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AN EXTENSION AND A GENERALIZATION OF DEDEKIND'S THEOREM

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Abstract. For any given finite abelian group, we give factorizations of the group determinant in the group algebra of any subgroups. The factorizations is an extension of Dedekind's theorem. The extension leads to a generalization of Dedekind's theorem.

1. Introduction

In this paper, we give factorizations of the group determinant for any given finite abelian group G in the group algebra of subgroups. The factorizations is an extension of Dedekind's theorem. The extension leads to a generalization of Dedekind's theorem and a simple expression for inverse elements in the group algebra.

The group determinant $\Theta(G)$ is the determinant of the matrix whose elements are independent variables x_g corresponding to $g \in G$. Dedekind gave the following theorem about the irreducible factorization of the group determinant for any finite abelian group.

Theorem 1.1 (Dedekind [4]). *Let G be a finite abelian group and $\widehat{G}$ the group of characters of G. Then we have*

$$\Theta(G) = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g.$$

Frobenius gave the following theorem about the irreducible factorization of the group determinant for any finite group, thus Frobenius gave a generalization of Dedekind's theorem.

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Theorem 1.2 (Frobenius [2])**.** *Let G be a finite group and $\widehat{G}$ a complete set of irreducible representations of G over $\mathbb{C}$. Then we have*

$$\Theta(G) = \prod_{\varphi \in \widehat{G}} \det \left(\sum_{g \in G} \varphi(g) x_g \right)^{\deg \varphi}.$$

The main results of this paper are an extension and a generalization of Dedekind's theorem that are different from Frobenius' theorem.

1.1. **Main results.** We give an extension and a generalization of Dedekind's theorem.

Let G be a finite abelian group, $\mathbb{C}G$ the group algebra of G over $\mathbb{C}$, $\mathbb{C}[x_g] = \mathbb{C}\left[x_g; g \in G\right]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$, $\mathbb{C}[x_g]G = \mathbb{C}[x_g] \otimes \mathbb{C}G = \left\{ \sum_{g \in G} A_g g \mid A_g \in \mathbb{C}[x_g] \right\}$ the group algebra of G over $\mathbb{C}[x_g]$, H a subgroup of G and $[G : H]$ the index of H in G. Then we have the following theorem that is an extension of Dedekind's theorem.

Theorem 1.3. *Let G be a finite abelian group, e the unit element of G, H a subgroup of G and $\widehat{H}$ the dual group of H. For every $h \in H$, there exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ such that $\deg A_h = [G : H]$ and*

$$\Theta(G)e = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Note that the equality in Theorem 1.3 is the equality in $\mathbb{C}[x_g]H$. Theorem 1.3 leads to the following theorem.

Theorem 1.4. *Let G be a finite abelian group and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ such that $\deg A_h = [G : H]$ and*

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Theorem 1.4 is a generalization of Dedekind's theorem. In fact, let $H = G$ and $A_h = x_h$ for each $h \in H$. Then we have Dedekind's theorem.

Moreover, we obtain the following formula for inverse elements in the group algebra $\mathbb{C}G$ by Theorem 1.3. However, only now the situation is assumed that x_g is a complex number for any $g \in G$. Hence we assumed that $\sum_{g \in G} x_g g \in \mathbb{C}G$ and $\Theta(G) = \det\left(x_{gh^{-1}}\right)_{g,h \in G} \in \mathbb{C}$.

Corollary 1.5. *Let G be a finite abelian group, χ_1 the trivial representation of G and $\sum_{g \in G} x_g g \in \mathbb{C}G$ such that $\Theta(G) \neq 0$. Then we have*

$$\left(\sum_{g \in G} x_g g \right)^{-1} = \frac{1}{\Theta(G)} \prod_{\chi \in \widehat{G} \setminus \{\chi_1\}} \left(\sum_{g \in G} \chi(g) x_g g \right).$$

2. Irreducible factorization of group determinant

In this section, we recall definition of group determinant and its irreducible factorization.

2.1. Irreducible factorization of group determinant. Let G be a finite group and $\{x_g \mid g \in G\}$ independent commuting variables. We define the group determinant $\Theta(G)$ of G.

Definition 2.1. *The group determinant $\Theta(G)$ of G is given by*

$$\Theta(G) = \det\left(x_{gh^{-1}}\right)_{g,h\in G}$$

where we give a numbering to the element of G.

Namely, the group determinant $\Theta(G)$ is a homogeneous polynomial of degree $|G|$ in $\{x_g \mid g \in G\}$ where $|G|$ is the order of G.

In general, the matrix $\left(x_{gh^{-1}}\right)_{g,h\in G}$ is a covariant under change of a numbering to the element of G. However, the group determinant $\Theta(G)$ is an invariant.

Example 2.2. *Let $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. Then we have*

$$\Theta(G) = \det\begin{bmatrix} x_0 & x_2 & x_1 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{bmatrix}.$$

Dedekind gave the following theorem about the irreducible factorization of the group determinant for any finite group.

Theorem 2.3 (Dedekind [4])**.** *Let G be a finite group and $\widehat{G}$ the group of characters of G. Then we have*

$$\Theta(G) = \prod_{\chi\in\widehat{G}} \sum_{g\in G} \chi(g)x_g.$$

Example 2.4. *Let $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. Then we have*

$$\begin{aligned}\Theta(G) &= \det\begin{bmatrix} x_0 & x_2 & x_1 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{bmatrix} \\ &= (x_0 + x_1 + x_2)(x_0 + x_1\omega + x_2\omega^2)(x_0 + x_1\omega^2 + x_2\omega)\end{aligned}$$

where ω is a primitive third root of unity.

3. An extension and a generalization of Dedekind's theorem

In this section, we give an extension and a generalization of Dedekind's theorem.

3.1. **Degree one representations.** In this subsection, we describe two lemmas needed later.

Let G be a finite group, $\overline{G}$ the set of degree one representations, H a subgroup of G and

$$\overline{G}_H = \left\{ \chi \in \overline{G} \mid \chi(h) = 1, h \in H \right\}.$$

Then, $\overline{G}_H$ is a subgroup of $\overline{G}$.

Let $\widehat{G}$ be a complete set of irreducible representation of G. If G is an abelian group, since the degree of irreducible representations of G is one, we have $\overline{G} = \widehat{G}$.

The following lemmas are well known.

Lemma 3.1. [3, lemma 2.22] *Let G be a finite group and H a normal subgroup of H such that G/H is an abelian group. Then we have*

$$\overline{G}_H = \left\{ \varphi \circ \pi \mid \varphi \in \widehat{G/H} \right\}$$

where $\pi : G \to G/H$ is a natural projection.

Lemma 3.2. [1, Corollary 3.5] *Let G be a finite group and H a normal subgroup of G such that G/H is an abelian group. If $g \notin H$, there exists $\chi \in \overline{G}_H$ such that $\chi(g) \neq 1$.*

3.2. **Operators on group algebras.** In this subsection, we define operators on group algebras that are used in the proof of the main theorem.

Definition 3.3. *Let G be a finite group and $\chi \in \overline{G}$. We define the map $T_\chi : \mathbb{C}[x_g]G \to \mathbb{C}[x_g]G$ by*

$$T_\chi \left(\sum_{g \in G} A_g g \right) = \sum_{g \in G} \chi(g) A_g g$$

where $A_g \in \mathbb{C}[x_g]$.

Let $\chi, \chi' \in \overline{G}$ and $\alpha, \beta \in \mathbb{C}[x_g]G$. It is easy to see that $T_\chi \circ T_{\chi'} = T_{\chi \circ \chi'}$ and $T_\chi(\alpha\beta) = T_\chi(\alpha) T_\chi(\beta)$ where $(\chi \circ \chi')(g) = \chi(g)\chi'(g)$.

We give a necessary and sufficient condition for T_χ-invariance for all $\chi \in \overline{G}_H$.

Lemma 3.4. *Let G be a finite group, H a normal subgroup of G such that G/H is an abelian group and $\alpha \in \mathbb{C}[x_g]G$. For all $\chi \in \overline{G}_H$, $T_\chi(\alpha) = \alpha$ if and only if $\alpha \in \mathbb{C}[x_g]H$.*

Proof. Let $\alpha \in \mathbb{C}[x_g]H$. Obviously, $T_\chi(\alpha) = \alpha$ for all $\chi \in \overline{G}_H$. Let $\alpha = \sum_{g \in G} A_g g$. If $T_\chi(\alpha) = \alpha$ for all $\chi \in \overline{G}_H$, then we have $\chi(g) A_g g = A_g g$ for all $g \in G$. From this condition and Lemma 3.2, if $g \notin H$, there exists $\chi \in \overline{G}_H$ such that $\chi(g) \neq 1$. Therefore, $A_g = 0$. Namely, $\alpha = \sum_{h \in H} A_h h$. This completes the proof. ☐

Let G be a finite group, S a subgroup of $\widehat{G}$ and $S|_H$ the set of restrictions of $\chi \in S$ to H.

Lemma 3.5. *Let G be a finite abelian group, H a subgroup of G and $\widehat{G} = \chi_1 \widehat{G}_H \sqcup \chi_2 \widehat{G}_H \sqcup \cdots \sqcup \chi_k \widehat{G}_H$. Then we have $k = |H|$ and $\widehat{H} = \{\chi_1, \chi_2, \ldots, \chi_k\}|_H$.*

Proof. First, we show that $k = |H|$. By $|G| = |\widehat{G}| = k|\widehat{G}_H|$ and Lemma 3.1, we have $|\widehat{G}_H| = |\widehat{G/H}| = \frac{|G|}{|H|}$. Therefore, $k = |H|$. We show that $\widehat{H} = \{\chi_1, \chi_2, \ldots, \chi_k\}|_H$. Since the restriction of elements of $\widehat{G}_H$ is the trivial representation on H, $\widehat{G}|_H = \{\chi_1, \chi_2, \ldots, \chi_k\}|_H \subset \widehat{H}$. By $|\widehat{H}| = |H|$, we show that $\chi_1, \chi_2, \ldots, \chi_k$ are different on H. If $\chi_i(h) = \chi_j(h)$ $(1 \leq i \neq j \leq k)$ for all $h \in H$, $\left(\chi_i^{-1} \circ \chi_j\right)(h) = 1$. Therefore, $\chi_i^{-1} \circ \chi_j \in \widehat{G}_H$. This is contradiction for left $\widehat{G}_H$-coset decomposition of $\widehat{G}$. Namely, We have $\chi_i \neq \chi_j$. This completes the proof. □

3.3. **An extension and a generalization of Dedekind's theorem.** In this subsection, we give an extension and generalization of Dedekind's theorem.

Lemma 3.6. *Let G be a finite abelian group, e the unit element of G and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ such that $\deg A_h = [G : H]$ and*

$$\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g = \sum_{h \in H} A_h h$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Proof. For all $\chi' \in \widehat{G}_H$,

$$\begin{aligned} T_{\chi'}\left(\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g\right) &= \prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \left(\chi' \circ \chi\right)(g) x_g g \\ &= \prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g. \end{aligned}$$

From Lemma 3.4, we have $\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \in \mathbb{C}[x_g]H$. Clearly, $\deg A_h = |\widehat{G}_H| = [G : H]$. If $H = G$, $\widehat{G}_H$ is the trivial group. This completes the proof. □

Definition 3.7. *Let $F : \mathbb{C}[x_g]G \to \mathbb{C}[x_g]$ be the $\mathbb{C}[x_g]$-algebra homomorphism such that $F(g) = 1$ for all $g \in G$. We call the map F the fundamental $\mathbb{C}[x_g]G$-function.*

We give factorizations of the group determinant for any given finite abelian group in the group algebra of subgroups. The factorizations is an extension of Dedekind's theorem.

Theorem 3.8. *Let G be a finite abelian group, e the unit element of G and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ such that $\deg A_h = [G : H]$ and*

$$\Theta(G)e = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Proof. Clearly,

$$T_\chi\left(\prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g\right) = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g$$

for all $\chi \in \widehat{G}$. From this, $\widehat{G} = \widehat{G}_{\{e\}}$ and Lemma 3.4, there exists $C \in \mathbb{C}[x_g]$ such that

$$\prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g = \prod_{\chi \in \widehat{G}_{\{e\}}} \sum_{g \in G} \chi(g) x_g g = Ce.$$

Let F be the fundamental $\mathbb{C}[x_g]G$-function. By applying F to this equation and Theorem 2.3, we have $C = \Theta(G)$. Namely, we have

$$\prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g = \Theta(G) e.$$

Let $\widehat{G} = \chi_1 \widehat{G}_H \sqcup \chi_2 \widehat{G}_H \sqcup \cdots \sqcup \chi_k \widehat{G}_H$. Then we have

$$\prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g = \prod_{i=1}^{k} \prod_{\chi \in \chi_i \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g = \prod_{i=1}^{k} T_{\chi_i} \left(\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \right).$$

There exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ for each $h \in H$ such that

$$\prod_{i=1}^{k} T_{\chi_i} \left(\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \right) = \prod_{i=1}^{k} T_{\chi_i|_H} \left(\sum_{h \in H} A_h h \right) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h h$$

by Lemma 3.5 and 3.6. This completes the proof. □

As a corollary, we obtain the following formula for inverse elements in the group algebra $\mathbb{C}G$. However, only now the situation is assumed that x_g is a complex number for any $g \in G$. Hence we assumed that $\sum_{g \in G} x_g g \in \mathbb{C}G$ and $\Theta(G) = \det(x_{gh^{-1}})_{g,h \in G} \in \mathbb{C}$.

Corollary 3.9. *Let G be a finite abelian group, χ_1 the trivial representation of G and $\sum_{g \in G} x_g g \in \mathbb{C}G$ such that $\Theta(G) \neq 0$. Then we have*

$$\left(\sum_{g \in G} x_g g \right)^{-1} = \frac{1}{\Theta(G)} \prod_{\chi \in \widehat{G} \setminus \{\chi_1\}} \left(\sum_{g \in G} \chi(g) x_g g \right).$$

We give factorizations of the group determinant for any given finite abelian group. The factorizations is a generalization of Dedekind's theorem.

Theorem 3.10. *Let G be a finite abelian group and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ such that $\deg A_h = [G : H]$ and*

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Proof. From Theorem 3.8 and the fundamental $\mathbb{C}[x_g]G$-function, we have

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h.$$

This completes the proof. □

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References

[1] K. Conrad, Characters of finite abelian groups, `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/charthy.pdf`.

[2] K. Conrad, The origin of representation theory, `http://www.math.uconn.edu/~kconrad/articles/groupdet.pdf`.

[3] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, London, 1976.

[4] B. L. van der Waerden, *A History of Algebra*, From al-Khwarizmi to Emmy Noether, Springer-Verlag, Berlin, 1985.

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