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DEGREE RESISTANCE DISTANCE OF UNICYCLIC GRAPHS

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ABSTRACT. Let G be a connected graph with vertex set $V(G)$. The degree resistance distance of G is defined as $D_R(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]R(u,v)$, where $d(u)$ is the degree of vertex u , and $R(u,v)$ denotes the resistance distance between u and v . In this paper, we characterize n -vertex unicyclic graphs having minimum and second minimum degree resistance distance.

1. Introduction

Graph invariants, based on the distances between the vertices of a graph [2], are widely used in theoretical chemistry to establish relations between the structure and the properties of molecules [7, 8].

Let $G = (V(G), E(G))$ be a simple undirected graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges. In this paper all graphs considered are assumed to be connected. The ordinary distance $d(v,u) = d(u,v|G)$ between the vertices v and u of the graph G is the length of a shortest path between v and u [2].

The *Wiener index* is the sum of distances between all unordered pairs of vertices

$$(1.1) \quad W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) .$$

This graph invariant is the oldest and one of the most popular molecular structure descriptors [7, 8], well correlated with many physical and chemical properties of a variety of classes of chemical compounds. For details on its mathematical properties, see the survey [4].

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A modified version of the Wiener index is the *degree distance* defined as [5]

$$(1.2) \quad D(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]d(u, v)$$

where $d(u) = d(u|G)$ is the degree (number of first neighbors) of the vertex u of the graph G . The same quantity was examined in the paper [6] under the name *Schultz index*.

If G is a tree on n vertices, then the Wiener index and the degree distance are related as $D(G) = 4W(G) - n(n - 1)$ (for details see [6]).

The degree distance of graphs was much studied in the literature. In [1, 13, 14] general graphs with minimum D were examined. Tomescu [12] determined the unicyclic and bicyclic graphs with minimum D -value. Yuan and An [18] determined the unicyclic graphs with maximum D -value. Some further mathematical results on degree distance can be found in [3, 9, 10].

In 1993 Klein and Randić [11] introduced a new distance function named *resistance distance*, based on the theory of electrical networks. They viewed G as an electrical network N by replacing each edge of G with a unit resistor. The resistance distance between the vertices u and v of the graph G , denoted by $R(u, v) = R(u, v|G)$, is then defined to be the effective resistance between the nodes u and v in N . This new kind of distance between vertices of a graph was eventually studied in detail (see [15, 16, 17] and the references cited therein).

For the following consideration it is important that $R(u, v) = R(v, u)$, $R(u, u) = 0$ and that [11] $d(u, v) \geq R(u, v)$ with equality if and only if there is a unique path linking the vertices u and v .

If in the expression for the Wiener index, Eq. (1.1), the ordinary distance is replaced by resistance distance, then we arrive at the *Kirchhoff index*

$$Kf(G) = \sum_{\{u,v\} \subseteq V(G)} R(u, v)$$

which also has been much studied in the mathematical literature.

If in the expression for the degree distance, Eq. (1.2), the ordinary distance is replaced by resistance distance, then we arrive at the invariant

$$D_R(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]R(u, v)$$

which we name *degree resistance distance* of the graph G .

If G is a tree, then $R(u, v) = d(u, v)$ for any two vertices u and v . Consequently, the Kirchhoff and Wiener indices of trees coincide, as well as their degree distances and degree resistance distances.

To our best knowledge, the degree resistance distance is being considered here for the first time. In this paper we study the degree resistance distance of the simplest connected non-tree graphs, namely of unicyclic graphs.

The paper is organized as follows. In Section 2 we state some preparatory results, whereas in Section 3 we determine the unicyclic graphs with minimum and second–minimum D_R -value.

2. Preliminary Results

Lemma 2.1. [11] *Let x be a cut vertex of a graph G , and let a and b be vertices belonging to different components which arise upon deletion of x . Then $R(a, b) = R(a, x) + R(x, b)$.*

Lemma 2.1 has the following important corollary. Let H be a connected graph and let u be its vertex. Denote

$$\sum_{v \in V(H)} R(v, u|H)$$

by $R(u) = R(u|H)$, and

$$\sum_{v \in V(H)} d(v) R(v, u|H)$$

by $S'(u) = S'(u|H)$.

Theorem 2.2. *Let G_1 and G_2 be connected graphs with disjoint vertex sets, with n_1 and n_2 vertices, and with m_1 and m_2 edges, respectively. Let $u_1 \in V(G_1)$, $u_2 \in V(G_2)$. Construct the graph G by identifying the vertices u_1 and u_2 , and denote the so obtained vertex by u . Then*

$$(2.1) \quad \begin{aligned} D_R(G) &= D_R(G_1) + D_R(G_2) + 2m_2 R(u_1|G_1) + 2m_1 R(u_2|G_2) \\ &+ (n_2 - 1)S'(u_1|G_1) + (n_1 - 1)S'(u_2|G_2) . \end{aligned}$$

Proof. Denote the abbreviations

$$V_1 = V(G_1) \setminus \{u_1\} \quad \text{and} \quad V_2 = V(G_2) \setminus \{u_2\} .$$

In view of the structure of the graph G , from the definition (1.2) of the degree resistance distance, we have

$$D_R(G) = \left[\sum_{\{x,y\} \subseteq V_1} + \sum_{\{x,y\} \subseteq V_2} + \sum_{\substack{x \in V_1 \\ y = u}} + \sum_{\substack{x \in V_2 \\ y = u}} + \sum_{\substack{x \in V_1 \\ y \in V_2}} \right] [d(x|G) + d(y|G)] R(x, y|G) .$$

Now,

$$(2.2) \quad \begin{aligned} \sum_{\{x,y\} \subseteq V_1} [d(x|G) + d(y|G)] R(x, y|G) &= D_R(G_1) - \sum_{x \in V_1} [d(x|G_1) + d(u_1|G_1)] R(x, u_1|G_1) \\ &= D_R(G_1) - \sum_{x \in V_1} d(x|G_1) R(x, u_1|G_1) - d(u_1|G_1) \sum_{x \in V_1} R(x, u_1|G_1) \\ &= D_R(G_1) - S'(u_1|G_1) - d(u_1) R(u_1|G_1) \end{aligned}$$

and analogously

$$(2.3) \quad \sum_{\{x,y\} \subseteq V_2} [d(x|G) + d(y|G)] R(x, y|G) = D_R(G_2) - S'(u_2|G_2) - d(u_2) R(u_2|G_2) .$$

Further,

$$\begin{aligned}
 \sum_{\substack{x \in V_1 \\ y = \bar{u}}} [d(x|G) + d(y|G)] R(x, y|G) &= \sum_{x \in V_1} [d(x|G_1) + d(u|G)] R(x, u_1|G_1) \\
 (2.4) \qquad \qquad \qquad &= S'(u_1|G_1) + d(u|G) R(u_1|G_1)
 \end{aligned}$$

and analogously

$$(2.5) \qquad \sum_{\substack{x \in V_2 \\ y = \bar{u}}} [d(x|G) + d(y|G)] R(x, y|G) = S'(u_2|G_2) + d(u|G) R(u_2|G_2) .$$

Finally,

$$\begin{aligned}
 \sum_{\substack{x \in V_1 \\ y \in V_2}} [d(x|G) + d(y|G)] R(x, y|G) &= \sum_{\substack{x \in V_1 \\ y \in V_2}} [d(x|G) + d(y|G)] [R(x, u_1|G_1) + R(y, u_2|G_2)] \\
 &= \sum_{\substack{x \in V_1 \\ y \in V_2}} d(x|G_1) R(x, u_1|G_1) + \sum_{\substack{x \in V_1 \\ y \in V_2}} d(x|G_1) R(y, u_2|G_2) \\
 &+ \sum_{\substack{x \in V_1 \\ y \in V_2}} d(y|G_2) R(x, u_1|G_1) + \sum_{\substack{x \in V_1 \\ y \in V_2}} d(y|G_2) R(y, u_2|G_2) \\
 &= (n_2 - 1) S'(u_1|G_1) + [2m_1 - d(u_1|G_1)] R(u_2|G_2) \\
 (2.6) \quad &+ (n_1 - 1) S'(u_2|G_2) + [2m_2 - d(u_2|G_2)] R(u_1|G_1)
 \end{aligned}$$

because for $i = 1, 2$, the vertex set V_i has $n_i - 1$ elements, and because $\sum_{x \in V_i} d(x) = 2m_i - d(u_i|G_i)$.

Adding Eqs. (2.2)–(2.6) and taking into account that $d(u|G) = d(u_1|G_1) + d(u_2|G_2)$, we obtain Eq. (2.1). \square

Let v be a vertex of degree $p + 1$ in a graph G , such that vv_1, vv_2, \dots, vv_p are pendent edges incident with v , and u is the neighbor of v distinct from v_1, v_2, \dots, v_p . We form a graph $G' = \sigma(G, v)$ by removing the edges vv_1, vv_2, \dots, vv_p and adding new edges uv_1, uv_2, \dots, uv_p . We say that G' is a σ -transform of G (see Fig. 1).

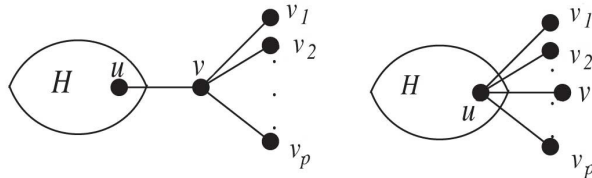


FIGURE 1. The σ -transformation at v .

Theorem 2.3. Let $G' = \sigma(G, v)$ be a σ -transform of the graph G , $d(u) \geq 1$ (see Fig. 1). Then

$$D_R(G) \geq D_R(G') .$$

Equality holds if and only if G is a star with v as its center.

Proof. Let $T = \{v, v_1, v_2, \dots, v_p\}$ and let H denote the subgraph of G induced by the vertex set $V(G) \setminus T$. From the definition of $D_R(G)$, we have

$$\begin{aligned} D_R(G) &= \left[\sum_{x,y \in V(H-u)} + \sum_{x,y \in T \setminus \{v\}} + \sum_{\substack{x \in V(H-u) \\ y \in T \setminus \{v\}}} \right] [d(x) + d(y)]R(x, y) \\ &+ \left[\sum_{x \in V(H-u)} [d(x) + d(u)]R(x, u) + \sum_{x \in T \setminus \{v\}} [d(x) + d(u)]R(x, u) \right] \\ &+ \left[\sum_{x \in V(H-u)} (d(x) + d(v))R(x, v) + \sum_{x \in T \setminus \{v\}} [d(x) + d(v)]R(x, v) \right] \\ &+ [d(u) + d(v)]R(u, v) . \end{aligned}$$

After the σ -transformation, the degree of the vertex u increases by p , while the degree of the vertex v decreases by p . During the transformation, for $x, y \in V(H-u)$ and $x, y \in T \setminus \{v\}$, $\sum_{x,y} [d(x) + d(y)]R(x, y)$ does not change.

In G ,

$$B_1 := \sum_{\substack{x \in V(H-u) \\ y \in T \setminus \{v\}}} [d(x) + d(y)]R(x, y) = p \sum_{x \in V(H-u)} [d(x) + 1][R(x, u) + 2]$$

while in G' ,

$$B_2 := \sum_{\substack{x \in V(H-u) \\ y \in T \setminus \{v\}}} [d(x) + d(y)][R(x, u) + 1] = p \sum_{x \in V(H-u)} [d(x) + 1][R(x, u) + 1] .$$

For the vertex u , in G ,

$$\begin{aligned} B_3 &:= \sum_{x \in V(H-u)} [d(x) + d(u)]R(x, u) + \sum_{x \in T \setminus \{v\}} [d(x) + d(u)]R(x, u) \\ &= \sum_{x \in V(H-u)} [d(x) + d(u)]R(x, u) + 2p[1 + d(u)] \end{aligned}$$

whereas for u in G' ,

$$\begin{aligned} B_4 &:= \sum_{x \in V(H-u)} [d(u) + p + d(x)]R(x, u) + \sum_{x \in T \setminus \{v\}} [d(u) + p + d(x)]R(x, u) \\ &= \sum_{x \in V(H-u)} [d(u) + p + d(x)]R(x, u) + p[d(u) + p + 1] . \end{aligned}$$

For the vertex v in G ,

$$\begin{aligned}
B_5 : &= \sum_{x \in V(H-u)} [d(x) + d(v)]R(x, v) + \sum_{x \in T \setminus \{v\}} [d(x) + d(v)]R(x, v) \\
&+ [d(u) + d(v)]R(u, v) \\
&= \sum_{x \in V(H-u)} [d(x) + p + 1][R(x, u) + 1] + p(p + 2) + [d(u) + p + 1]
\end{aligned}$$

whereas for v in G' ,

$$B_6 : = \sum_{x \in V(H-u)} (d(x) + 1)[R(x, u) + 1] + 5p + d(u) + 1 .$$

From the above relations it follows

$$\begin{aligned}
&B_1 - B_2 + B_3 - B_4 + B_5 - B_6 \\
&= p \sum_{x \in V(H-u)} [d(x) + 1][R(x, u) + 2] - \left[p \sum_{x \in V(H-u)} (d(x) + 1)(R(x, u) + 1) \right] \\
&+ \left[\sum_{x \in V(H-u)} [d(x) + d(u)]R(x, u) + 2p[1 + d(u)] \right] \\
&- \left[\sum_{x \in V(H-u)} [d(u) + p + d(x)]R(x, u) + p[d(u) + p + 1] \right] \\
&+ \left[\sum_{x \in V(H-u)} [d(x) + p + 1][R(x, u) + 1] + p(p + 2) + [d(u) + p + 1] \right] \\
&- \left[\sum_{x \in V(H-u)} (d(x) + 1)[R(x, u) + 1] + 5p + d(u) + 1 \right] \\
&= p \sum_{x \in V(H-u)} [d(x) + 2] + pd(u) - p \geq 0 .
\end{aligned}$$

The equality holds if and only if H consists of only one vertex u . This completes the proof. \square

Theorem 2.4. *Let G be a unicyclic graph. Let u be one vertex on G such that there are s pendent vertices u_1, u_2, \dots, u_s attached at u . Let v be a vertex on G such that there are t pendent vertices v_1, v_2, \dots, v_t attached at v . Assume that*

$$G_1 = G - \{vv_1, vv_2, \dots, vv_t\} + \{uv_1, uv_2, \dots, uv_t\}$$

and

$$G_2 = G - \{uu_1, uu_2, \dots, uu_s\} + \{vu_1, vu_2, \dots, vu_s\} .$$

Then either $D_R(G) > D_R(G_1)$ or $D_R(G) > D_R(G_2)$.

Proof. Let $A = \{u_1, u_2, \dots, u_s\}$, $B = \{v_1, v_2, \dots, v_t\}$ and let H be the subgraph induced by $V(G) \setminus \{A, B\}$. Further, let $R(u, v) = \ell$.

In the transformation $G \rightarrow G_1$ for any pair of vertices x, y satisfying either $x, y \in V(H-u-v)$, or $x, y \in A$, or $x, y \in B$, or $x \in A, y \in V(H-u-v)$, the term $\sum_{x,y} [d(x)+d(y)]R(x,y)$ does not change. Then

$$\begin{aligned}
D_R(G) &= \left[\sum_{x,y \in V(H-u-v)} + \sum_{x,y \in A} + \sum_{x,y \in B} + \sum_{\substack{x \in A \\ y \in V(H-u-v)}} \right] [d(x) + d(y)]R(x, y) \\
&+ \sum_{\substack{x \in A \\ y \in B}} [d(x) + d(y)]R(x, y) + \sum_{\substack{x \in V(H-u-v) \\ y \in B}} [d(x) + d(y)]R(x, y) \\
&+ \left[\sum_{x \in V(H-u-v)} [d(x) + d(u)]R(x, u) + \sum_{x \in A} [d(x) + d(u)]R(x, u) \right. \\
&+ \left. \sum_{x \in B} [d(x) + d(u)]R(x, u) \right] + \left[\sum_{x \in V(H-u-v)} [d(x) + d(v)]R(x, v) \right. \\
&+ \left. \sum_{x \in A} [d(x) + d(v)]R(x, v) + \sum_{x \in B} [d(x) + d(v)]R(x, v) \right] + [d(u) + d(v)]R(u, v) \\
&= \left[\sum_{x,y \in V(H-u-v)} + \sum_{x,y \in A} + \sum_{x,y \in B} + \sum_{\substack{x \in A \\ y \in V(H-u-v)}} \right] [d(x) + d(y)]R(x, y) \\
&+ 2(\ell + 2)st + t \sum_{x \in V(H-u-v)} [d(x) + 1][R(x, v) + 1] \\
&+ \sum_{x \in V(H-u-v)} [d(x) + s + 2]R(x, u) + s(s + 3) + (s + 3)(\ell + 1)t \\
&+ \sum_{x \in V(H-u-v)} [d(x) + t + 2]R(x, v) + t(t + 3) + (\ell + 1)s(t + 3) + (s + 2 + t + 2)\ell
\end{aligned}$$

and analogously,

$$D_R(G_1) = \left[\sum_{x,y \in V(H-u-v)} + \sum_{x,y \in A} + \sum_{x,y \in B} + \sum_{\substack{x \in A \\ y \in V(H-u-v)}} \right] [d(x) + d(y)]R(x, y)$$

$$\begin{aligned}
& + 4st + t \sum_{x \in V(H-u-v)} [d(x) + 1][R(x, u) + 1] \\
& + \left[\sum_{x \in V(H-u-v)} [d(x) + s + t + 2]R(x, u) + s(s + t + 3) + t(s + t + 3) \right] \\
& + \left[\sum_{x \in V(H-u-v)} [d(x) + 2]R(x, v) + 3(\ell + 1)t + 3(\ell + 1)s \right] + (s + t + 2 + 2)\ell .
\end{aligned}$$

So we get

$$D_R(G) - D_R(G_1) = t \left[4\ell s + \sum_{x \in V(H-u-v)} [d(x) + 2][R(x, v) - R(x, u)] \right] .$$

By a similar reasoning one arrives at

$$D_R(G) - D_R(G_2) = s \left[4\ell t + \sum_{x \in H-u-v} (d(x) + 2)[R(x, u) - R(x, v)] \right] .$$

Hence, if $D_R(G) - D_R(G_i) > 0$ for $i = 1, 2$, then the result follows. If at least one difference is negative, say $D_R(G) - D_R(G_1) < 0$, then $\sum_{x \in H-u-v} [d(x) + 2][R(x, u) - R(x, v)] > 4\ell s$ and therefore $D_R(G) - D_R(G_2) > s(4\ell t + 4\ell s) > 0$. This completes the proof. \square

3. The Minimum and Second Minimum Degree Resistance Distance of Unicyclic Graphs

In this section, for convenience, we represent a unicyclic graph G with the unique cycle $C_k = v_1v_2 \dots v_kv_1$ as $G = U(C_k; T_1, T_2, \dots, T_k)$, where T_i is the component of $G - E(C_k)$ containing v_i , $1 \leq i \leq k$. Obviously, T_i is a tree rooted at v_i . We say that T_i is trivial if it consists of an isolated vertex. We denote by $H_{n,k}$ the graph obtained from C_k by adding $n - k$ pendent vertices to a vertex of C_k .

Theorem 3.1. *Let G be a unicyclic graph of order n with girth k . Then $D_R(G) \geq D_R(H_{n,k})$, with equality if and only if $G \cong H_{n,k}$.*

Proof. Let $G = U(C_k; T_1, T_2, \dots, T_k)$ as described above. By Theorem 2.3, T_i ($1 \leq i \leq k$) is a star with center v_i . From Theorem 2.4, there exists only one non-trivial star attached at C_k , and this implies the result. \square

Let $C_k = v_1v_2 \dots v_kv_1$ be a cycle on k vertices. Then, for $1 \leq i < j \leq k$,

$$R(v_i, v_j) = \frac{(j-i)(k-j+i)}{k} \geq \frac{k-1}{k}$$

and

$$R(v_1|C_k) = \sum_{x \in V(C_k - v_1)} R(x, v_1) = \frac{k^2 - 1}{6} \quad , \quad Kf(C_k) = \frac{k^3 - k}{12} .$$

For the graph $H_{n,k}$, in view of Theorem 2.2, let $G_1 = C_k$, $G_2 = H_{n,k} - C_k + u$, where u is the only vertex on C_k with degree greater than 2. It is easy to see that

$$D_R(G_1) = 4Kf(C_k) = \frac{k^3 - k}{3} .$$

Note that, when $r = n - k + 1$, then $W(K_{1,r-1}) = (r - 1)^2$. It follows that $D_R(K_{1,r-1}) = D'(K_{1,r-1}) = 4W(K_{1,r-1}) - r(r - 1) = (r - 1)(3r - 4)$ and hence

$$D_R(G_2) = (n - k)(3n - 3k - 1) .$$

It is also easy to see that

$$R(u|G_1) = \frac{k^2 - 1}{6} \quad , \quad R(u|G_2) = n - k \quad , \quad S'(u|G_2) = n - k \quad , \quad S'(u|G_1) = \frac{k^2 - 1}{3} .$$

Therefore it follows that

$$\begin{aligned} D_R(H_{n,k}) &= D_R(G_1) + D_R(G_2) + 2kR(u|G_2) + 2(n - k)R(u|G_1) \\ &+ (n - k)S'(u|G_1) + (k - 1)S'(u|G_2) \\ &= \frac{k^3 - k}{3} + (n - k)(3n - 3k - 1) + 2(n - k)\frac{k^2 - 1}{6} + 2k(n - k) \\ &+ (n - k)\frac{k^2 - 1}{3} + (k - 1)(n - k) \\ &= \frac{1}{3}(9n^2 + (2k^2 - 9k - 8)n - k^3 + 7k) . \end{aligned}$$

Corollary 3.2. *Let G be a unicyclic graph of order $n \geq 7$. Then*

$$D_R(G) \geq \frac{1}{3}(9n^2 - 17n - 6) .$$

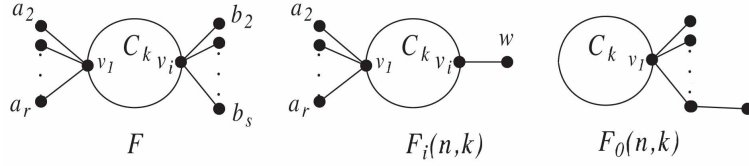
The equality holds if and only if $G \cong H_{n,3}$.

Proof. It is easy to see that for $3 \leq k \leq n$,

$$D_R(H_{n,k}) - D_R(H_{n,3}) = \frac{1}{3}(k - 3)(2kn - 3n - k^2 - 3k - 2) := f(n, k) .$$

For $k = n, n - 1, n - 2, 4, 5, 6$, it can be checked that $f(n, k) > 0$. For $7 \leq k \leq n - 3$, $f(n, k) \geq f(k + 3, k) = \frac{1}{3}(k - 3)(k^2 - 11) > 0$. Therefore, $f(n, k) \geq 0$, with equality holding if and only if $k = 3$. This implies the result. \square

Lemma 3.3. *Let $F_i(n, k)$ and $F_0(n, k)$ be two graphs as depicted in Fig. 2, $n \geq 8$. Then $D_R(F_i(n, k)) \geq D_R(F_2(n, k))$, $D_R(F_2(n, k)) \geq D_R(F_2(n, 3))$, and $D_R(F_0(n, k)) > D_R(F_2(n, k))$.*

FIGURE 2. *Graphs mentioned in Lemma 7 and Theorem 8*

Proof. For the graph $F_i(n, k)$, let $G_1 = K_{1, r-1}$, $G_2 = H_{k+1, k}$ with common vertex v_1 and $r = n - k$. Assume that the pendent vertex of G_2 is w . It is easy to see that

$$D_R(K_{1, r-1}) = (r - 1)(3r - 4) \quad , \quad D_R(H_{k+1, k}) = \frac{1}{3}(k^3 + 2k^2 + 8k + 1) .$$

In view of Theorem 2.2,

$$\begin{aligned} R(v_1|G_1) &= \sum_{x \in V(G_1 - v_1)} R(x, v_1) = n - k - 1 , \\ R(v_1|G_2) &= \sum_{x \in V(G_2 - v_1) \setminus \{v_i\}} R(x, v_1) + R(v_i, v_1) + R(w, v_1) \\ &= \sum_{x \in V(G_2 - v_1) \setminus \{v_i\}} R(x, v_1) + R(v_i, v_1) + R(v_i, v_1) + 1 \\ &= \sum_{x \in V(C_k)} R(x, v_1) + R(v_i, v_1) + 1 \\ &= R(v_1|C_k) + R(v_i, v_1) + 1 , \\ S'(v_1|G_1) &= \sum_{x \in V(G_1 - v_1)} d(x)R(x, v_1) = n - k - 1 , \\ S'(v_1|G_2) &= \sum_{x \in V(G_2 - v_1)} d(x)R(x, v_1) \\ &= \sum_{x \in V(G_2 - v_1) \setminus \{v_i\}} d(x)R(x, v_1) + d(v_i)R(v_i, v_1) + d(w)R(w, v_1) \\ &= \sum_{x \in V(G_2 - v_1) \setminus \{v_i\}} 2 R(x, v_1) + 3R(v_i, v_1) + R(v_i, v_1) + 1 \\ &= 2 \sum_{x \in V(C_k) \setminus \{v_i\}} R(x, v_1) + 2R(v_i, v_1) + 1 \\ &= 2R(v_1|C_k) + 2R(v_i, v_1) + 1 . \end{aligned}$$

Therefore

$$\begin{aligned}
D_R(F_i(n, k)) &= D_R(G_1) + D_R(G_2) + 2(k+1)R(v_1|G_1) + 2(n-k-1)R(v_1|G_2) \\
&+ kS'(v_1|G_1) + (n-k-1)S'(v_1|G_2) \\
&= (n-k-1)(3n-3k-4) + \frac{1}{3}(k^3 + 2k^2 + 8k + 1) \\
&+ 2(k+1)(n-k-1) + 2(n-k-1) \left[R(v_1|C_k) + R(v_i, v_1) + 1 \right] \\
&+ k(n-k-1) + (n-k-1) \left[2R(v_1|C_k) + 2R(v_i, v_1) + 1 \right] \\
&= (n-k-1)(3n-3k-4) + \frac{1}{3}(k^3 + 2k^2 + 8k + 1) \\
&+ (3k+1)(n-k-1) + 4(n-k-1)R(v_1|C_k) \\
&+ 4(n-k-1)R(v_i, v_1) + 3(n-k-1) \\
&= 3n(n-k-1) + \frac{1}{3}(k^3 + 2k^2 + 8k + 1) \\
&+ 4(n-k-1)R(v_1|C_k) + 4(n-k-1)R(v_i, v_1) \\
&\geq 3n(n-k-1) + \frac{1}{3}(k^3 + 2k^2 + 8k + 1) \\
&+ 4(n-k-1)R(v_1|C_k) + 4(n-k-1)R(v_2, v_1) \\
&= 3n(n-k-1) + \frac{1}{3}(k^3 + 2k^2 + 8k + 1) \\
&+ 4(n-k-1)\frac{k^2-1}{6} + 4(n-k-1)\frac{k-1}{k} \\
&= \frac{1}{3k} (12 + 3k - 2k^2 - k^4 - 12n + kn - 9k^2n + 2k^3n + 9kn^2) \\
&= D_R(F_2(n, k))
\end{aligned}$$

i. e., $D_R(F_2(n, 3)) = \frac{1}{3}(9n^2 - 12n - 26)$.

It follows that $D_R(F_i(n, k)) \geq D_R(F_2(n, k))$ and $F_2(n, k) \cong F_k(n, k)$. Therefore

$$D_R(F_2(n, k)) - D_R(F_2(n, 3)) = \frac{1}{3k}(k-3)(-4 - 11k - 3k^2 - k^3 + 4n - 3kn + 2k^2n).$$

Let $f(k) = -4 - 11k - 3k^2 - k^3 + 4n - 3kn + 2k^2n$. Then $f(k) > 0$ for $n \geq 8$ and $k = 4, 5, 6, 7, 8$.

Note that, for $k \geq 8$ and $n \geq k+1$, one has

$$\frac{df(k)}{dk} = -11 - 6k - 3k^3 - 3n + 4kn$$

$$\geq -11 - 6k - 3k^3 + (4k - 3)(k + 1) = (k + 2)(k - 7) > 0 .$$

It follows that $f(k) \geq f(8) > 0$.

From above, we have $D_R(F_2(n, k)) - D_R(F_2(n, 3)) \geq 0$, with equality if and only if $k = 3$.

For $F_0(n, k)$, let $G_1 = F_0(n, k) - C_k + v_1$, $G_2 = C_k$. Assume that the vertex of degree 2 in G_1 is u and its pendent neighbor is w .

In view of Theorem 2.2, we have

$$D_R(G_1) = 4W(G_1) - r(r - 1) = 3r^2 - 3r - 8 \quad , \quad \text{where } r = n - k + 1 \quad ,$$

and

$$R(v_1|G_1) = n - k + 1 \quad , \quad S'(v_1|G_1) = n - k + 2 \quad , \quad S'(v_1|G_2) = 2R(v_1|G_2) = \frac{k^2 - 1}{3} .$$

Hence

$$\begin{aligned} D_R(F_0(n, k)) &= \frac{k^3 - k}{3} + 3(n - k + 1)^2 - 3(n - k + 1) - 8 \\ &+ 2k(n - k + 1) + 2(n - k) \frac{k^2 - 1}{6} \\ &+ (k - 1)(n - k + 2) + (n - k) \frac{k^2 - 1}{3} \\ &= \frac{1}{3} (-30 + 7k - k^3 + 4n - 9kn + 2k^2n + 9n^2) . \end{aligned}$$

which finally yields

$$D_R(F_0(n, k)) - D_R(F_2(n, k)) = \frac{1}{k} (-4 - 11k + 3k^2 + 4n + kn) > 0 .$$

This proves the result. □

Theorem 3.4. *Let $G \neq H_{n,k}$ be a unicyclic graph of order n (≥ 8). Then $D_R(G) \geq D_R(F_2(n, k))$. Equality holds if and only if $G \cong F_2(n, k)$.*

Proof. Suppose that G has the second minimal degree resistance distance among all n -vertex unicyclic graphs. Suppose that the girth of G is k . Then G has the form $U(C_k; T_1, T_2, \dots, T_k)$ as described above.

First, we claim that at most two of T_1, T_2, \dots, T_k are not trivial. Assume the contrary, namely that T_1, T_2, T_3 are not trivial. By Theorem 2.3, they must be stars with centers v_1, v_2, v_3 , respectively. Let $V(T_1) = \{v_1, a_2, a_3, \dots, a_r\}$, $V(T_2) = \{v_2, b_2, b_3, \dots, b_s\}$, $V(T_3) = \{v_3, c_2, c_3, \dots, c_t\}$. Then by Theorem 2.4, $D_R(G) > \min\{D_R(G - v_2b_2 + v_1b_2)$ and $D_R(G - v_1a_2 + v_2a_2)\} > D_R(H_k)$. This contradicts to the choice of G .

Next, if exactly two of T_1, T_2, \dots, T_k are not trivial, then without loss of generality we may assume that these are T_1 and T_i , $2 \leq i \leq k$. Then by Theorems 2.3 and 2.4 these are stars with centers v_1, v_i , respectively. In other words, G is the graph of the form F as shown in Figure 2. Let $V(T_1) = \{v_1, a_2, a_3, \dots, a_r\}$, $V(T_i) = \{v_i, b_2, b_3, \dots, b_s\}$, where $r + s + k = n + 2$, $r \geq 2$ and $s \geq 2$. From Lemma

3.3, we have $r = 2$ or $s = 2$. Without loss of generality, assume that $s = 2$, i. e., that $G = F_i(n, k)$ is the graph shown in Figure 2. Then $r + k = n$. From Lemma 3.3, we have $i = 2$ or $i = k$.

If exactly one of T_1, T_2, \dots, T_k is not trivial, then without loss of generality, we assume that it is T_1 . Since $G \neq H_{n,k}$ and T_1 is not a star, from Theorem 2.3 it follows that G must be the graph $F_0(n, k)$ as shown in Figure 2.

From Lemma 3.3, $D_R(F_0(n, k)) > D_R(F_2(n, k))$, so we get the result. \square

Corollary 3.5. *Let $G \neq H_{n,3}$ be a unicyclic graph of order $n \geq 12$. Then $D_R(G) \geq \frac{1}{3}(9n^2 - 12n - 26)$. Equality holds if and only if $G \cong F_2(n, 3)$.*

Proof. From Theorem 3.4, we have $D_R(G) \geq D_R(F_2(n, k))$. By Lemma 3.3, $D_R(F_2(n, k)) \geq D_R(F_2(n, 3))$, which implies the result. \square

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