ON THE SEMI COVER-AVOIDING PROPERTY AND $\mathcal{F}$-SUPPLEMENTATION

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Abstract. In this paper, we investigate the influence of some subgroups of Sylow subgroups with semi cover-avoiding property and $\mathcal{F}$-supplementation on the structure of finite groups and generalize a series of known results.

1. Introduction

Throughout the paper, all groups are finite. We use conventional notions and notation, as in Huppert [17]. $G$ always denotes a group, $|G|$ is the order of $G$, $O_p(G)$ is the maximal normal $p$-subgroup of $G$ and $\Phi(G)$ is the Frattini subgroup of $G$.

Let $L/K$ be a normal factor of a group $G$. A subgroup $H$ of $G$ is said to cover $L/K$ if $HL = HK$, and $H$ is said to avoid $L/K$ if $H \cap L = H \cap K$. If $H$ covers or avoids every chief factor of $G$, then $H$ is said to have the cover-avoiding property in $G$. This conception was first studied by Gaschütz (see [5]) to study the solvable groups, later by Gillam (see [6]) and Ezquerro (see [3]), et al. More recently, in Fan et al. (see [4]) introduced the semi cover-avoiding property, which is the generalization not only of the cover-avoiding property but also of $c$-normality (see [22]). A subgroup $H$ of a group $G$ is said to have the semi cover-avoiding property in $G$, if there exists a chief series of $G$ such that $H$ either covers or avoids every $G$-chief factor of this series. The results in Guo and Shum (see [10]) and Wang (see [22]) were extended with the requirement that the certain subgroups of $G$ have the semi cover-avoiding property (see [9] and [16]). More recently, many authors presented some conditions for a group to be $p$-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup have the semi cover-avoiding property (see [15], [19] and [27]).


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A subgroup $H$ of a group $G$ is said to be complemented in $G$ if $G$ has a subgroup $K$ such that $G = HK$ and $H \cap K = 1$. A subgroup $H$ of a group $G$ is said to be supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$. Obviously, a complemented subgroup is a special supplemented subgroup. Recently, by considering some other special supplemented subgroups, many authors obtained a series of new characterization theorems for soluble groups and supersolvable groups. For example, Wang introduced the concept of $c$-supplemented subgroup [21] (a subgroup $H$ of a group $G$ is said to be a $c$-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G$ is the maximal normal subgroup of $G$ contained in $H$). In 2007, A. Y. Alsheik Ahmad, et al., introduced the concept of $U_c$-normal subgroup [1] (A subgroup $H$ of a group $G$ is called $U_c$-normal in $G$ if there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $U$-hypercenter $Z^U(G/H_G)$, where $U$ is the class of the finite supersoluble groups). As a promotion of above a series of subgroups, W. Guo introduced the concept of $F$-supplemented subgroup [8] (A subgroup $H$ of a group $G$ is $F$-supplemented in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $F$-hypercenter $Z^F(G/H_G)$, where $F$ is a formation of finite groups). In [8], by using some $F$-supplemented subgroups, W. Guo has given some conditions under which a finite group belongs to some formations.

A subgroup that satisfies the cover-avoiding property does not necessarily need to be $F$-supplemented and vice-versa. In this paper, we will try an attempt to unify the two concepts and establish the structure of groups under the assumption that all maximal subgroups of a Sylow subgroup either have the semi cover-avoiding property or are $F$-supplemented subgroups. Some new results are obtained and a series of previously known results are generalized, such as in [9], [11], [12], [13], [14], [18], [19], [21], [23], [24] and [25].

2. Preliminaries

In this section, we list some lemmas which will be useful for the proofs of our main results.

**Lemma 2.1.** [9, Lemmas 2.5 and 2.6] Suppose that $H$ has the semi cover-avoiding property in $G$.

1. If $H \leq L \leq G$, then $H$ has the semi cover-avoiding property in $L$.
2. If $N \triangleleft G$ and $N \leq H \leq G$, then $H/N$ has the semi cover-avoiding property in $G/N$.
3. If $H$ is a $\pi$-subgroup and $N$ is a normal $\pi'$-subgroup of $G$, then $HN/N$ has the semi cover-avoiding property in $G/N$.

**Lemma 2.2.** [8, Lemma 2.2] Let $G$ be a group and $H \leq K \leq G$. Then

1. If $H$ is $F$-supplemented in $G$ and $F$ is $s$-closed, then $H$ is $F$-supplemented in $K$.
2. Suppose that $H$ is normal in $G$. Then $K/H$ is $F$-supplemented in $G/H$ if and only if $K$ is $F$-supplemented in $G$.
3. Suppose that $H$ is normal in $G$. Then, for every $F$-supplemented subgroup $E$ in $G$ satisfying $(|H|, |E|) = 1$, $HE/H$ is $F$-supplemented in $G/H$. 


(4) $H$ is $\mathcal{F}$-supplemented in $G$ if and only if there exists a subgroup $T$ of $G$ such that $G = HT$, $H_G \leq T$ and $(H/H_G) \cap (T/H_G) \leq Z^G_{\infty}(G/H_G)$.

**Lemma 2.3.** [9, Lemma 3.1] Let $p$ be a prime dividing the order of the group $G$ with $(|G|, p - 1) = 1$ and let $P$ be a $p$-Sylow subgroup of $G$. If there is a maximal subgroup $P_1$ of $P$ such that $P_1$ has the semi cover-avoiding property in $G$, then $G$ is $p$-solvable.

**Lemma 2.4.** [21, Lemma 2.8] Let $M$ be a maximal subgroup of $G$ and $P$ a normal $p$-subgroup of $G$ such that $G = PM$, where $p$ is a prime. Then $P \cap M$ is a normal subgroup of $G$.

**Lemma 2.5.** [26, Lemma 2.7] Let $G$ be a group and $p$ a prime dividing $|G|$ with $(|G|, p - 1) = 1$.

1. If $N$ is normal in $G$ of order $p$, then $N \leq Z(G)$.

2. If $G$ has cyclic Sylow $p$-subgroup, then $G$ is $p$-nilpotent.

3. If $M \leq G$ and $|G : M| = p$, then $M \leq G$.

**Lemma 2.6.** [7, Main Theorem] Suppose that $G$ has a Hall $\pi$-subgroup where $\pi$ is a set of odd primes. Then all Hall $\pi$-subgroups of $G$ are conjugate.

**Lemma 2.7.** [18, Lemma 2.6] Let $H \neq 1$ be a solvable normal subgroup of a group $G$. If every minimal normal subgroup of $G$ which is contained in $H$ is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of $H$ is the direct product of minimal normal subgroups of $G$ which are contained in $H$.

**Lemma 2.8.** [21, Lemma 2.16] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $N$ such that $G/N \in \mathcal{F}$. If $N$ is cyclic, then $G \in \mathcal{F}$.

### 3. Main results

**Theorem 3.1.** Let $p$ be a prime dividing the order of a group $G$ with $(|G|, p - 1) = 1$ and $H$ a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent. Suppose that there exists a Sylow $p$-subgroup $P$ of $H$ such that every maximal subgroup of $P$ either has the semi cover-avoiding property or is $\mathcal{N}_p$-supplemented in $G$, where $\mathcal{N}_p$ is the class of all $p$-nilpotent groups. Then $G$ is $p$-nilpotent.

**Proof.** We distinguish two cases:

Case I. $H = G$.

Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

1. $O_{\alpha'}(G) = 1$.

Assume that $O_{\alpha'}(G) \neq 1$. Then $PO_{\alpha'}(G)/O_{\alpha'}(G)$ is a Sylow $p$-subgroup of $G/O_{\alpha'}(G)$. Suppose that $M/O_{\alpha'}(G)$ is a maximal subgroup of $PO_{\alpha'}(G)/O_{\alpha'}(G)$. Then there exists a maximal subgroup $P_1$ of $P$ such that $M = P_1O_{\alpha'}(G)$. By the hypothesis of the theorem, $P_1$ either has the semi cover-avoiding property or is $\mathcal{N}_p$-supplemented in $G$. Then $M/O_{\alpha'}(G) = P_1O_{\alpha'}(G)/O_{\alpha'}(G)$ either has the semi cover-avoiding property or is $\mathcal{N}_p$-supplemented in $G/O_{\alpha'}(G)$ by Lemmas 2.1 and 2.2. It is clear
that \(|G/O_p(G)|, p - 1) = 1\). The minimal choice of \(G\) implies that \(G/O_p(G)\) is \(p\)-nilpotent, and so \(G\) is \(p\)-nilpotent, a contradiction. Therefore, we have \(O_p(G) = 1\).

(2) \(O_p(G) \neq 1\).

If not, suppose that \(O_p(G) = 1\). If there is a maximal subgroup of \(P\) which has the semi cover-avoiding property in \(G\), then \(G\) is \(p\)-solvable by Lemma 2.3. Since \(O_p(G) = 1\) by Step (1), we have \(O_p(G) \neq 1\), a contradiction. Thus we may assume that all maximal subgroups of \(P\) are \(M_p\)-supplemented in \(G\). Let \(L\) be an arbitrary maximal subgroup of \(P\). Then \(G\) has a subgroup \(T\) of \(G\) such that \(G = LT\) and \((L \cap T)L_G/L_G\) is contained in the \(N_p\)-hypercenter \(Z_p^\infty(G/L_G)\). Since \(O_p(G) = 1\), obviously \(L_G = 1\). It follows that \(L \cap T \leq Z_p^\infty(G)\). If \(Z_p^\infty(G) \neq 1\), we can take a minimal normal \(N\) of \(G\) which contained in \(Z_p^\infty(G)\). By Step (1), \(N\) is not a \(p'\)-group. Consequently, \(N\) is a \(p\)-group and so \(O_p(G) \neq 1\), a contradiction. Therefore we have \(Z_p^\infty(G) = 1\) and so every maximal subgroup of \(P\) is complemented in \(G\). If \(p \neq 2\), then \(G\) is odd from the assumption that \(|G|, p - 1) = 1\). By the Feit-Thompson Theorem, \(G\) is solvable. It follows that \(O_p(G) \neq 1\) by Step (1), a contradiction. If \(p = 2\), we get also \(G\) is solvable by [2] Lemma 3, the same contradiction.

(3) If \(N \leq O_p(G)\), then \(G/N\) is \(p\)-nilpotent. Consequently, \(G\) is solvable.

Suppose that \(M/N\) is a maximal subgroup of \(P/N\). Then \(M\) is a maximal subgroup of \(P\). By the hypothesis of the theorem, \(M\) either has the semi cover-avoiding property or is \(M_p\)-supplemented in \(G\). Then \(M/N\) either has the semi cover-avoiding property or is \(M_p\)-supplemented in \(G/N\) by Lemmas 2.1 and 2.2. Therefore \(G/N\) satisfies the hypothesis of the theorem. The minimal choice of \(G\) implies that \(G/N\) is \(p\)-nilpotent. If \(p\) is odd, then \(|G|, p - 1) = 1\) implies that \(G\) is odd order, hence \(G\) is solvable. If \(p = 2\), then \(G/N\) is solvable, and so \(G\) is solvable.

(4) \(O_p(G)\) is the unique minimal normal subgroup of \(G\).

Let \(N\) be a minimal normal subgroup of \(G\). Since \(G\) is solvable by Step (3), \(N\) is an elementary abelian subgroup. Note that \(O_p(G) = 1\), then we have \(N\) is a \(p\)-subgroup and so \(N \leq O_p(G)\). Step (3) implies that \(G/O_p(G)\) is \(p\)-nilpotent. Since the class of all \(p\)-nilpotent groups is a saturated formation, \(N\) is a unique minimal normal subgroup of \(G\) and \(N \not\leq \Phi(G)\). Choose \(M\) to be a maximal subgroup of \(G\) such that \(G = NM\). Obviously, \(G = O_p(G)M\) and so \(O_p(G) \cap M\) is normal in \(G\) by Lemma 2.4. The uniqueness of \(N\) yields \(N = O_p(G)\).

(5) The final contradiction.

By the proof in Step (4), \(G\) has a maximal subgroup \(M\) such that \(G = MO_p(G)\) and \(G/O_p(G) \cong M\) is \(p\)-nilpotent. Clearly, \(P = O_p(G)(P \cap M)\). Furthermore, \(P \cap M < P\). Thus, there exists a maximal subgroup \(V\) of \(P\) such that \(P \cap M \leq V\). Hence, \(P = O_p(G)V\). By the hypothesis, \(V\) either has the semi cover-avoiding property or is \(M_p\)-supplemented in \(G\).

First, we assume that \(V\) has the semi cover-avoiding property in \(G\). Since \(O_p(G)\) is the unique minimal normal subgroup of \(G\), \(V\) covers or avoids \(O_p(G)/1\). If \(V\) covers \(O_p(G)/1\), then \(V O_p(G) = V\), i.e., \(O_p(G) \leq V\). It follows that \(P = O_p(G)V = V\), a contradiction. If \(V\) avoids \(O_p(G)/1\), then \(V \cap O_p(G) = 1\). Since \(V \cap O_p(G)\) is a maximal subgroup of \(O_p(G)\), we have that \(O_p(G)\) is of order \(p\) and so \(O_p(G)\) lies in \(Z(G)\) by Lemma 2.5. By Step (3), we have \(G/O_p(G)\) is \(p\)-nilpotent. Then \(G/Z(G)\) is \(p\)-nilpotent, and so \(G\) is \(p\)-nilpotent, a contradiction.
Now, we may assume that \( V \) is \( \mathcal{N}_p \)-supplemented in \( G \). Then there is a subgroup \( T \) of \( G \) such that \( G = VT \) and \((V \cap T)V_G/V_G \) is contained in the \( \mathcal{N}_p \)-hypercenter \( Z^\mathcal{N}_p \infty (G/V_G) \). If \( V_G \neq 1 \), then \( O_p(G) = V_G \leq V \). It follows that \( P = O_p(G)V = V \), a contradiction. Thus we may assume \( V_G = 1 \). Consequently, we have \( V \cap T \leq Z^\mathcal{N}_p \infty (G) \). If \( Z^\mathcal{N}_p \infty (G) \neq 1 \), then \( O_p(G) \leq Z^\mathcal{N}_p \infty (G) \) and \(|O_p(G)| = p \). It follows that \( G \) is \( p \)-nilpotent as above, a contradiction. Now assume that \( Z^\mathcal{N}_p \infty (G) = 1 \). Then \( V \cap T = 1 \), and so \(|T|_p = p \). By Lemma 2.5, \( T \) is \( p \)-nilpotent. Let \( T \) be the normal \( p \)-complement of \( T \). Since \( M \) is \( p \)-nilpotent, we may suppose \( M \) has a normal Hall \( p' \)-subgroup \( M_p' \) and \( M \leq N_G(M_p') \leq G \). The maximality of \( M \) implies that \( M = N_G(M_p') \) or \( M_G(M_p') = G \). If the latter holds, then \( M_p' \leq G \) and \( M_p' \) is actually the normal \( p \)-complement of \( G \), which is contrary to the choice of \( G \). Hence, we may assume \( M = N_G(M_p') \). By applying Lemma 2.6 and the Feit-Thompson Theorem, there exists \( g \in G \) such that \( T^g_p = M_p' \). Hence, \( T^g \leq N_G(T^g_p) = N_G(M_p') = M \). However, \( T_p' \) is normalized by \( T \), so \( g \) can be considered as an element of \( V \). Thus, \( G = VT^g = VM \) and \( P = V(P \cap M) = V \), a contradiction.

Case II. \( H < G \).

By Lemmas 2.1 and 2.2, every maximal subgroup of \( P \) has the semi cover-avoiding property or is \( \mathcal{N}_p \)-supplemented in \( H \). By Case I, \( H \) is \( p \)-nilpotent. Now, let \( H_p' \) be the normal \( p \)-complement of \( H \). Then \( H_p' \leq G \). Assume \( H_p' \neq 1 \) and consider \( G/H_p' \). Applying Lemmas 2.1 and 2.2, it is easy to see that \( G/H_p' \) satisfies the hypotheses for the normal subgroup \( H/H_p' \). Therefore, by induction \( G/H_p' \) is \( p \)-nilpotent and so \( G \) is \( p \)-nilpotent. Hence, we may assume \( H_p' = 1 \) and so \( H = P \) is a \( p \)-group. Since \( G/H \) is \( p \)-nilpotent, we can let \( K/H \) be the normal \( p \)-complement of \( G/H \). By The Schur-Zassenhaus Theorem, there exists a Hall \( p' \)-subgroup \( K_p' \) of \( K \) such that \( K = HK_p' \). A new application of Case I yields \( K \) is \( p \)-nilpotent and so \( K = H \times K_p' \). Hence, \( K_p' \) is a normal \( p \)-complement of \( G \) and \( G \) is \( p \)-nilpotent.  

\[\square\]

**Corollary 3.2.** Let \( P \) be a Sylow \( p \)-subgroup of a group \( G \), where \( p \) is the smallest prime divisor of \(|G|\). If every maximal subgroup of \( P \) either has the semi cover-avoiding property or is \( \mathcal{N}_p \)-supplemented in \( G \), then \( G \) is \( p \)-nilpotent.

**Proof.** It is clear that \((|G|, p - 1) = 1 \) if \( p \) is the smallest prime dividing the order of \( G \) and so the corollary follows immediately from Theorem 3.1.  
\[\square\]

**Corollary 3.3.** Suppose that every maximal subgroup of any Sylow subgroup of a group \( G \) either has the semi cover-avoiding property or is \( \mathcal{U} \)-supplemented in \( G \), where \( \mathcal{U} \) is the class of all supersolvable groups. Then \( G \) is a Sylow tower group of supersolvable type.

**Proof.** Let \( p \) be the smallest prime dividing \(|G|\) and \( P \) a Sylow \( p \)-subgroup of \( G \). By Corollary 3.2, \( G \) is \( p \)-nilpotent. Let \( T \) be the normal \( p \)-complement of \( G \). By Lemmas 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of \( T \) has the semi cover-avoiding property or is \( \mathcal{U} \)-supplemented in \( T \). Thus \( T \) satisfies the hypothesis of the Corollary. It follows by induction that \( T \), and hence \( G \) is a Sylow tower group of supersolvable type.  
\[\square\]
Corollary 3.4. [19] Theorem 3.3] Let $G$ be a group, $p$ a prime dividing the order of $G$, and $P$ a Sylow $p$-subgroup of $G$. If $(|G|, p - 1) = 1$ and every maximal subgroup of $P$ has the semi cover-avoiding property in $G$, then $G$ is $p$-nilpotent.

Corollary 3.5. [9] Theorem 3.2] Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is the smallest prime divisor of $|G|$. If $P$ is cyclic or every maximal subgroup of $P$ has the semi cover-avoiding property in $G$, then $G$ is $p$-nilpotent.

Proof. If $P$ is cyclic, by Lemma 2.5 we have $G$ is $p$-nilpotent. Thus we may assume that every maximal subgroup of $P$ has the semi cover-avoiding property in $G$. By Corollary 3.2, $G$ is $p$-nilpotent. □

Corollary 3.6. [11] Theorem 3.4] Let $G$ be a group and $P$ a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing $|G|$. If all maximal subgroups of $P$ are $c$-normal in $G$, then $G$ is $p$-nilpotent.

Corollary 3.7. [12] Theorem 3.2] Let $G$ be a group and $P$ a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing $|G|$. If all maximal subgroups of $P$ are $c$-supplemented in $G$, then $G$ is $p$-nilpotent.

Corollary 3.8. [13] Theorem 3.1] Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. If every maximal subgroup of $P$ is $c$-supplemented in $G$, then $G$ is $p$-nilpotent.

Corollary 3.9. [21] Theorem 3.1] Let $p$ be a prime dividing the order of a group $G$ with $(|G|, p - 1) = 1$. Suppose that every maximal subgroup of $P$ is $c$-supplemented in $G$ and $G \in C_{p'}$, then $G/O_p(G)$ is $p$-nilpotent and $G \in D_{p'}$.

Theorem 3.10. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup $H$ of $G$ such that $G/H \in \mathcal{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of $H$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G$.

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let $G$ be a counterexample of minimal order.

(1) $G$ has a minimal normal subgroup $N \leq H$ and $N$ is an elementary abelian $p$-group, where $p$ is the largest prime in $\pi(H)$.

By the hypothesis of the theorem, every maximal subgroup of any noncyclic Sylow subgroup of $H$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G$. Consequently, by Lemmas 2.1 and 2.2 every one also either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $H$. Applying Corollary 3.3, $H$ is a Sylow tower group of supersolvable type. Let $p$ be the largest prime divisor of $|H|$ and $P$ a Sylow $p$-subgroup of $H$. Then $P$ is normal in $H$. Obviously, $P$ is normal in $G$. Therefore, $G$ has a minimal normal subgroup $N \leq H$ and $N$ is an elementary abelian $p$-group.

(2) $G/N \in \mathcal{F}$ and $N = P$ is the Sylow $p$-subgroup of $H$.

First, we want to prove that $G/N$ satisfies the hypothesis of the theorem. In fact, $(G/N)/(H/N) \cong G/H \in \mathcal{F}$. Let $P_1/N$ be a maximal subgroup of the Sylow $p$-subgroup $P/N$ of $H/N$. Then $P_1$ is a
maximal subgroup of the Sylow \( p \)-subgroup \( P \) of \( H \). If \( P/N \) is noncyclic, then \( P \) is also noncyclic. By the hypothesis of the theorem, \( P_1 \) either has the semi cover-avoiding property or is \( \mathcal{U} \)-supplemented in \( G \). By Lemmas 2.1 and 2.2, \( P_1/N \) either has the semi cover-avoiding property or is \( \mathcal{U} \)-supplemented in \( G/N \). Let \( M_1/N \) be a maximal subgroup of the noncyclic Sylow \( q \)-subgroup \( QN/N \) of \( H/N \), where \( q \neq p \) and \( Q \) is a noncyclic Sylow \( q \)-subgroup of \( H \). It is clear that \( M_1 = Q_1N \), where \( Q_1 \) is a maximal subgroup of \( Q \). By the hypothesis of the theorem, \( Q_1 \) either has the semi cover-avoiding property or is \( \mathcal{U} \)-supplemented in \( G \). Hence \( M_1/N \) either has the semi cover-avoiding property or is \( \mathcal{U} \)-supplemented in \( G/N \) by Lemmas 2.1 and 2.2. We now have proved that \( G/N \) satisfies the hypothesis of the theorem. By the minimal choice of \( G \), we have \( G/N \in \mathcal{F} \). Since \( \mathcal{F} \) is a saturated formation, \( N \) is the unique minimal normal subgroup of \( G \) contained in \( P \) and \( N \notin \Phi(G) \). By Lemma 2.7, it follows that \( P = F(P) = N \).

(3) The final contradiction.

Let \( M \) be a maximal subgroup of \( N \). By the hypothesis, \( M \) either has the semi cover-avoiding property or is \( \mathcal{U} \)-supplemented in \( G \). First we assume that \( M \) is \( \mathcal{U} \)-supplemented in \( G \). Then \( G \) has a subgroup \( T \) of \( G \) such that \( G = MT \) and \( T \cap M \leq Z_\mathcal{U}^\infty(G) \). Thus \( G = NT \) and \( N = N \cap MT = M(N \cap T) \). This implies that \( N \cap T \neq 1 \). Since \( N \cap T \) is normal in \( G \) and \( N \) is a minimal normal subgroup of \( G \), \( N \cap T = N \). It follows that \( T = G \), and so \( M \leq N \cap Z_\mathcal{U}^\infty(G) \). By the minimality of \( N \), \( Z_\mathcal{U}^\infty(G) \cap N = 1 \) or \( N \leq Z_\mathcal{U}^\infty(G) \). If the latter holds, then \(|N| = p \). By Step (2), \( G/N \in \mathcal{F} \). Applying Lemma 2.8, \( G \in \mathcal{F} \), a contradiction. Therefore \( Z_\mathcal{U}^\infty(G) \cap N = 1 \). It follows that \( M = 1 \) and \(|N| = p \), the same contradiction as above.

Now we assume that \( M \) has the semi cover-avoiding property in \( G \). Then there exists a chief series of \( G \)

\[
1 = G_0 < G_1 < \cdots < G_{n-1} < G_n = G
\]
such that \( M \) covers or avoids every factor \( G_j/G_{j-1} \). Since \( N \) is minimal normal in \( G \), there exists \( j \) such that \( G_j \cap N = N \) and \( G_{j-1} \cap N = 1 \). If \( M \) covers \( G_j/G_{j-1} \), then \( MG_j = MG_{j-1} \) and so \( MG_j \cap N = MG_{j-1} \cap N \). Hence \( M(G_j \cap N) = M(G_{j-1} \cap N) \), i.e., \( MN = M \), a contradiction. If \( M \) avoids \( G_j/G_{j-1} \), then \( M \cap G_j = M \cap G_{j-1} \) and so \( M \cap G_j \cap N = M \cap G_{j-1} \cap N \), i.e., \( M = 1 \). It follows that the same contradiction as above.

**Corollary 3.11.** [19, Theorem 3.6] Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{U} \). If there is a normal Hall subgroup \( H \) of \( G \) such that \( G/H \in \mathcal{F} \) and every maximal subgroup of any Sylow subgroup of \( H \) has the semi cover-avoiding property in \( G \), then \( G \in \mathcal{F} \).

**Corollary 3.12.** [18, Theorem 3.3] Let \( H \) be a normal subgroup of a group \( G \) such that \( G/H \) is supersolvable. If every maximal subgroup of any Sylow subgroup of \( H \) is \( c \)-normal in \( G \), then \( G \) is supersolvable.

**Corollary 3.13.** [12, Theorem 4.2] Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{U} \). If there is a normal subgroup \( H \) of \( G \) such that \( G/H \in \mathcal{F} \) and every maximal subgroup of any Sylow subgroup of \( H \) is \( c \)-supplemented in \( G \), then \( G \in \mathcal{F} \).
Corollary 3.14. [23, Theorem 4.1] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup $H$ of $G$ such that $G/H \in \mathcal{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of $H$ is $\mathcal{F}$-supplemented in $G$, then $G \in \mathcal{F}$.

Theorem 3.15. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $N$ such that $G/N \in \mathcal{F}$. If every maximal subgroup of each non-cyclic Sylow subgroup of $F(N)$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G$, then $G \in \mathcal{F}$.

Proof. Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We distinguish two cases.

(1) $\Phi(G) \cap N \neq 1$.

Since $\Phi(G) \cap N \neq 1$, then there exists a prime $p$ dividing the order of $\Phi(G) \cap N$. Let $P_0$ be the Sylow $p$-subgroup of $\Phi(G) \cap N$. Then $P_0 \leq G$. Since $(G/P_0)/(N/P_0) \cong G/N$, it follows that $(G/P_0)/(N/P_0) \in \mathcal{F}$. By [1, p.270 Satz 3.5], $F(N/P_0) = F(N)/P_0$. Let $P_1/P_0$ be a maximal subgroup of the Sylow $p$-subgroup $P/P_0$ of $F(N)/P_0$. Then $P_1$ is a maximal subgroup of the Sylow $p$-subgroup $P$ of $F(N)$. If $P/P_0$ is non-cyclic, then $P$ is non-cyclic. By the hypothesis, $P_1$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G$. Hence $P_1/P_0$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G/P_0$ by Lemmas 2.1 and 2.2. Set $Q_1/P_0$ be a maximal subgroup of the non-cyclic Sylow $p$-subgroup of $F(N)/P_0$, where $p \neq q$. It is clear that $Q_1 = Q_1/P_0$, where $Q_1$ is a maximal subgroup of the non-cyclic Sylow $p$-subgroup of $F(N)$. Then $Q_1$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G$. Hence $Q_1/P_0$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G/P_0$ by Lemmas 2.1 and 2.2. Now we have proved that $G/P_0$ satisfies the hypotheses of the theorem. Therefore $G/P_0 \in \mathcal{F}$ by minimal choice of $G$. Since $P_0 \leq \Phi(G)$ and $\mathcal{F}$ is a saturated formation, we have that $G \in \mathcal{F}$, a contradiction.

(2) $\Phi(G) \cap N = 1$.

If $N = 1$, nothing need to be proved. So assume $N \neq 1$. Then $F(N) \neq 1$ by the solvability of $N$. By Lemma 2.7, $F(N)$ is the direct product of some minimal normal subgroups of $G$. Let $P$ be the Sylow $p$-subgroup of $F(N)$. We can denote $P = R_1 \times R_2 \times \cdots \times R_m$, where every $R_i$ is a minimal normal subgroup of $G$. We will show that $|R_i| = p$ ($i = 1, 2, \cdots, m$). If not, then there exists an index $i$ such that $|R_i| > p$. Without loss of generality, suppose that $i = 1$. Since $R_1 \nleq \Phi(G)$, there exist a maximal subgroup $M$ of $G$ such that $G = R_1M$ and $R_1 \cap M = 1$. Then $G_p = R_1M_p$. Pick a maximal subgroup $G_p^*$ of $G_p$ containing $M_p$. Then $|R_1 : G_p^* \cap R_1| = |R_1G_p^*/G_p^*| = |G_p : G_p^*| = p$. Hence $R_1 = G_p^* \cap R_1$ is a maximal subgroup of $R_1$. This implies that $P^* = R_1^1R_2 \cdots R_m$ is a maximal subgroup of $P$. Obviously, $P$ is not cyclic. By the hypothesis, $P^*$ either has the semi cover-avoiding property or is $\mathcal{U}$-supplemented in $G$. Let $K = R_2 \times \cdots \times R_m$.

First, we assume that $P^*$ has the semi cover-avoiding property in $G$. By Lemma 2.1, $P^*/K$ has the semi cover-avoiding property in $G/K$. Suppose that $P^*/K$ cover-avoids a chief series $1 = K \triangleleft G_1/K = G_1 \triangleleft \cdots \triangleleft G/K = G_n$ of $G/K$. Let $i$ be the smallest number in $\{1, 2, \cdots, n-1\}$ such that $G_{i+1}/G_i$ was covered by $P^*/K$ in above chief series. Then we have $G_i \cap P^* = K$ and $G_{i+1} \leq G_iP^* = G_iR_i^*$. Hence
\[ G_{i+1} = G_i(R_i^* \cap G_{i+1}) \text{ and } R_i^* \cap G_{i+1} > 1. \] Since \( R_1 \) is a minimal normal subgroup of \( G \), we have \( R_1 \leq G_{i+1} \text{ and } R_1 \cap G_i = 1. \) Hence \( |R_1| = |G_{i+1}/G_i| = |R_i^* \cap G_{i+1}| < |R_1| \), a contraction. Therefore, \( P^*/K \) does not cover any chief factor in above chief series. It follows that \( P^*/K = 1 \) and \( |R_1| = p \), a contraction.

We now assume that \( P^* \) is \( \mathcal{U} \)-supplemented in \( G \). By Lemma 2.11 (4), there exists a subgroup \( T \) of \( G \) such that \( G = P^*T \) and \( P^*/P_G \cap T/P_G \leq Z_{\infty}(G/P_G) \). Obviously, \( P_G = K \) and so \( P^*/K \cap T/K \leq Z_{\infty}(G/K) \). If \( P^*/K \cap T/K = 1 \), then \( P^* \cap T = K \) and so \( G = P^*T = R_i^*KT = R_iT = R_iT \).

It is easy to see that \( R_1 \cap T \triangleleft G \). Since \( R_1 \) is a minimal normal in \( G \), we have \( R_1 \cap T = 1 \) or \( R_1 \cap T = R_1 \). If the latter holds, then \( T = G \) and \( K = P^* \). In this case, \( R_i^* = 1 \) and \( R_1 \) is of order \( p \), a contraction. Therefore we have \( R_1 \cap T = 1 \). It follows that \( R_i^* \cap T = 1 \). Then \( |G| = |R_i^*||T| = |R_i||T| \) and \( |R_i^*| = |R_1| \). This contraction shows that \( P^*/K \cap T/K \neq 1 \). Let \( Z_{\infty}(G/K) = V/K \). Then \( P/K \cap V/K \triangleleft G/K \). Since \( P \cap V \supseteq P^* \cap T \cap V = P^* \cap T > K \), we have \( P/K \cap V/K \neq 1 \). By the \( G \)-isoformation \( P/K \cong R_1 \), we see that \( P/K \) is a chief factor of \( G \) contained in \( V/K \). Therefore, \( P/K \) is of order \( p \) and so \( |R_1| = p \), a contraction.

From above discussion, we can let \( F(N) = L_1 \times L_2 \times \cdots \times L_n \), where every \( L_i \) is a normal subgroup of prime order. Obviously \( G/C_G(L_i) \) is abelian. Since \( C_G(F(N)) = \bigcap_{i=1}^n C_G(L_i) \), \( G/C_G(F(N)) \) is abelian. Hence \( G/C_G(F(N)) \in \mathcal{U} \subseteq \mathcal{F} \). By the assumption, \( G/N \in \mathcal{F} \), it implies \( G/N \cap C_G(F(N)) = G/C_N(F(N)) \in \mathcal{F} \) by the properties of formations. Since \( N \) is solvable, \( C_N(F(N)) \leq F(N) \). Again, \( F(N) \) is abelian, so \( F(N) \leq C_N(F(N)) \). Thus \( F(N) = C_N(F(N)) \) and \( G/F(N) \in \mathcal{F} \). By Theorem 3.10 \( G \in \mathcal{F} \), a contradiction.

**Corollary 3.16.** [25 Theorem 1] Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{U} \), the class of all supersolvable groups. Suppose that \( G \) is a group with a solvable normal subgroup \( H \) such that \( G/H \in \mathcal{F} \). If all maximal subgroups of all Sylow subgroups of \( F(H) \) are c-normal in \( G \), then \( G \in \mathcal{F} \).

**Corollary 3.17.** [24 Theorem 4.5] Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{U} \), the class of all supersolvable groups. Suppose that \( G \) is a group with a solvable normal subgroup \( H \) such that \( G/H \in \mathcal{F} \). If all maximal subgroups of all Sylow subgroups of \( F(H) \) are c-supplemented in \( G \), then \( G \in \mathcal{F} \).

**Corollary 3.18.** [14 Theorem 1.6] Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{U} \), the class of all supersolvable groups. Suppose that \( G \) is a group with a solvable normal subgroup \( H \) such that \( G/H \in \mathcal{F} \). If all maximal subgroups of all Sylow subgroups of \( F(H) \) are complemented in \( G \), then \( G \in \mathcal{F} \).

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