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A CHARACTERIZATION OF SOLUBLE GROUPS IN WHICH NORMALITY IS A TRANSITIVE RELATION

GIOVANNI VINCENZI

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ABSTRACT. A subgroup X of a group G is said to be an \mathcal{H} -subgroup if $N_G(X) \cap X^g \leq X$ for each element g belonging to G . In [M. Bianchi and e.a., On finite soluble groups in which normality is a transitive relation, *J. Group Theory*, **3** (2000) 147–156.] the authors showed that finite groups in which every subgroup has the \mathcal{H} -property are exactly soluble groups in which normality is a transitive relation. Here we extend this characterization to groups without simple sections.

1. Introduction

A group G is called a \mathcal{T} -group if normality is a transitive relation or, equivalently, if all subnormal subgroups of G are normal in G . The study of finite groups of these types can be traced back to the 1940s, and in their seminal papers G. Zacher [23] (1952) and W. Gaschütz (1957) [4] described the structure of finite soluble \mathcal{T} -groups. It turns out that a finite group G is a soluble \mathcal{T} -group if and only if G is a $\overline{\mathcal{T}}$ -group (that is, a group whose subgroup is a \mathcal{T} -group). The structure of infinite soluble \mathcal{T} -group is more complicated, and example of soluble \mathcal{T} -group that are not $\overline{\mathcal{T}}$ -group can be constructed (see for example [15] or [13]). We refer to [15] for results concerning the structure of soluble \mathcal{T} -groups.

Later, many authors studied finite groups in which normality is transitive, and more characterizations were obtained (see for instance [1, 2, 8, 9, 14]) in terms of properties of subgroups. More recently, it has been shown that, most of these characterizations also hold for large classes of infinite groups (see [18, 21, 19, 10, 3]).

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Following [2] a subgroup X of a group G is said to be \mathcal{H} -subgroup or that it has the \mathcal{H} -property in G , if $N_G(X) \cap X^g \leq X$ for all elements g of G . It is easy to see that every \mathcal{H} -subgroup of a group G that is also subnormal in G , is normal in G (see [2, Lemma 5]). It follows that if every subgroup of a group G has the \mathcal{H} -property, then G is a $\overline{\mathcal{F}}$ -group.

In 2000, M. Bianchi, A. Gillio, M. Herzog and L. Verardi, (see [2, Theorem 10]) showed also the converse for finite groups. Thus: A finite group G is a (soluble) $\overline{\mathcal{F}}$ -group if and only if every subgroup of G is an \mathcal{H} -subgroup. At the moment extensions to the infinite case of this result are unknown.

In this paper we show that this characterization can be extended to groups without simple sections. The statement of the main result is the following:

Theorem 3.2 *Let G be a group without infinite simple sections. Then G is a $\overline{\mathcal{F}}$ -group if and only if every subgroup of G satisfies the \mathcal{H} -property in G . When these conditions hold, G is metabelian.*

Moreover, we will see in Theorem 3.1, that for periodic groups, the above characterization also holds for larger classes of infinite groups.

Most of our notation is standard and can for instance be found in [16].

2. The \mathcal{H} -norm

For a deeper understanding of the \mathcal{H} -property, we may introduce the notion of \mathcal{H} -norm. We will say that an element g of a group G \mathcal{H} -normalizes a subgroup X , if $N_G(X) \cap X^g \leq X$. We define \mathcal{H} -norm of G the set $H(G)$ consisting of all elements of G that \mathcal{H} -normalize every subgroup of G . Actually it is unknown if the \mathcal{H} -norm of a group is a subgroup or not, but it will be an useful tool for our purposes. In this section we show that for finite groups with nilpotent commutator subgroup the \mathcal{H} -norm is a subgroup (Corollary 2.6), and we prove a crucial result (Lemma 2.5) for Theorem 3.1 in section 3.

Lemma 2.1. *Let G be a group, and let N be a normal π -subgroup of G and X be a π' -subgroup of G (where π is a set of primes). If g is an element of G that \mathcal{H} -normalizes XN then g even \mathcal{H} -normalizes X .*

Proof. Let $y \in N_G(X) \cap X^g$, then by hypothesis $y \in N_G(XN) \cap (XN)^g \leq XN$. It follows that y is a π' -element of XN , so that $X\langle y \rangle$ is a π' -group, as y lies in $N_G(X)$. On the other hand X is a Sylow π' -subgroup of XN , so that $y \in X$. \square

Recall that a subgroup H of a group G is said to be *ascendant* if there is an ascending series between H and G . Following the notation introduced in [6], we will denote by $\tau(G)$ the intersection of all the normalisers of ascendant subgroups of G . Clearly every subnormal subgroup is also ascendant, so that for any group G the subgroup $\tau(G)$ is contained in the intersection of all the normalisers of subnormal subgroups of G , that is the *Wielandt* subgroup of G , $\omega(G)$. Clearly, if G is a polycyclic-by-finite group, ascendant and subnormal subgroups of G coincide, and hence $\tau(G) = \omega(G)$. It was proved by Romano and Vincenzi that $\tau(G) = \omega(G)$ also for others large classes of groups ([18, Lemma 3.5]). On

the other hand examples show that $\tau(G)$ can be properly contained in $\omega(G)$, for instance the infinite locally dihedral 2-group.

Lemma 2.2. *Let G be a periodic soluble group, and let E be a finite p -subgroup of G (p prime). If N is a normal subgroup of G such that $N \leq \tau(EN)$, then E is an \mathcal{H} -subgroup of EN .*

Proof. Put $K = EN$. Then N is contained in $\tau(K)$ and hence $K = E\tau(K)$. As the Fitting subgroup V of $\tau(K)$ is obviously nilpotent, we have $V = F \cap \tau(K)$, where F is the Fitting subgroup of K . Then F is nilpotent-by-finite, and so even nilpotent. Application of [5, Lemma 3.2] yields that $[\tau(K), K]$ is contained in V , so that K/V is a nilpotent group. In particular, the subgroup EV is subnormal in K . Consider now the normal subgroup $L = O_{p'}(V)$ of K . Clearly EV/L is a nilpotent-by-finite p -group, so that it is hypercentral and hence EL is ascendant in EV , and so also in K . Thus EL is a normal subgroup of $K = E\tau(K)$, and it follows from Lemma 2.1 that E is an \mathcal{H} -subgroup of K . \square

In order to study the \mathcal{H} -norm of a group G we recall the definitions of two useful descending normal series of G , related to the subgroups $\omega(G)$ and $\tau(G)$.

Let G be a group. The *lower Wielandt series* of G is the descending normal series whose terms $\omega_\alpha(G)$ are defined inductively by positions

$$\omega_0(G) = G, \omega_{\alpha+1} = \bigcap_{K \in \Omega_\alpha(G)} \omega(K),$$

where $\Omega_\alpha(G)$ is the set of all subgroups of G containing $\omega_\alpha(G)$, and

$$\omega_\lambda(G) = \bigcap_{\beta < \lambda} \omega_\beta(G)$$

if λ is a limit ordinal. The last term of the lower Wielandt series of G will be denoted by $\bar{\omega}(G)$. The *lower τ -series* of G is the descending normal series obtained by replacing in the above definition the Wielandt subgroup $\omega(X)$ by the subgroup $\tau(X)$ for each group X . The last term of the lower τ -series of G will be denoted by $\bar{\tau}(G)$. Clearly $\omega_1(G) = \omega(G)$ and $\tau_1(G) = \tau(G)$. The last term of the lower τ -series of G is a characteristic subgroup of G .

Corollary 2.3. *Let G be a periodic soluble group, and let E be a finite p -subgroup of G (p prime). Then E is \mathcal{H} -normalized by every element of $\bar{\tau}(G)$.*

Proof. Put $N = \bar{\tau}(G)$. Then $N \leq \tau(EN)$ by [5, Lemma 4.3]. An application of lemma 2.2 shows that E is an \mathcal{H} -subgroup of EN . \square

Lemma 2.4. *Let G be a periodic group, and let X be a subnormal subgroup of G . Then $H(G)$ is contained in $N_G(X)$.*

Proof. By hypothesis the descending normal series of X in G stops after finitely many steps, say n :

$$X = X^{G,n} \triangleleft X^{G,n-1} \dots \triangleleft X^G \triangleleft G.$$

Consider an element $g \in H(G)$. We prove that g normalizes X . Clearly we may assume $n \geq 1$. Proceeding by induction we may suppose that $X^{G,n-1}$ is normalized by each element of $H(G)$, in

particular $X^g \leq X^{G, n-1}$. It follows that $X^g = X^{G, n-1} \cap X^g \leq N_G(X) \cap X^g \leq X$. By hypothesis G is periodic, therefore:

$$X \geq X^g \geq X^{g^2} \geq \dots \geq X^{g^m} = X, \text{ for a suitable positive integer } m.$$

□

Recall that a subgroup X of a group G is called *pronormal* if for every element g of G the subgroups X and X^g are conjugate in $\langle X, X^g \rangle$. In the proof of the following lemma, we will use some properties on pronormal subgroups, connected with the subgroup $\bar{\tau}(G)$. In fact, it is relevant in our context that every cyclic subgroup C of a periodic metabelian group G is pronormal in $C\bar{\tau}(G)$ (see [5, Theorem 4.6]).

Lemma 2.5. *Let G be a periodic group with abelian commutator subgroup. Then $\bar{\tau}(G) \subseteq H(G) \subseteq \bar{\omega}(G)$.*

Proof. First we will show that every finite subgroup X of G is an \mathcal{H} -subgroup of $X\bar{\tau}(G)$. Let $g \in \bar{\tau}(G)$, and prove that $N_G(X) \cap X^g \leq X$. Suppose that X is abelian. Let y be a p -element of $N_G(X) \cap X^g$, and let P be the p -component of X (i. e. the p -Sylow subgroup of X). For every element $x \in P$, the cyclic subgroup $\langle x \rangle$ is subnormal in $N_{X\bar{\tau}(G)}(X)$ and pronormal in $X\bar{\tau}(G)$ by [5, Theorem 4.6], so that $\langle x \rangle$ is normal in $N_{X\bar{\tau}(G)}(X)$. The choice of x shows that the normalizer $N_{X\bar{\tau}(G)}(X)$ is contained in $N_{X\bar{\tau}(G)}(P)$. Clearly $N_{X\bar{\tau}(G)}(X) \cap X^g = N_G(X) \cap X^g$ and P^g is the p -component of X^g . It follows that $y \in N_{X\bar{\tau}(G)}(P) \cap P^g$ which is contained in P by Corollary 2.3. In this way we have proved that every primary element of $N_G(X) \cap X^g$ lies in X , and hence $N_G(X) \cap X^g \leq X$.

Suppose now that X is not abelian. Note that X' is subnormal in G and it is an \mathcal{H} -subgroup of $X\bar{\tau}(G)$ by the first part of the proof, thus X' is normal in $X\bar{\tau}(G)$. Moreover, the factor group X/X' is \mathcal{H} -normalized by every element of $\bar{\tau}(X\bar{\tau}(G)/X')$, again by the first part of the proof. On the other hand by [5, Lemmas 4.4 and 4.3] we have:

$$\bar{\tau}(X\bar{\tau}(G)/X') \geq \bar{\tau}(X\bar{\tau}(G))X'/X' \geq \bar{\tau}(G)X'/X'.$$

Therefore X/X' is an \mathcal{H} -subgroup of $X\bar{\tau}(G)/X'$ and X is an \mathcal{H} -subgroup of $X\bar{\tau}(G)$.

Let now X be a whichever (infinite) subgroup of G and $g \in \bar{\tau}(G)$. For every element $y \in N_G(X) \cap X^g$ put $E = \langle g, y \rangle$ and $L = (X \cap E)\bar{\tau}(G)$. Clearly y belongs both to $N_G(X \cap E)$ and $X^g \cap E$, so that

$$y \in N_{(X \cap E)^g}(X \cap E) \cap (X \cap E)^g \leq N_L(X \cap E) \cap (X \cap E)^g.$$

By hypothesis E is finite, and the first part of the proof yields that $X \cap E$ is an \mathcal{H} -subgroup of L . Then $N_L(X \cap E) \cap (X \cap E)^g \leq X \cap E$, and y belongs to X . We have proved that X is an \mathcal{H} -subgroup of $X\bar{\tau}(G)$.

In order to show that $H(G)$ is contained in $\bar{\omega}(G)$, we argue by contradiction. Let α be the minimum ordinal number, such that $H(G)$ is not contained in $\omega_\alpha(G)$. By Lemma 2.4 α is greater than 1, and clearly it is not limit. Let now K be a subgroup of G containing $\omega_{\alpha-1}(G)$. By the choice of α , the subgroup K also contains $H(G)$, so that $H(G) \leq H(K)$, and every subnormal subgroup of K is

normalized by each element of $H(G)$, again by Lemma 2.4. In particular $H(G)$ is contained in $\omega(K)$. This means by definition, that $H(G)$ is contained in $\omega_\alpha(G)$. A contradiction. \square

In [5, Corollary 4.13], the authors pointed out that the pronorm $P(G)$ of a group G , that is the set of the elements that pronormalize every subgroup of G , coincides with $\omega(G)$, whenever G is polycyclic with nilpotent commutator subgroup. Therefore if G is a finite metabelian group the \mathcal{H} -norm of the group G coincides with the pronorm of G , by Lemma 2.5 and [5, Theorem 4.11]. The following result can be read as an extension of [2, Theorem 10].

Corollary 2.6. *Let G be a finite metabelian group. Then $H(G) = \omega(G) = P(G)$.*

3. Main results

The structure of soluble \mathcal{T} -groups has been investigated by Robinson [15] in the general case. In particular, it turns out that soluble groups with the property \mathcal{T} are metabelian and that any finitely generated soluble \mathcal{T} -group either is finite or abelian. Clearly, every subnormal subgroup of a \mathcal{T} -group is likewise a \mathcal{T} -group, but the class of \mathcal{T} -groups is not subgroup closed. As we have recalled, all finite $\overline{\mathcal{T}}$ -groups are soluble, and hence they are metabelian. An easy argument (see [6, Lemma 3.1]) shows that this latter result can be extended to the class of periodic locally graded groups. Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index. This class of groups is considered to avoid pathologies such as Tarsky groups. It is easy to see that the class of groups without simple sections is contained in the class of locally graded groups.

For periodic locally graded groups we have the following characterization in terms of \mathcal{H} -subgroups.

Theorem 3.1. *Let G be a periodic locally graded group. The following are equivalent:*

- i) G is a $\overline{\mathcal{T}}$ -group.
- ii) Every subgroup is an \mathcal{H} -subgroup of G .
- iii) G is locally finite and every cyclic subgroup is an \mathcal{H} -subgroup of G .

When these conditions hold, G is metabelian.

Proof. $i) \Rightarrow ii)$ Clearly G is metabelian, and hence $G = \tau(G) = \bar{\tau}(G)$ by [5, Theorem 3.6]. It follows from lemma 2.5 that $G=H(G)$, and every subgroup of G satisfies the \mathcal{H} -property in G .

$ii) \Rightarrow iii)$ If every subgroup has the \mathcal{H} -property in G , then G is a $\overline{\mathcal{T}}$ -group. It follows that G is metabelian (see Lemma 3.1 [6]) and hence it is locally finite.

$iii) \Rightarrow i)$ Let X be a cyclic p -subgroup of G . Then X is pronormal in every finite subgroup of G containing X , by [1, Lemma 2] (see also [7, Lemma 2.10]). By hypothesis G is locally finite, so that X is pronormal in G . Let now K be a cyclic subgroup of G . The primary component of K are pronormal in G , thus K is pronormal in G , by a result of J.S. Rose [20] (see also [5, Lemma 2.2]). It follows by [22, Lemma 2.2] that G is a $\overline{\mathcal{T}}$ -group. \square

The third condition of the above theorem appears as the weaker. Indeed if we remove the preliminary hypothesis of periodicity, it is easy to show that the results doesn't hold. The consideration of the infinite Dihedral group, D_∞ shows that may exist non periodic soluble groups, whose cyclic subgroups have the \mathcal{H} -property, but that are not $\overline{\mathcal{T}}$ -groups.

For those concerning non periodic groups, we remark that it is an open question (see [11, Question 14.36]), whether a non-periodic locally graded $\overline{\mathcal{T}}$ -group is soluble or, equivalently, abelian in virtue of [15, Theorem 6.1.1].

We will prove our main theorem for the class of groups without infinite simple sections, that is a large class of generalized soluble groups.

Theorem 3.2. *Let G be a group without infinite simple sections. Then G is a $\overline{\mathcal{T}}$ -group if and only if every subgroup of G satisfies the \mathcal{H} -property in G .*

Proof. Every group whose subgroups satisfy the \mathcal{H} -property is a $\overline{\mathcal{T}}$ -group. On the other hand if G is a non-periodic $\overline{\mathcal{T}}$ -group, then it is abelian ([6, Lemma 3.3]). By hypothesis G has no infinite simple sections, thus G is locally graded. Therefore if G is periodic the assert follows from Theorem 3.1. \square

For finite groups G , the \mathcal{H} -property is stronger (see [1, Remark 1]) than H being *weakly normal*, i.e., satisfying: $H^g \leq N_G(H)$ implies $g \in N_G(H)$.

In view of the characterization of $\overline{\mathcal{T}}$ -groups without infinite simple sections, in terms of weakly normal subgroups (see [21, Corollary 4] and [19, Theorem 2.8]) we have:

Corollary 3.3. *Let G be a group without infinite simple sections. Then every subgroup of G satisfies the \mathcal{H} -property in G if and only if every subgroup of G is weakly normal.*

Note that non periodic locally graded groups whose subgroup are pronormal (resp. weakly normal) are abelian ([17, Theorem E] and [19, Theorem 2.8]). Thus the following questions arise:

Question 1: *If G is a locally graded non periodic group whose subgroup satisfy the \mathcal{H} -property in G , can we say that G is abelian?*

Question 2: *Let G be a group, and let X be an \mathcal{H} -subgroup of G . Can we say that X is weakly normal in G ?*

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Giovanni Vincenzi

Department of Mathematics, University of Salerno, via Giovanni Paolo II, 132–84084, Fisciano, Salerno, Italy

Email: vincenzi@unisa.it