Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 1 No. 3 (2012), pp. 21-38.

www.combinatorics.ir

www.ui.ac.ir

THE $L(2,1)$-CHOOSABILITY OF CYCLE

H. ZHOU, W. C. SHIU* AND P. C. B. LAM

Communicated by Ivan Gutman

Abstract. For a given graph $G = (V, E)$, let $\mathscr{L}(G) = \{L(v) : v \in V\}$ be a prescribed list assignment. G is $\mathscr{L}$-$L(2,1)$-colorable if there exists a vertex labeling f of G such that $f(v) \in L(v)$ for all $v \in V$; $|f(u) - f(v)| \geq 2$ if $d_G(u, v) = 1$; and $|f(u) - f(v)| \geq 1$ if $d_G(u, v) = 2$. If G is $\mathscr{L}$-$L(2,1)$-colorable for every list assignment $\mathscr{L}$ with $|L(v)| \geq k$ for all $v \in V$, then G is said to be k-$L(2,1)$-choosable. In this paper, we prove all cycles are 5-$L(2,1)$-choosable.

1. Introduction

Let $G = (V, E)$ be a graph of order n. Sometimes, we use $V(G)$ and $E(G)$ to denote V and E, respectively.

As a variation of Hale's channel assignment problem [5], the $L(2,1)$-labeling of a simple graph with a condition at distance two was first proposed and studied by Griggs and Yeh [1]. An $L(2,1)$-labeling of a graph G is a function f from the vertex set of G to the set of nonnegative integers such that $|f(u) - f(v)| \geq 2$ if $d_G(u, v) = 1$; and $|f(u) - f(v)| \geq 1$ if $d_G(u, v) = 2$. If no label of an $L(2,1)$-labeling is greater than k, then the labeling is called a k-$L(2,1)$-labeling. The $L(2,1)$-labeling number $\lambda(G)$ of a graph G is the smallest number k such that G has a k-$L(2,1)$-labeling. Griggs and Yeh [1] determined the exact values of $\lambda(P_n)$, $\lambda(C_n)$ and $\lambda(W_n)$. In addition to obtaining bounds on the λ-numbers of graphs in such classes as trees and n-cubes, they considered the relationship between $\lambda(G)$ and invariants $\chi(G)$ (the chromatic number), $\Delta(G)$ (the maximum degree) and $|V(G)|$. They showed that $\lambda(G) \leq \Delta(G)^2 + 2\Delta(G)$ and conjecture that $\lambda(G) \leq \Delta(G)^2$ for $\Delta(G) \geq 2$. Chang and Kuo [2] improved the bound to $\lambda(G) \leq \Delta(G)^2 + \Delta(G)$. Other researchers have considered various aspects or

MSC(2010): Primary: 05C78; Secondary: 05C15.
Keywords:$L(2,1)$-labeling, Choosability, cycle, path
Received: 27 July 2012, Accepted: 11 October 2012.
*Corresponding author.

variations of the $L(2,1)$-labeling problem. Vizing [6] and Erdős et al. [7] generalized the graph coloring problem and introduced the list coloring problem independently more than three decades ago. We shall consider a new variation of the $L(2,1)$-labeling problem, the *list-$L(2,1)$-labeling* problem. In the same way as list-coloring problem helps to obtain solutions to some coloring problems, we can consider list-$L(2,1)$-coloring problem, hopefully, that will help us to solve some $L(2,1)$-coloring problems.

Let $\mathbb{N}$ be the set of all non-negative integers. A list coloring of a graph G is an assignment of labels (colors) to the vertices such that each vertex v receives a label from a prescribed list $L(v) \subseteq \mathbb{N}$ and adjacent vertices receive distinct labels. $\mathscr{L}(G) = \{L(v) : v \in V(G)\}$ is called a *list assignment* of G. G is called k-choosable if G admits a list coloring for all list assignments $\mathscr{L}$ with at least k labels in each list. For list coloring of plane graphs, some results have obtained. All 2-choosable graphs have been characterized by Erdős et al. [7]. Thomassen [8] proved that every plane graph is 5-choosable, whereas Voigt [14] presented examples of plane graphs which are not 4-choosable.

Let $\mathscr{L}(G) = \{L(v) : v \in V(G)\}$ be a list assignment of a graph $G = (V,E)$. G is *$\mathscr{L}$-$L(2,1)$-colorable* if there exists a vertex labeling f of G such that $f(v) \in L(v)$ for all $v \in V$; $|f(u) - f(v)| \ge 2$ if $d_G(u,v) = 1$; and $|f(u) - f(v)| \ge 1$ if $d_G(u,v) = 2$. Such labeling f is called a *$\mathscr{L}$-$L(2,1)$-labeling* of G. If G is $\mathscr{L}$-$L(2,1)$-colorable for every list assignment $\mathscr{L}$ with $|L(v)| \ge k$ for all $v \in V$, then G is said to be *k-$L(2,1)$-choosable.*

In this paper, we denote the path P_n by $v_1 \cdots v_n$. A list assignment $\mathscr{L}$ of P_n is of *order* $(a_1,\dots,a_n)$ if $|L(v_i)| \ge a_i$, for all $i = 1,\dots,n$. P_n is said to be *$(a_1,\dots,a_n)$-$L(2,1)$-choosable* if P_n is $\mathscr{L}$-$L(2,1)$-colorable for every list assignment $\mathscr{L}$ of order $(a_1,\dots,a_n)$. An $\mathscr{L}$-$L(2,1)$-labeling f of P_n is said to be *strictly* if $f(v_1) \ne f(v_n)$. P_n is called *strictly $\mathscr{L}$-$L(2,1)$-colorable* if there exists a strictly $\mathscr{L}$-$L(2,1)$-labeling of P_n. P_n is said to be *$[a_1,\dots,a_n]$-$L(2,1)$-choosable* if for every list assignment $\mathscr{L}$ of order $(a_1,\dots,a_n)$, P_n is strictly $\mathscr{L}$-$L(2,1)$-colorable.

From now on, $L(v_i) = (\ell_i^1,\dots,\ell_i^{a_i})$ denotes the set of labels available for vertex v_i, where all the labels are in descending order.

Let $\mathscr{L} = \{L(v) : v \in V\}$ be a list assignment of $G = (V,E)$. Let $M = \max\{\cup_{v\in V} L(v)\}$ and $m = \min\{\cup_{v\in V} L(v)\}$. A vertex $v \in V$ is called an *M-vertex* or *m-vertex* (with respect to $\mathscr{L}$) if $M \in L(v)$ or $m \in L(v)$, respectively. Also, a vertex $v \in V$ is called an *M^*-vertex* or *m^*-vertex* (with respect to $\mathscr{L}$) if $\{M, M-1\} \subseteq L(v)$ or $\{m, m+1\} \subseteq L(v)$, respectively. Unless there is possibility of confusion, the phrase "with respect to $\mathscr{L}$" will be omitted. Clearly, an M^*-vertex (m^*-vertex) must also be an M-vertex (m-vertex). A vertex which is not an M-vertex is called a *non-M-vertex*. Definitions for non-m-vertex, non-M^*-vertex and non-m^*-vertex are similarly. In this paper we shall establish the following main result:

Theorem 1.1. *For $n \ge 3$, the cycle C_n is 5-$L(2,1)$-choosable.*

We shall prove the theorem in Section 3. All the results in Section 3 are based on the $L(2,1)$-choosability of paths which are listed in Section 2.

2. **Supporting Lemmas**

Lemma 2.1. *P_2 is $(2,3)$-$L(2,1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_2 of order $(2,3)$.

If $\ell_1^1 \geq \ell_2^1$, then $\ell_1^1 - \ell_2^3 \geq 2$. Hence we label v_1 with ℓ_1^1 and v_2 with ℓ_2^3. If $\ell_1^1 < \ell_2^1$, then $\ell_2^1 - \ell_1^2 \geq 2$. Hence we label v_2 with ℓ_2^1 and v_1 with ℓ_1^2. □

For convenience, we use $(a, b, c^{[n]}, d)$ to denote the sequence $(a, b, \overbrace{c, \ldots, c}^{n}, d)$. Suppose $\mathscr{L} = \{L(v) : v \in V(G)\}$ is a list assignment for a graph G. Suppose a vertex v has been labeled by $\ell \in L(v)$. Then ℓ cannot be used to label the vertex u with $d(u,v) \leq 2$ and $\ell \pm 1$ cannot be used to label the vertex w with $d(w,v) = 1$. In this case, we remove those corresponding labels from the list of the vertices of distance at most 2 from v. That is, $L'(u) = L(u) \backslash \{\ell\}$ if $d(u,v) = 2$ and $L'(w) = L(w) \backslash \{\ell, \ell+1, \ell-1\}$ if $d(w,v) = 1$. The resulting list assignment $\mathscr{L}' = \{L'(x) : x \neq v\}$ is called the *residual list assignment* (*RLA* for abbreviation) for the graph $G - v$. This concept can be extended to more vertices have been labeled.

Lemma 2.2. *P_{n+3} is $(2, 4, 5^{[n]}, 3)$-$L(2,1)$-choosable, $n \geq 0$.*

Proof: Firstly, we prove the lemma is true for $n = 0$. Let $\mathscr{L}$ be a list assignment of P_3 of order $(2,4,3)$.

Case 1. $\ell_1^1 > \ell_2^1$.

We label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_2v_3$ is $(3,2)$. By Lemma 2.1, P_3 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. $\ell_1^1 \leq \ell_2^1 - 2$.

If $\ell_3^1 \leq \ell_2^1$, we label v_2 with ℓ_2^1 and v_3 with ℓ_3^3. There is at least one label left in the residual list of v_1. So P_3 is $\mathscr{L}$-$L(2,1)$-colorable. If $\ell_3^1 > \ell_2^1$, we label v_3 with ℓ_3^1. Then the order of the RLA of $P = v_1v_2$ is $(2,3)$. By Lemma 2.1, P_3 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 3. $\ell_1^1 = \ell_2^1 - 1$ with $\ell_3^1 \neq \ell_2^1$.

If $\ell_3^1 < \ell_2^1$, then we label v_2 with ℓ_2^1 and v_1 with ℓ_1^2. There is at least one label left in the residual list of v_3. So P_3 is $\mathscr{L}$-$L(2,1)$-colorable. If $\ell_3^1 > \ell_2^1$, then we label v_3 by ℓ_3^1. The order of the RLA of $P = v_1v_2$ is $(2,3)$. Hence P_3 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 4. $\ell_1^1 = \ell_2^1$ with $\ell_3^1 \neq \ell_2^1$.

If $\ell_3^1 < \ell_2^1$, then we label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3$ is $(2,3)$. Hence P_3 is $\mathscr{L}$-$L(2,1)$-colorable. If $\ell_3^1 > \ell_2^1$, then we label v_3 with ℓ_3^1. The order of the RLA of $P = v_1v_2$ is $(2,3)$. Hence P_3 is $\mathscr{L}$-$L(2,1)$-colorable.

So we have shown that P_3 is $\mathscr{L}$-$L(2,1)$-colorable except the cases when $\ell_1^1 = \ell_2^1 = \ell_3^1$ or $\ell_1^1 + 1 = \ell_2^1 = \ell_3^1$.

When we consider the reverse ordering of the natural ordering of each list. By the same argument, we can also show that P_3 is $\mathscr{L}$-$L(2,1)$-colorable except the cases when $\ell_1^2 = \ell_2^4 = \ell_3^3$ or $\ell_1^2 - 1 = \ell_2^4 = \ell_3^3$.

Combining both considerations, the case which has not been considered is when (i) $\ell_1^1 = \ell_2^1 = \ell_3^1$ or $\ell_1^1 + 1 = \ell_2^1 = \ell_3^1$ and (ii) $\ell_1^2 = \ell_2^4 = \ell_3^3$ or $\ell_1^2 - 1 = \ell_2^4 = \ell_3^3$. Now, we have $\ell_2^1 - \ell_1^2 \geq 2$, $\ell_1^1 - \ell_2^4 \geq 2$ and $3 \leq \ell_2^1 - \ell_2^4 = \ell_2^1 - \ell_3^2 + \ell_3^2 - \ell_2^4$. From the last inequality, we have $\ell_2^1 - \ell_3^2 \geq 2$ or $\ell_3^2 - \ell_2^4 \geq 2$.

If $\ell_2^1 - \ell_3^2 \geq 2$, then we label v_1 and v_2 with ℓ_1^2 and ℓ_2^1 respectively. There is at least one label left in the residual list of v_3. If $\ell_3^2 - \ell_2^4 \geq 2$, then we label v_1 and v_2 with ℓ_1^1 and ℓ_2^4. There is also at least one label left in the residual list of v_3. So P_3 is $\mathscr{L}$-$L(2,1)$-colorable.

Remark 2.3. It is very often to use the symmetry with respect to the ordering of the list for proving the choosability of paths and cycles in this paper.

Secondly, we prove the lemma for $n = 1$. Let $\mathscr{L}$ be a list assignment of P_4 of order $(2,4,5,3)$. Suppose $\ell_1^1 \geq \ell_2^1$. We label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_2v_3v_4$ is $(2,4,3)$. Since we have proved P_3 is $(2,4,3)$-$L(2,1)$-choosable, we get P_4 is $\mathscr{L}$-$L(2,1)$-colorable. So we assume that $\ell_1^1 < \ell_2^1$ in the following.

Case 1. $\ell_3^1 < \ell_2^1$.

We label v_2 with ℓ_2^1 and v_1 with ℓ_1^2. Then the order of the RLA of $P = v_3v_4$ is $(3,2)$. By Lemma 2.1, P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. $\ell_3^1 > \ell_2^1$.

If $\ell_4^1 \geq \ell_3^1$, then we label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(2,4,3)$. Hence P_4 is $\mathscr{L}$-$L(2,1)$-colorable. If $\ell_4^1 < \ell_3^1$, then we label v_3 with ℓ_3^1. The order of the RLA of $P = v_1v_2$ is $(2,3)$. By Lemma 2.1, we may label P. Now there is at least one label left in the residual list of v_4. So P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 3. $\ell_3^1 = \ell_2^1$ with $\ell_4^1 \neq \ell_3^1$.

If $\ell_4^1 > \ell_3^1$, then we label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(2,4,4)$. Hence P_4 is $\mathscr{L}$-$L(2,1)$-colorable. If $\ell_4^1 < \ell_3^1$, then we label v_2 with ℓ_2^1 and v_1 with ℓ_1^2. The order of the RLA of $P = v_3v_4$ is $(2,3)$. Hence P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

So we have proved that P_4 is $\mathscr{L}$-$L(2,1)$-colorable except the case that $\ell_1^1 < \ell_2^1 = \ell_3^1 = \ell_4^1$. By symmetry, we also know that P_4 is $\mathscr{L}$-$L(2,1)$-colorable except the case that $\ell_1^2 > \ell_2^4 = \ell_3^5 = \ell_4^3$.

So now we assume that $\ell_1^1 < \ell_2^1 = \ell_3^1 = \ell_4^1$ and $\ell_1^2 > \ell_2^4 = \ell_3^5 = \ell_4^3$. Then $4 \leq \ell_3^1 - \ell_3^5 = \ell_3^1 - \ell_4^2 + \ell_4^2 - \ell_3^5$. It implies that either $\ell_3^1 - \ell_4^2 \geq 2$ or $\ell_4^2 - \ell_3^5 \geq 2$.

If $\ell_3^1 - \ell_4^2 \geq 2$, then we label v_1, v_2, v_3 and v_4 with ℓ_1^1, ℓ_2^4, ℓ_3^1 and ℓ_4^2, respectively. If $\ell_4^2 - \ell_3^5 \geq 2$, then we label v_1, v_2, v_3 and v_4 with ℓ_1^2, ℓ_2^1, ℓ_3^5 and ℓ_4^2, respectively. So P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Thirdly, we prove the lemma for $n = 2$. Let $\mathscr{L}$ be a list assignment of P_5 of order $(2,4,5,5,3)$.

Suppose $\ell_1^1 \geq \ell_2^1$. We label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_2v_3v_4v_5$ is $(2,4,5,3)$. Since we have proved P_4 is $(2,4,5,3)$-$L(2,1)$-choosable, we get P_5 is $\mathscr{L}$-$L(2,1)$-colorable. Suppose $\ell_1^1 < \ell_2^1$. If $\ell_3^1 \leq \ell_2^1$, then we label v_2 with ℓ_2^1 and v_1 with ℓ_1^2. Hence the order of the RLA of $P = v_3v_4v_5$ is $(2,4,3)$. Then P_5 is $\mathscr{L}$-$L(2,1)$-colorable. So we have to deal with the case when $\ell_1^1 < \ell_2^1 < \ell_3^1$.

Case 1. If $\ell_4^1 < \ell_3^1$, then we label v_3 with ℓ_3^1. The orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(2,3)$ and $(4,2)$, respectively. By Lemma 2.1 we may label P. Then the order of the RLA of P' becomes $(3,2)$. Hence P_5 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. If $\ell_4^1 > \ell_3^1$ and $\ell_4^1 \geq \ell_5^1$, then label v_4 with ℓ_4^1 and v_5 with ℓ_5^3. The order of the RLA of $P = v_1v_2v_3$ is $(2,4,3)$. Hence P_5 is $\mathscr{L}$-$L(2,1)$-colorable.
Case 3. If $\ell_4^1 > \ell_3^1$ and $\ell_4^1 < \ell_5^1$, then label v_5 with ℓ_5^1. The order of the RLA of $P = v_1v_2v_3v_4$ is $(2,4,5,4)$. Hence P_5 is $\mathscr{L}$-$L(2,1)$-colorable.
Case 4. If $\ell_4^1 = \ell_3^1$ and $\ell_4^1 > \ell_5^1$, then label v_3 with ℓ_3^1. The orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(2,3)$ and $(3,3)$, respectively. By Lemma 2.1 we may label P. The order of the RLA of P' becomes $(2,3)$. Hence P_5 is $\mathscr{L}$-$L(2,1)$-colorable.
Case 5. If $\ell_4^1 = \ell_3^1$ and $\ell_4^1 < \ell_5^1$, then label v_5 with ℓ_5^1. The order of the RLA of $P = v_1v_2v_3v_4$ is $(2,4,5,4)$. Hence P_5 is $\mathscr{L}$-$L(2,1)$-colorable.

So we have proved that P_5 is $\mathscr{L}$-$L(2,1)$-colorable except the case that $\ell_1^1 < \ell_2^1 < \ell_3^1 = \ell_4^1 = \ell_5^1$. By symmetry, we also know that P_5 is $\mathscr{L}$-$L(2,1)$-colorable except the case that $\ell_1^2 > \ell_2^4 > \ell_3^5 = \ell_4^5 = \ell_5^3$.

Now we have to deal with the case that $\ell_1^1 < \ell_2^1 < \ell_3^1 = \ell_4^1 = \ell_5^1$ and $\ell_1^2 > \ell_2^4 > \ell_3^5 = \ell_4^5 = \ell_5^3$. Similar to the proof for $n = 1$, either $\ell_4^1 - \ell_5^2 \geq 2$ or $\ell_5^2 - \ell_4^5 \geq 2$. For the first case, we label v_1, v_2, v_3, v_4 and v_5 by $\ell_1^2, \ell_2^1, \ell_3^5$, ℓ_4^1 and ℓ_5^2, respectively. For the second case, we label v_1, v_2, v_3, v_4 and v_5 by $\ell_1^1, \ell_2^4, \ell_3^1$, ℓ_4^5 and ℓ_5^2, respectively. So P_5 is $\mathscr{L}$-$L(2,1)$-colorable.

Finally, we prove the lemma for $n \geq 3$ using induction on n. The lemma is true when $n = 0, 1, 2$. Assume that the lemma holds when $n < k$ for $k \geq 3$. Now we consider $n = k$. Let $\mathscr{L}$ be a list assignment of P_{k+3} of order $(2,4,5^{[k]},3)$. Assume i is the smallest index such that v_i is an M-vertex.
Case 1. $i = 1$.

We label v_1 with ℓ_1^1. The order of the RLA $\mathscr{L}'$ of $P = v_2v_3\cdots v_{k+3}$ is $(2,4,5^{[k-1]},3)$. By the induction hypothesis, P is $\mathscr{L}'$-$L(2,1)$-colorable. Hence P_{k+3} is $\mathscr{L}$-$L(2,1)$-colorable.
Case 2. $i = 2$.

We label v_2 with ℓ_2^1 and v_1 with ℓ_1^2. The order of the RLA $\mathscr{L}'$ of $P = v_3v_4\cdots v_{k+3}$ is $(2,4,5^{[k-2]},3)$. By the induction hypothesis P is $\mathscr{L}'$-$L(2,1)$-colorable. Hence P_{k+3} is $\mathscr{L}$-$L(2,1)$-colorable.
Case 3. $i = 3$.

We label v_3 with ℓ_3^1. The orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5\cdots v_{k+3}$ are $(2,3)$ and $(3,4,5^{[k-3]},3)$, respectively. By Lemma 2.1 we may label P. The order of the RLA $\mathscr{L}'$ of P' becomes $(2,4,5^{[k-3]},3)$. By the induction hypothesis P' is $\mathscr{L}'$-$L(2,1)$-colorable. Hence P_{k+3} is $\mathscr{L}$-$L(2,1)$-colorable.
Case 4. $4 \leq i \leq k$ when $k \geq 4$.

We label v_i with ℓ_i^1. The orders of the RLAs of $P = v_1v_2\cdots v_{i-1}$ and $P' = v_{i+1}v_{i+2}\cdots v_{k+3}$ are $(2,4,5^{[i-4]},4)$ and $(3,4,5^{[k-i]},3)$, respectively. By the induction hypothesis, P can be labeled. After that the order of the RLA $\mathscr{L}'$ of P' is $(2,4,5^{[k-i]},3)$. By the induction hypothesis P' is $\mathscr{L}'$-$L(2,1)$-colorable. Hence P_{k+3} is $\mathscr{L}$-$L(2,1)$-colorable.
Case 5. $i = k + 1$.

We label v_{k+1} with ℓ_{k+1}^1. The orders of the RLAs of $P = v_1v_2\cdots v_k$ and $P' = v_{k+2}v_{k+3}$ are $(2,4,5^{[k-3]},4)$ and $(3,2)$, respectively. By Lemma 2.1, we may label P'. Then the order of the RLA $\mathscr{L}'$ of P is $(2,4,5^{[k-3]},3)$. By the induction hypothesis P is $\mathscr{L}'$-$L(2,1)$-colorable. Hence P_{k+3} is $\mathscr{L}$-$L(2,1)$-colorable.

Case 6. $i = k + 2$.

We label v_{k+2} with ℓ^1_{k+2} and v_{k+3} with ℓ^3_{k+3}. The order of the RLA $\mathscr{L}'$ of $P = v_1v_2 \cdots v_{k+1}$ is $(2, 4, 5^{[k-2]}, 3)$. Similar to the above cases, P_{k+3} is $\mathscr{L}$-$L(2,1)$-colorable.

Case 7. $i = k + 3$.

We label v_{k+3} with ℓ^1_{k+3}. The order of the RLA of $P = v_1v_2 \cdots v_{k+2}$ is $(2, 4, 5^{[k-1]}, 4)$. Similar to the above cases, P_{k+3} is $\mathscr{L}$-$L(2,1)$-colorable. □

Following we only list the statements of supporting lemmas. The proofs of these lemmas are shown in Appendix since they are all technical.

Lemma 2.4. *P_3 is $\mathscr{L}$-$L(2,1)$-colorable for $\mathscr{L} = \{L(v) : v \in V(P_3)\}$ of order $(3, 3, 3)$ and $\ell^1_1 \neq \ell^1_3$.*

Lemma 2.5. *P_3 is $(3, 3, 4)$-$L(2,1)$-choosable.*

Lemma 2.6. *P_4 is $(2, 5, 4, 4)$-$L(2,1)$-choosable.*

Lemma 2.7. *P_4 is $(3, 4, 4, 4)$-$L(2,1)$-choosable.*

Lemma 2.8. *P_5 is $(3, 4, 5, 4, 4)$-$L(2,1)$-choosable.*

Corollary 2.9. *P_2 is 3-$L(2,1)$-choosable, P_3 and P_4 are 4-$L(2,1)$-choosable, P_n is 5-$L(2,1)$-choosable for $n \geq 5$.*

Following we study the choosability of a path requested that the labels of the end vertices are different.

Lemma 2.10. *P_3 is $[3, 4, 3]$-$L(2,1)$-choosable.*

Lemma 2.11. *P_4 is $[3, 4, 4, 4]$-$L(2,1)$-choosable.*

Lemma 2.12. *P_5 is $[4, 4, 5, 4, 4]$-$L(2,1)$-choosable.*

Lemma 2.13. *For $n \geq 2$, P_{n+3} is $[3, 4, 5^{[n]}, 4]$-$L(2,1)$-choosable.*

3. Proof of Main Theorem

Chen [13] proved C_n is 5-$L(2,1)$-choosable for $3 \leq n \leq 5$. Due to the thesis may not be easily accessed, we provide a different proof. Also we prove Theorem 1.1 is true for C_n with $n \geq 6$.

We let $\mathscr{L} = \{L(v) : v \in V(C_n)\}$, with $|L(v)| \geq 5$ throughout this section. We always denote $C_n = (v_1 \cdots v_n)$.

Proof of the Main Theorem for $3 \leq n \leq 4$.

Without loss of generality, we assume v_1 is an M-vertex. For C_3, we label v_1 with ℓ^1_1, then the order of the RLA of $P = v_2v_3$ is $(3, 3)$. So we can label P properly according to Lemma 2.1. Hence the result follows. For C_4, we label v_1 with ℓ^1_1, then the order of the RLA of $P = v_2v_3v_4$ is $(3, 4, 3)$. Note that v_2 and v_4 must be labeled differently. We can label P properly according to Lemma 2.10. Hence the result follows. □

Proof of the Main Theorem for $n = 5$.

Suppose there exists a non-M-vertex. Without loss of generality, we assume v_2 is a non-M-vertex and v_1 is an M-vertex. We label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_2v_3v_4v_5$ is $(4, 4, 4, 3)$. We can label P properly according to Lemma 2.11 since v_2 and v_5 must be labeled differently. So now we assume all vertices of C_5 are M-vertices.

Suppose there exists a non-M^*-vertex. Without loss of generality, we assume v_2 is a non-M^*-vertex. Since v_1 is an M-vertex, we label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_2v_3v_4v_5$ is $(4, 4, 4, 3)$. We can label P properly according to Lemma 2.11. Now we assume all vertices of C_5 are M^*-vertices.

By symmetry, we can conclude C_5 is $\mathscr{L}$-$L(2, 1)$-colorable if there exists a non-m^*-vertex. So we assume all vertices of C_5 are M^*-vertices and m^*-vertices. Now we label v_1, v_2, v_3, v_4 and v_5 with ℓ_1^1, ℓ_2^3, ℓ_3^5, ℓ_4^2 and ℓ_5^4, respectively. This completes the proof. □

Finally, we prove the Main Theorem for $n \geq 6$. This will be done in a series of lemmas. We shall first show that if at least one vertex of C_n is a non-M-vertex, then C_n is $\mathscr{L}$-$L(2, 1)$-colorable. It follows by symmetry that if at least one vertex of C_n is a non-m-vertex, then C_n is $\mathscr{L}$-$L(2, 1)$-colorable. We shall also show that if at least one vertex of C_n is a non-M^*-vertex, and similarly a non-m^*-vertex, then C_n is $\mathscr{L}$-$L(2, 1)$-colorable.

Lemma 3.1. *For $n \geq 6$, if C_n contains two adjacent non-M-vertices, then C_n is $\mathscr{L}$-$L(2, 1)$-colorable.*

Proof: By renumbering the vertices we may denote the longest path with all non-M-vertices as $P = v_{n-i+1} \cdots v_{n-1}v_n$, where $i \geq 2$. Now v_{n-1} and v_n are two non-M-vertices, and v_1 is an M-vertex. We label v_1 with ℓ_1^1. This action will not eliminate any label from $L(v_{n-1})$, it will eliminate at most one label from $L(v_n)$, two labels from $L(v_2)$ and one label from $L(v_3)$. The order of the RLA of the path $v_2v_3 \cdots v_n$ is $(3, 4, 5^{[n-4]}, 4)$. Since the labels of v_2 and v_n must be different, the lemma is proved if the path $v_2v_3 \cdots v_n$ is $[3, 4, 5^{[n-4]}, 4]$-$L(2, 1)$-choosable. It is clear that the lemma follows from Lemma 2.13. □

Lemma 3.2. *For $n \geq 6$, if C_n contains a non-M-vertex, then C_n is $\mathscr{L}$-$L(2, 1)$-colorable.*

Proof: By Lemma 3.1, we only need to consider cases in which the two neighbors of every non-M-vertex of C_n are M-vertices. We renumber the vertices of C_n as follows:

For $n \geq 8$, if no two non-M-vertices of C_n are at distance 4, we number the vertices of C_n such that v_2 is a non-M-vertex, and consequently v_{n-2} is an M-vertex. If there exist two non-M-vertices at distance 4, we number the vertices such that v_n and v_4 are non-M-vertices. For $n = 6$ or 7, we can always number the vertices such that v_2 is a non-M-vertex. By our assumption, v_1 is an M-vertex.

Now we make the following marking process for vertices of C_n:

We first mark v_1. Suppose that v_i has been marked, for some $i \leq n - 2$. If $i \leq n - 6$, then we mark the vertex v_{i+3} if it is an M-vertex, otherwise we mark v_{i+4}. Note that by our assumption, if v_{i+3} is a non-M-vertex, then v_{i+4} must be an M-vertex. Repeat this process until the subscript of the newly marked vertex is greater than $n - 6$. If this case occurs, then let the newly marked vertex be v_k and do the following process.

Case 1. If $k = n - 3$ or $k = n - 2$, then stop.

Case 2. If $k = n - 4$, and v_{n-3} is a non-M-vertex, then stop.

Case 3. If $k = n - 4$ and v_{n-3} is an M-vertex, then remove the mark from v_{n-4}, mark v_{n-3} instead and stop.

Case 4. If $k = n - 5$ and v_{n-2} is an M-vertex, then mark v_{n-2} and stop.

Case 5. If $k = n - 5$, and v_{n-2} is a non-M-vertex, then stop.

Note that all marked vertices are M-vertices. We draw the cycle C_n on the plane such that v_1, v_2, $\ldots$, v_n are placed clockwise.

This lemma is proved if we can obtain a proper $\mathscr{L}$-$L(2,1)$-labeling function of C_n. To do that, we first label all marked M-vertices by the largest label M in their lists. After that, remove all marked vertices. It dissects C_n into disjoint paths which are called *links*. We call the link containing v_2 the first link, and the link containing v_n the last link. A link containing x vertices is called an *x-link*. With the possible exception of the last two links, each link contains 2 or 3 vertices. When we label a marked vertex v by M. It will eliminate at most two labels from an M-vertex adjacent to v, at most one label from a non-M-vertex adjacent to v. It will also eliminate at most one label or zero label from an M-vertex or a non-M-vertex at distance two from v, respectively. Consequent orders of the RLAs of these 2-links and 3-links are $(3,3)$ and $(3,4,3)$, respectively.

Consider the first link. If v_2 is a non-M-vertex, then the RLA of this link is of the order $(4,3)$ or $(4,4,4)$. If v_2 is an M-vertex, then v_4 must be a non-M-vertex. The RLA of this link is of the order $(3,4,4)$.

We consider $n \geq 10$ first.

For Case 1 or Case 4, each link is either a 2-link or a 3-link. The order of the RLA of the first link is $(4,3)$ or $(3,4,4)$ and that of the remaining links are $(3,3)$ or $(3,4,3)$. We shall label the links clockwise. By Lemma 2.1 or Lemma 2.2, we may label the second link properly first. After that, the order of the RLA of the next link becomes $(2,3)$ or $(2,4,3)$. By Lemma 2.1 or Lemma 2.2 again, we may label it properly. Hence the procedure can be continued up to the last link. After labeling the last link, the order of the RLA of the first link may be reduced from $(4,3)$ or $(3,4,4)$ to $(3,2)$ or $(2,4,3)$, respectively. By Lemma 2.1 or Lemma 2.2, it can be labeled properly.

For Case 2, we will label the links anticlockwise. The last link is $v_n v_{n-1} v_{n-2} v_{n-3}$ and v_{n-3} is a non-M-vertex. The order of its RLA $\mathscr{L}'$ is $(3,4,4,4)$. By Lemma 2.7, the last link is $\mathscr{L}'$-$L(2,1)$-colorable. So we first label this link. After that, the order of the RLA of the next link becomes $(2,3)$ or $(2,4,3)$. By Lemma 2.1 or Lemma 2.2, we may label it properly. Hence the procedure can be continued up to the second link. After the second link has been labeled, the order of the RLA of the first link v_3v_2 or $v_4v_3v_2$ may be reduced from $(3,4)$ or $(4,4,3)$ to $(2,3)$ or $(3,4,2)$, respectively. By Lemma 2.1 or Lemma 2.2, it can be labeled properly.

For Case 3, the last link is $P = v_n v_{n-1} v_{n-2}$. The second last link Q is either $v_{n-4}v_{n-5}v_{n-6}$ or $v_{n-4}v_{n-5}v_{n-6}v_{n-7}$ where v_{n-4} is an M-vertex. For the latter case, v_{n-5} is a non-M-vertex. After labeling all marked vertices, the order of the RLA of Q is either $(3,4,3)$ or $(3,5,4,3)$, and that of P is

$(3,4,3)$. By Lemma 2.2 we may label Q first. After that the order of the RLA of P may be reduced to $(3,4,2)$. By Lemma 2.2 we may label P. Now we label the remaining links from the third last link anticlockwise. Similar to the previous cases, the labeling can be continued up to the second link. Finally we may label the first link.

For Case 5, since v_{n-2} is a non-M-vertex, by our numbering C_n contains two non-M-vertices of distance 4. By our numbering again, v_n and v_4 are non-M-vertices. So the order of the RLA of the last link $v_nv_{n-1}v_{n-2}v_{n-3}v_{n-4}$ is $(4,4,5,4,3)$. Hence we may label the last link by means of Lemma 2.8. Now we can continue as before to label the links anticlockwise one by one until the first link is labeled.

Finally, we consider $6 \le n \le 9$.

Case A. $n=6$. If v_4 is an M-vertex, then it is referred to Case 1. So we assume v_4 is a non-M-vertex and consequently v_3 and v_5 are M-vertices. If v_6 is an M-vertex, then we mark v_3 and v_6, and label them by M. Now the orders of the RLAs of v_1v_2 and v_5v_4 are both $(3,4)$. We can first label v_1v_2 by means of Lemma 2.1. Then the order of the RLA of v_5v_4 is reduced to $(2,3)$. By means of Lemma 2.1, we can complete the labeling of C_6. If v_6 is a non-M-vertex, then we mark v_1 and label it by M. Then the order of the RLA of the link $P_5 = v_2v_3v_4v_5v_6$ is $(4,4,5,4,4)$. By Lemma 2.12, P_5 is $[4,4,5,4,4]$-$L(2,1)$-choosable. We can complete the labeling of the cycle.

Case B. $n=7$. It is referred to Case 1.

Case C. $n=8$. There are three subcases. If v_4 is a non-M-vertex, then mark v_1 and v_5. It is referred to Case 1. If v_4 is an M-vertex but v_5 is not, then it is referred to Case 2. Finally, if both v_4 and v_5 are M-vertices, then we mark v_1 and v_5, and label them by M. By our numbering, v_2 is a non-M-vertex. So the orders of the RLAs of the first and the second links are $(4,4,3)$ and $(3,4,3)$ respectively. We may label the second link first and then label the first link to complete the labeling.

Case D. $n=9$. Suppose there exist two non-M-vertices of distance 4. By our numbering v_4 and v_9 are non-M-vertices. Hence v_5 is an M-vertex. As a special case, we mark v_1 and v_5 and label them by M. We have to consider two links $P = v_6v_7v_8v_9$ and $Q = v_2v_3v_4$. The orders of the RLAs of P and Q are $(3,4,4,4)$ and $(3,4,4)$, respectively. By Lemma 2.7 we may label P first. After that the order of the RLA of Q is reduced to $(2,4,3)$. By Lemma 2.2 we may label Q.

Suppose there are no two non-M-vertices of distance 4. So by our numbering v_2 is a non-M-vertex and v_7 is an M-vertex. If v_4 is an M-vertex, then it is referred to Case 4. If v_4 is a non-M-vertex, then v_3 and v_8 are both M-vertices. As a special case, we will mark v_3 and v_8, and label them by M. We consider two links, namely $P = v_4v_5v_6v_7$ and $Q = v_9v_1v_2$. The orders of the RLAs of P and Q are $(4,4,4,3)$ and $(3,4,4)$, respectively. The labeling method is similar as above.

This completes the proof of the lemma. □

It is straight forward to obtain the following:

Corollary 3.3. *For $n \geq 6$, if C_n contains a non-m-vertex, then C_n is $\mathscr{L}$-$L(2,1)$-colorable.*

Lemma 3.4. *For $n \geq 6$, if C_n contains a non-M^*-vertex, then C_n is $\mathscr{L}$-$L(2,1)$-colorable.*

Proof: By Lemma 3.2, we may assume that all vertices of C_n are M-vertices. Suppose v_2 is a non-M^*-vertex. Let $n = 3k + j$, where $k \geq 2$ and $0 \leq j \leq 2$. We shall first mark v_1 and then mark the remaining vertices of C_n in the following manner:

(1) If $j = 0$, then mark v_{3i+1}, $1 \leq i \leq k-1$.

(2) If $j = 1$, then mark v_{3i+2}, $1 \leq i \leq k-1$.

(3) If $j = 2$, then mark v_5. If $n = 8$, stop. If $n > 8$, then mark v_{3i}, $3 \leq i \leq k$.

As the proof of Lemma 3.2, we label all marked vertices by M. After removing all marked vertices, C_n is dissected into some 2- or 3-links. The first link is either v_2v_3 or $v_2v_3v_4$. Because v_2 is a non-M^*-vertex, the order of the RLA of the first link is $(4,3)$ or $(4,4,3)$. The order of the RLA of each other link is either $(3,3)$ or $(3,4,3)$. Similar to the proof of Lemma 3.2, we can label all the links clockwise starting from the second link to the last link. After that, the order of the RLA of the first link is reduced to $(3,2)$ or $(3,4,2)$. Finally, we can label it. Hence, the cycle is labeled properly. □

Corollary 3.5. *For $n \geq 6$, if C_n contains a non-m^*-vertex, then C_n is $\mathscr{L}$-$L(2,1)$-colorable.*

Proof of the Main Theorem for $n \geq 6$.

From Lemma 3.4 and Corollary 3.5, it suffices to consider the case that all vertices of C_n are M^*-vertices and m^*-vertices. We give a $\mathscr{L}$-$L(2,1)$-labeling f of C_n as following:

(1) If $n \equiv 0 \pmod 3$, then define

$$f(v_i) = \begin{cases} m & \text{if } i \equiv 1 \pmod 3, \\ \ell^3_{v_i} & \text{if } i \equiv 2 \pmod 3, \\ M & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

(2) If $n \equiv 1 \pmod 3$, then redefine the above f at $v_{n-3}, \ldots, v_n$ as

$$f(v_i) = \begin{cases} m & \text{if } i = n-3, \\ M-1 & \text{if } i = n-2, \\ m+1 & \text{if } i = n-1, \\ M & \text{if } i = n. \end{cases}$$

(3) If $n \equiv 2 \pmod 3$, then redefine the f in (1) at v_{n-1} and v_n as

$$f(v_i) = \begin{cases} m+1 & \text{if } i = n-1, \\ M-1 & \text{if } i = n. \end{cases}$$

It is easy to check that f is a proper $\mathscr{L}$-$L(2,1)$-labeling function of G. Hence we conclude that the cycle $C_n, n \geq 6$, is 5-$L(2,1)$-choosable.

4. Appendix: Proofs of Supporting Lemmas

Lemma 2.4 *P_3 is $\mathscr{L}$-$L(2,1)$-colorable for $\mathscr{L} = \{L(v) : v \in V(P_3)\}$ of order $(3,3,3)$ and $\ell^1_1 \neq \ell^1_3$.*

Proof: Without loss of generality, we assume $\ell_1^1 > \ell_3^1$. If $\ell_2^1 \geq \ell_1^1$, we label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. There is at least one label left in the residual list of v_3 to complete the labeling. If $\ell_2^1 < \ell_1^1$, we label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3$ is $(2,3)$. By Lemma 2.1, P can be labeled properly. □

Lemma 2.5 *P_3 is $(3,3,4)$-$L(2,1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_3 of order $(3,3,4)$.

Case 1. $\ell_1^1 > \ell_2^1$.

We label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3$ is $(2,3)$. By Lemma 2.1, P_3 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. $\ell_1^1 < \ell_2^1$.

We label v_2 with ℓ_2^1. There are at least two labels left in the residual list of v_1 and at least one label left in that of v_3. We first label v_3. Then there is at least one label left in the residual list of v_1. So we can label v_1.

Case 3. $\ell_1^1 = \ell_2^1$.

If $\ell_3^1 > \ell_2^1$, then we label v_3 with ℓ_3^1. The order of the RLA of $P = v_1v_2$ is $(3,2)$. By Lemma 2.1, P_3 is $\mathscr{L}$-$L(2,1)$-colorable.

If $\ell_3^1 \leq \ell_2^1$, then we label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. There is at least one label left in the residual list of v_3 to complete the labeling. □

Lemma 2.6 *P_4 is $(2,5,4,4)$-$L(2,1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_4 of order $(2,5,4,4)$.

Case 1. $\ell_1^1 \geq \ell_2^1$.

We label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3v_4$ is $(3,3,4)$. The result follows by Lemma 2.5.

Case 2. $\ell_3^1 < \ell_4^1$.

We label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(2,4,3)$. The result follows by Lemma 2.2.

We assume that $\ell_1^1 < \ell_2^1$ and $\ell_3^1 \geq \ell_4^1$ in the remaining cases.

Case 3. $\ell_2^1 \leq \ell_3^1$.

We label v_3 with ℓ_3^1. The order of the RLA of $P = v_1v_2$ is $(2,3)$ and there are at least two labels left in the residual list of v_4. By Lemma 2.1, P can be labeled. After labeling P, there is at least one label left in the residual list of v_4. So P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 4. $\ell_2^1 > \ell_3^1$.

We label v_2 with ℓ_2^1 and v_1 with ℓ_1^2. Then the order of the RLA of $P = v_3v_4$ is $(2,4)$. The result follows by Lemma 2.1. □

Lemma 2.7 *P_4 is $(3,4,4,4)$-$L(2,1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_4 of order $(3,4,4,4)$.

Suppose $\ell_1^1 > \ell_2^1$. Then we label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3v_4$ is $(3,3,4)$. By Lemma 2.5, we can label P and hence P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Suppose $\ell_1^1 < \ell_2^1$. If $\ell_3^1 > \ell_2^1$, then we label v_3 with ℓ_3^1. There is at least one label left in the residual list of v_4. So we label v_4 by an available label. The order of the RLA of $P = v_1v_2$ becomes $(3, 2)$. By Lemma 2.1, we know that P_4 is $\mathscr{L}$-$L(2,1)$-colorable. If $\ell_3^1 \le \ell_2^1$, then we label v_2 with ℓ_2^1. The order of the RLA of $P = v_3v_4$ is $(2, 3)$ and there are at least two labels left in the residual list of v_1. By Lemma 2.1 we may label $P = v_3v_4$. Then there is at least one label left in the residual list of v_1 to complete the labeling.

Now we assume that $\ell_1^1 = \ell_2^1$.

Case 1. Suppose $\ell_3^1 > \ell_2^1$. By the same proof as the case when $\ell_1^1 < \ell_2^1 < \ell_3^1$, we get that P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. Suppose $\ell_3^1 < \ell_2^1$. We label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. The order of the RLA of $P = v_3v_4$ is $(2, 3)$. By Lemma 2.1, P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 3. Suppose $\ell_3^1 = \ell_2^1$ and $\ell_4^1 > \ell_3^1$. We label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(3, 4, 3)$. By Lemma 2.2, P_4 is $\mathscr{L}$-$L(2,1)$-colorable.

Up to now, the remaining case we have to deal with is $\ell_1^1 = \ell_2^1 = \ell_3^1 \ge \ell_4^1$. By symmetry, we also know that P_4 is $\mathscr{L}$-$L(2,1)$-colorable except the case that $\ell_1^3 = \ell_2^4 = \ell_3^4 \le \ell_4^4$.

So we have to deal with the case that $\ell_1^1 = \ell_2^1 = \ell_3^1 \ge \ell_4^1$ and $\ell_1^3 = \ell_2^4 = \ell_3^4 \le \ell_4^4$. In this case, either $\ell_2^1 - \ell_1^2 \ge 2$ or $\ell_1^2 - \ell_2^4 \ge 2$. For the former case, we label v_1, v_2, v_3 and v_4 by ℓ_1^2, ℓ_2^1, ℓ_3^4 and ℓ_4^2, respectively. For the latter case, we label v_1, v_2, v_3 and v_4 by ℓ_1^2, ℓ_2^4, ℓ_3^1 and ℓ_4^3, respectively. So P_4 is $\mathscr{L}$-$L(2,1)$-colorable. □

Lemma 2.8 *P_5 is $(3, 4, 5, 4, 4)$-$L(2,1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_5 of order $(3, 4, 5, 4, 4)$.

Case 1. $\ell_1^1 > \ell_2^1$.

We label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3v_4v_5$ is $(3, 4, 4, 4)$. The result follows by Lemma 2.7.

Case 2. $\ell_1^1 < \ell_2^1$.

Case 2.1. If $\ell_2^1 \ge \ell_3^1$, then we label v_2 with ℓ_2^1. The order of the RLA $\mathscr{L}'$ of $P = v_3v_4v_5$ is $(3, 3, 4)$ and there are at least two labels left in the residual list of v_1. So by Lemma 2.5, P is $\mathscr{L}'$-$L(2,1)$-colorable. After labeling P, there is at least one label left in the residual list of v_1. So P_5 is $\mathscr{L}$-$L(2,1)$-colorable.

Case 2.2. If $\ell_2^1 < \ell_3^1$ and $\ell_4^1 \le \ell_3^1$, then we label v_3 with ℓ_3^1. The orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(3, 3)$ and $(2, 3)$, respectively. We first label $P' = v_4v_5$ according to Lemma 2.1. Then the order of the RLA of $P = v_1v_2$ becomes $(3, 2)$. The result follows by Lemma 2.1.

Case 2.3. If $\ell_2^1 < \ell_3^1$ and $\ell_4^1 > \ell_3^1$, then we label v_4 with ℓ_4^1. There is at least one label left in the residual list of v_5. Then we label v_5. After that, the order of the RLA of $P = v_1v_2v_3$ is $(3, 4, 3)$. So the result follows by Lemma 2.2.

Case 3. $\ell_1^1 = \ell_2^1$.

Case 3.1. $\ell_3^1 < \ell_2^1$.

We label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. The order of the RLA of $P = v_3v_4v_5$ is $(3, 3, 4)$. The result follows by Lemma 2.5.

Case 3.2. $\ell_3^1 > \ell_2^1$.

If $\ell_4^1 \le \ell_3^1$, then by the same proof of Case 2.2, we get the result.

If $\ell_4^1 > \ell_3^1$, then by the same proof of Case 2.3, we get the result.

Case 3.3. $\ell_3^1 = \ell_2^1$.

If $\ell_4^1 > \ell_3^1$, then by the same proof of Case 2.3, we have the result.

If $\ell_4^1 < \ell_3^1$, then we label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. The order of the RLA of $P = v_3v_4v_5$ is $(2, 4, 4)$. So the result follows by Lemma 2.2.

If $\ell_4^1 = \ell_3^1$ and $\ell_5^1 > \ell_4^1$, then we label v_5 with ℓ_5^1. The order of the RLA of $P = v_1v_2v_3v_4$ is $(3, 4, 5, 3)$. So the result follows by Lemma 2.2.

Up to now, the remaining case we need to deal with is the case that $\ell_1^1 = \ell_2^1 = \ell_3^1 = \ell_4^1 \ge \ell_5^1$. By symmetry we can also show that P_5 is $\mathscr{L}$-$L(2,1)$-colorable except the case that $\ell_1^3 = \ell_2^4 = \ell_3^5 = \ell_4^4 \le \ell_5^4$.

Now suppose $\ell_1^1 = \ell_2^1 = \ell_3^1 = \ell_4^1 \ge \ell_5^1$ and $\ell_1^3 = \ell_2^4 = \ell_3^5 = \ell_4^4 \le \ell_5^4$. In this case, either $\ell_2^1 - \ell_1^2 \ge 2$ or $\ell_1^2 - \ell_2^4 \ge 2$. For the former case, we label v_2 with ℓ_2^1. Then the order of the RLA of $P = v_3v_4v_5$ is $(3, 3, 4)$ and there are at least two labels left in the residual list of v_1. We first label P according to Lemma 2.5. Then there is at least one label left in the residual list of v_1 to complete the labeling. For the latter case, we label v_2 with ℓ_2^4. Then the order of the RLA of $P = v_3v_4v_5$ is $(3, 3, 4)$ and there are at least two labels left in the residual list of v_1. We first label P according to Lemma 2.5. Then there is at least one label left in the residual list of v_1 to complete the labeling. □

Lemma 2.10 *P_3 is $[3, 4, 3]$-$L(2, 1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_3 of order $(3, 4, 3)$.

Case 1. Both v_1 and v_3 are non-M-vertices.

Then v_2 must be an M-vertex, and we label v_2 by ℓ_2^1. There are both at least two labels left in the residual lists of v_1 and v_3. Now we can label v_1 and v_3 by two distinct labels.

Case 2. Only one of v_1 and v_3 is an M-vertex.

By symmetry we may assume that v_1 is an M-vertex and v_3 is not. We label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_2v_3$ is $(2, 3)$. By Lemma 2.1, we can label P.

Case 3. Both v_1 and v_3 are M-vertices but v_2 is not.

We label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_2v_3$ is $(3, 2)$. By Lemma 2.1, we can label P.

Note that the only case not covered by Cases 1 to 3 is the case that v_1, v_2 and v_3 are all M-vertices.

By symmetry, we can also show that P_3 is strictly $\mathscr{L}$-$L(2,1)$-colorable except when v_1, v_2 and v_3 are all m-vertices.

So we assume that v_1, v_2 and v_3 are M-vertices and m-vertices. If $\ell_2^1 - \ell_1^2 \ge 2$, then we label v_1 v_2 and v_3 with ℓ_1^2, ℓ_2^1 and ℓ_3^3 respectively; otherwise $\ell_1^2 - \ell_2^4 \ge 2$, then we label v_1 v_2 and v_3 with ℓ_1^2, ℓ_2^4 and ℓ_3^1 respectively. □

Lemma 2.11 *P_4 is $[3, 4, 4, 4]$-$L(2, 1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_4 of order $(3, 4, 4, 4)$.

Suppose $\ell_1^1 > \ell_3^1$. If $\ell_1^1 < \ell_2^1$, then we label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. The order of the RLA of $P = v_3v_4$ is $(3, 2)$. By Lemma 2.1, we can label P. If $\ell_1^1 \geq \ell_2^1$, then we label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3v_4$ is $(2, 4, 3)$. By Lemma 2.2, we can label P.

Suppose $\ell_1^1 \leq \ell_3^1 - 2$. If $\ell_2^1 \geq \ell_3^1$, then we label v_2 with ℓ_2^1. The order of the RLA of $P = v_3v_4$ is $(2, 3)$ and there are at least three labels left in the residual list of v_1. By Lemma 2.1, we can label P. Now there is still one label left in the residual list of v_1 to complete the labeling. If $\ell_2^1 < \ell_3^1$ and $\ell_4^1 \geq \ell_3^1$, then we label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(3, 4, 2)$. By Lemma 2.2, we can label P. If $\ell_2^1 < \ell_3^1$ and $\ell_4^1 < \ell_3^1$, then we label v_3 with ℓ_3^1. The order of the RLA of $P = v_1v_2$ is $(3, 3)$ and there are at least three labels left in the residual list of v_4. By Lemma 2.1, we can label P. Now there is still one label left in the residual list of v_4 to complete the labeling.

Suppose $\ell_1^1 = \ell_3^1 - 1$. If $\ell_2^1 > \ell_3^1$, then we label v_2 with ℓ_2^1. The order of the RLA of $P = v_3v_4$ is $(3, 3)$ and there are at least three labels left in the residual list of v_1. By Lemma 2.1, we can label P. Now there is still one label left in the residual list of v_1 to complete the labeling. If $\ell_2^1 \leq \ell_3^1$ and $\ell_4^1 > \ell_3^1$, then we label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(3, 4, 3)$. By Lemma 2.2, we can label P. If $\ell_2^1 \leq \ell_3^1$ and $\ell_4^1 \leq \ell_3^1$, then we label v_3 with ℓ_3^1 and v_1 with ℓ_1^1. There are at least one label left in the residual list of v_2 and two labels left in that of v_4. Now we can label v_2 and v_4 with two distinct labels.

Now we assume that $\ell_1^1 = \ell_3^1$.

Case 1. $\ell_2^1 \geq \ell_3^1 + 2$. We label v_2 with ℓ_2^1. The order of the RLA of $P = v_3v_4$ is $(4, 3)$ and there are at least three labels left in the residual list of v_1. By Lemma 2.1, we can label P. Now there is still one label left in the residual list of v_1 to complete the labeling.

Case 2. $\ell_2^1 \leq \ell_3^1 - 2$. We label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3v_4$ is $(4, 3, 3)$. By Lemma 2.5, we can label P.

Case 3. $\ell_2^1 = \ell_3^1 + 1$. If $\ell_4^1 > \ell_3^1$, we label v_4 with ℓ_4^1. Then at the worst ℓ_2^1 and ℓ_3^1 will be eliminated. The order of the RLA of $P = v_1v_2v_3$ is $(3, 3, 3)$ with $\ell_1^1 > \ell_3^2$. By Lemma 2.4, P can be labeled. If $\ell_4^1 \leq \ell_3^1$, we label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. Then the order of the RLA of $P = v_3v_4$ is $(2, 3)$. By Lemma 2.1, we can label P.

Case 4. $\ell_2^1 = \ell_3^1 - 1$. If $\ell_4^1 > \ell_3^1$, we label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(3, 4, 3)$. By Lemma 2.2, P can be labeled. If $\ell_4^1 < \ell_3^1$, we label v_1 with ℓ_1^1. Then the order of the RLA of $P = v_1v_2v_3$ is $(3, 3, 4)$. By Lemma 2.5, we can label P. If $\ell_4^1 = \ell_3^1$, we label v_1 with ℓ_1^1. Then at the worst ℓ_2^1, ℓ_3^1 and ℓ_4^1 will be eliminated. The order of the RLA of $P = v_2v_3v_4$ is $(3, 3, 3)$ which may not be labeled only when $\ell_2^2 = \ell_4^2$ according to Lemma 2.4. In this case, we label v_3 with ℓ_3^1. Then at the worst ℓ_1^1, ℓ_2^1 and ℓ_4^1 will be eliminated. The order of the RLA of $P = v_1v_2$ is $(2, 3)$ and there are at least three labels left in the residual list of v_4. By Lemma 2.1, we can label P. Now there is still one label left in the residual list of v_4 to complete the labeling.

Case 5. $\ell_2^1 = \ell_3^1$. If $\ell_4^1 > \ell_3^1$, we label v_4 with ℓ_4^1. The order of the RLA of $P = v_1v_2v_3$ is $(3, 4, 3)$. By Lemma 2.2, P can be labeled.

Up to now, the remaining case we have to deal with is $\ell_1^1 = \ell_2^1 = \ell_3^1 \geq \ell_4^1$. By symmetry, we also know that P_4 is strictly $\mathscr{L}$-$L(2, 1)$-colorable except the case that $\ell_1^3 = \ell_2^4 = \ell_3^4 \leq \ell_4^4$.

So we have to deal with the case that $\ell_1^1 = \ell_2^1 = \ell_3^1 \geq \ell_4^1$ and $\ell_1^3 = \ell_2^4 = \ell_3^4 \leq \ell_4^4$.

We first consider the subcase that $\ell_1^1 = \ell_2^1 = \ell_3^1 > \ell_4^1$. We know either $\ell_2^1 - \ell_1^2 \geq 2$ or $\ell_1^2 - \ell_2^4 \geq 2$. If $\ell_2^1 - \ell_1^2 \geq 2$, we label v_1, v_2 and v_3 with ℓ_1^2, ℓ_2^1 and ℓ_3^4 respectively. There is still one label left in the residual list of v_4 to complete the labeling. If $\ell_1^2 - \ell_2^4 \geq 2$, we label v_1, v_2 and v_3 with ℓ_1^2, ℓ_2^4 and ℓ_3^1 respectively. Now there is still one label left in the residual list of v_4 to complete the labeling. By symmetry, we know P_4 is strictly $\mathscr{L}$-$L(2,1)$-colorable when $\ell_1^3 = \ell_2^4 = \ell_3^4 < \ell_4^4$.

So we only have to deal with the subcase that $\ell_1^1 = \ell_2^1 = \ell_3^1 = \ell_4^1$ and $\ell_1^3 = \ell_2^4 = \ell_3^4 = \ell_4^4$.

In this subcase, without loss of generality, we assume that $\ell_2^1 - \ell_1^2 \geq 2$. We label v_1, v_2 and v_3 with ℓ_1^2, ℓ_2^1 and ℓ_3^4 respectively. Now v_4 can not be labeled only when $\ell_4^3 - \ell_4^4 = 1$ and $\ell_1^2 = \ell_4^2$. While this indicate that $\ell_1^2 - \ell_2^4 = \ell_4^2 - \ell_4^4 \geq 2$. So we can label v_1, v_2, v_3 and v_4 with $\ell_1^2, \ell_2^4, \ell_3^1$ and ℓ_4^3 respectively.

□

Lemma 2.12 *P_5 is $[4,4,5,4,4]$-$L(2,1)$-choosable.*

Proof: Let $\mathscr{L}$ be a list assignment of P_5 of order $(4,4,5,4,4)$.

Case 1. Both v_1 and v_5 are non-M-vertices.

If v_2 is an M-vertex, then we label v_2 by ℓ_2^1. The order of the RLA of $P = v_3v_4v_5$ is $(3,3,4)$ and there are at least three labels left in the residual list of v_1. By Lemma 2.5, P can be labeled. After that there is at least one label left in the residual list of v_1 to complete the labeling. If v_4 is an M-vertex, then by symmetry it is similar as above.

Now we consider the case when v_2 and v_4 are both non-M-vertices. Obviously v_3 is the unique M-vertex. So we label v_3 with ℓ_3^1. The orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(4,3)$ and $(3,4)$, respectively. We label P according to Lemma 2.1. Now the order of the RLA of P' is $(2,3)$. Hence P_5 is strictly $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. Only one of v_1 and v_5 is an M-vertex.

By symmetry we may assume that v_1 is an M-vertex and v_5 is not. If v_3 is not an M-vertex, then we can label v_1 with ℓ_1^1. The order of the RLA of $P = v_2v_3v_4v_5$ is $(2,5,4,4)$. By Lemma 2.6, we can label P.

So we assume v_3 is an M-vertex. If v_4 is an M-vertex, then we label v_1 and v_4 by ℓ_1^1 and ℓ_4^1, respectively. The order of the RLA of $P = v_2v_3$ is $(2,3)$ and there are at least three labels remaining in the residual list of v_5. By Lemma 2.1, we can label P with its RLA, and after that at least two labels are available to label v_5. If v_4 is a non-M-vertex, then we can label v_3 by ℓ_3^1. The orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(3,2)$ and $(3,4)$, respectively. By Lemma 2.1, we can label P first and after that P' can also be labeled.

Case 3. Both v_1 and v_5 are M-vertices but v_3 is not.

If v_4 is an M-vertex, then we can label v_1 and v_4 by ℓ_1^1 and ℓ_4^1, respectively. The order of the RLA of $P = v_2v_3$ is $(2,4)$ and there are at least two labels remaining in the residual list of v_5. By Lemma 2.1, we can label P with its RLA, and after that at least one label is available to label v_5. If v_2 is an M-vertex, then by symmetry it is similar as above.

Now we consider the case when both v_2 and v_4 are non-M-vertices. We label v_5 by ℓ_5^1. The order of the RLA of $P = v_1v_2v_3v_4$ is $(3,4,5,3)$. By Lemma 2.2, we can label P with its RLA.

Case 4. v_1, v_3 and v_5 are M-vertices.

If v_4 is an M-vertex, then we label v_1 and v_4 by ℓ_1^1 and ℓ_4^1, respectively. The order of the RLA of $P = v_2v_3$ is $(2,3)$ and there are at least two labels remaining in the residual list of v_5. By Lemma 2.1, we can label P with its RLA, and after that at least one label is available to label v_5. The labeling method is similar when v_2 is an M-vertex.

Note that the only case not covered by Cases 1 to 4 is the case that v_1, v_3 and v_5 are M-vertices but v_2 and v_4 are not.

By symmetry, we can also show that P_5 is strictly $\mathscr{L}$-$L(2,1)$-colorable except when v_1, v_3 and v_5 are m-vertices but v_2 and v_4 are not.

So we assume that (i) v_1, v_3 and v_5 are M-vertices and m-vertices and (ii) v_2 and v_4 are non-M-vertices and are non-m-vertices. We label v_3 and v_5 by ℓ_3^5 and ℓ_5^1, respectively. After that ℓ_4^2 and ℓ_4^3 in $L(v_4)$ are not eliminated. The order of the RLA of $P = v_1v_2$ is $(2,3)$. Again by Lemma 2.1, we can label P. After that, there is still one label available to label v_4. □

Lemma 2.13 *For $n \geq 2$, P_{n+3} is $[3,4,5^{[n]},4]$-$L(2,1)$-choosable.*

Proof: First we prove the lemma is true when $n = 2$. Let $\mathscr{L}$ be a list assignment of P_5 of order $(3,4,5,5,4)$. Assume that i is the smallest index such that v_i is an M-vertex.

Suppose $i = 1$. We label v_1 with ℓ_1^1. Since the label of v_5 must differ from ℓ_1^1, the order of the RLA of $P = v_2v_3v_4v_5$ is $(2,4,5,3)$. By Lemma 2.2 P has a proper $L(2,1)$-labeling and hence P_5 is strictly $\mathscr{L}$-$L(2,1)$-colorable.

Suppose $i = 2$. We label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. The order of the RLA of $P = v_3v_4v_5$ is $(2,4,3)$. The result follows by Lemma 2.2.

Suppose $i = 4$. We label v_4 with ℓ_4^1 and v_5 with ℓ_5^4. The order of the RLA of $P = v_1v_2v_3$ is $(2,4,3)$. The result follows by Lemma 2.2.

Suppose $i = 5$. We label v_5 with ℓ_5^1. The order of the RLA of $P = v_1v_2v_3v_4$ is $(3,4,5,4)$. The result follows by Lemma 2.2.

Suppose $i = 3$. We consider the following three cases:

Case 1. $\ell_3^1 - \ell_2^1 \geq 2$.

We label v_3 with ℓ_3^1. Then the orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(3,4)$ and $(3,3)$, respectively. According to Lemma 2.1, we may label P' first. Then the order of the RLA of P is $(2,3)$. P_5 is strictly $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. $\ell_3^1 - \ell_2^1 = 1$ and $\ell_3^1 > \ell_4^1$.

We label v_3 with ℓ_3^1. Then the orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(3,3)$ and $(4,3)$, respectively. We label $P = v_1v_2$ according to Lemma 2.1. Then the order of the RLA of P' becomes $(3,2)$. So we can label P' properly and hence the result follows.

Case 3. $\ell_3^1 - \ell_2^1 = 1$ and $\ell_3^1 = \ell_4^1 > \ell_5^1$.

We label v_3 with ℓ_3^1. Then the orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5$ are $(3,3)$ and $(3,4)$, respectively. We label $P = v_1v_2$ by means of Lemma 2.1. Then the order of the RLA of P' becomes $(2,3)$. So we can label P' properly. Hence the result follows.

Now the only case unsolved is when $\ell_1^1 < \ell_2^1 + 1 = \ell_3^1 = \ell_4^1 = \ell_5^1$. By considering the symmetric cases as above, we conclude that the only case unsolved is when $\ell_1^1 < \ell_2^1 + 1 = \ell_3^1 = \ell_4^1 = \ell_5^1$ and $\ell_1^3 > \ell_2^4 - 1 = \ell_3^5 = \ell_4^5 = \ell_5^4$. In this case, we label v_1, v_2, v_3, v_4, v_5 by ℓ_1^1, ℓ_2^4, ℓ_3^1, ℓ_4^3, ℓ_5^4, respectively. Hence P_5 is strictly $\mathscr{L}$-$L(2,1)$-colorable.

Then we prove the lemma for $n \geq 3$. Let $\mathscr{L}$ be a list assignment of P_{n+3} of order $(3, 4, 5^{[n]}, 4)$. Assume that i is the smallest index such that v_i is an M-vertex.

Case 1. $i = 1$.

We label v_1 with ℓ_1^1. The order of the RLA $\mathscr{L}'$ of $P = v_2v_3 \cdots v_{n+3}$ is $(2, 4, 5^{[n-1]}, 3)$. By Lemma 2.2, P is $\mathscr{L}'$-$L(2,1)$-colorable. Hence P_{n+3} is strictly $\mathscr{L}$-$L(2,1)$-colorable.

Case 2. $i = 2$.

We label v_2 with ℓ_2^1 and v_1 with ℓ_1^3. The order of the RLA of $P = v_3v_4 \cdots v_{n+3}$ is $(2, 4, 5^{[n-2]}, 3)$. The result follows by Lemma 2.2.

Case 3. $i = 3$.

We label v_3 with ℓ_3^1. The orders of the RLAs of $P = v_1v_2$ and $P' = v_4v_5 \cdots v_{n+3}$ are $(3,3)$ and $(3, 4, 5^{[n-3]}, 4)$, respectively. We label $P = v_1v_2$ by means of Lemma 2.1. Then the order of the RLA of P' becomes $(2, 4, 5^{[n-3]}, 3)$. The result follows by Lemma 2.2.

Case 4. $4 \leq i \leq n$ when $n \geq 4$.

We label v_i with ℓ_i^1. The orders of the RLAs of $P = v_1v_2 \cdots v_{i-1}$ and $P' = v_{i+1}v_{i+2} \cdots v_{n+3}$ are $(3, 4, 5^{[i-4]}, 4)$ and $(3, 4, 5^{[n-i]}, 4)$, respectively. We label P by means of Lemma 2.2. Then the order of the RLA of P' becomes $(2, 4, 5^{[n-i]}, 3)$. The result follows by Lemma 2.2.

Case 5. $i = n + 1$.

We label v_{n+1} with ℓ_{n+1}^1. The orders of the RLAs of $P = v_1v_2 \cdots v_n$ and $P' = v_{n+2}v_{n+3}$ are $(3, 4, 5^{[n-3]}, 4)$ and $(3,3)$, respectively. We label P' by means of Lemma 2.1. Then the order of the RLA of P is $(2, 4, 5^{[n-3]}, 3)$. The result follows by Lemma 2.2.

Case 6. $i = n + 2$.

We label v_{n+2} with ℓ_{n+2}^1 and v_{n+3} with ℓ_{n+3}^4. The order of the RLA of $P = v_1v_2 \cdots v_{n+1}$ is $(2, 4, 5^{[n-2]}, 3)$. The result follows by Lemma 2.2.

Case 7. $i = n + 3$.

We label v_{n+3} with ℓ_{n+3}^1. The order of the RLA of $P = v_1v_2 \cdots v_{n+2}$ is $(3, 4, 5,^{[n-1]}, 4)$. The result follows by Lemma 2.2. □

Acknowledgments

This work is supported by Faculty Research Grant, Hong Kong Baptist University.

References

[1] J. R. Griggs and R. K. Yeh, Labeling graphs with a condition at distance two, *SIAM J. Discrete Math.*, **5** (1992) 586-595.

[2] G. J. Chang and D. Kuo, The $L(2,1)$-labeling problem on graphs, *SIAM J. Discrete Math.*, **9** (1996) 309-316.

[3] G. J. Chang, W. T. Ke, D. Kuo, D. F. Liu and R. K. Yeh, On $L(d,1)$-labelings of graphs, *Discrete Math.*, **220** (2000) 57-66.

[4] F. S. Roberts, *Private communication with J. R. Griggs*, (1988).

[5] W. K. Hale, Frequency assignment: theory and applications, *Proc. IEEE*, **68** (1980) 1497-1514.

[6] V. G. Vizing, Vertex coloring with given colors (in Russian), *Diskret. Analiz*, **29** (1976) 3-10.

[7] P. Erdős, A. L. Rubin and H. Taylor, Choosability in graphs. *Congres. Numer.*, **26** (1979) 125-157.

[8] C. Thomassen, Every planar graph is 5-choosable, *Journal of Combinatorial Theory, Ser. B,* **62** (1994) 180-181.

[9] C. Thomassen, 3-list-coloring planar graphs of girth 5, *Journal of Combinatorial Theory, Ser. B*, **64** (1995) 101-107.

[10] P. C. B. Lam, B. G. Xu and J. Z. Liu, The 4-choosability of plane graphs without 4-cycles, *J. of Combin. Theory, Ser. B*, **76** (1999) 117-126.

[11] P. C. B. Lam, W. C. Shiu and B. G. Xu, On structure of some plane graphs with applications to choosability, *J. of Combin. Theory, Ser. B*, **82** (2001) 285-296.

[12] P. C. B. Lam, W. C. Shiu and Z. M. Song, The 3-choosability of plane graphs of girth 4, *Discrete Math.*, **294** (2005) 297-301.

[13] Y-J. Chen, The list-$L(2,1)$-labeling problem of graphs, *M. Phil. Thesis*, (2004), Dept. of Applied Math., National Dong Hwa University, Taiwan.

[14] M. Voigt and B. Wirth, On 3-colorable non-4-choosable planar graphs, *J. Graph Theory*, **24** no. 3 (1997) 233-235.

[15] M. Voigt, A not 3-choosable planar graph without 3-cycles, *Discrete Math.*, **146** (1995) 325-328.

Haiying Zhou

Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China;

Division of Science and Technology, BNU-HKBU United International College, Zhuhai, China.

`Email: 09467262@life.hkbu.edu.hk`

Wai Chee Shiu

Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China.

`Email: wcshiu@hkbu.edu.hk`

Peter Che Bor Lam

Division of Science and Technology, BNU-HKBU United International College, Zhuhai, China.

`Email: cblam@uic.edu.hk`