtained by gluing of $S(n_1, p_1) \in \overline{S(n_1, p_1)}$ and the graph W_{k-1} on a bridge whisker g. Let $(G)| = V_1 + V_2$ that $V_1 = |V(S(n_1, p_1))| - 1$ and $V_2 = |V(W_{k-1})| - 1$. Considering the following exact sequence:

$$0 \longrightarrow R_G/J_{G-g} : f_g(-2) \longrightarrow R_G/J_{G-g} \longrightarrow R_G/J_G \longrightarrow 0$$

the Hilbert series of R_G/J_G will be $H(R_G/J_{G-g}, t) - t^2 H(R_G/J_{G-g} : f_g, t)$.

By Lemma 3.1 and induction, the Hilbert series of $H(R_G/J_{G-g}, t)$ and also $H(R_G/J_{G-g} : f_g, t)$ is:

$$(1-t^2)^{p_1-1} H(R_G/J_{K_{n_1}}, t)(1-t)^{2V_2}(1-t^2)^{\alpha-p_1} \prod_{i \in \Gamma \ n_1} H(R_G/J_{K_i}, t)(1-t)^{2(k-2)V_2}(1-t)^{2V_1(k-1)}$$

So,

$$\begin{aligned} H(R_G/J_G, t) &= (1-t^2)(1-t^2)^{p_1-1} H(R_G/J_{K_{n_1}}, t)(1-t)^{2V_2}(1-t^2)^{\alpha-p_1} \\ &\prod_{i \in \Gamma_1} H(R_G/J_{K_i}, t)(1-t)^{2(k-2)V_2}(1-t)^{2V_1(k-1)} \\ &= (1-t)^{2(k-1)|V(G)|}(1-t^2)^{\alpha} \prod_{i \in \Gamma} H(R_G/J_{K_i}, t) \end{aligned}$$

□

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