CHROMATIC AND CLIQUE NUMBERS OF A CLASS OF PERFECT GRAPHS

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ABSTRACT. Let \( p \) be a prime number and \( n \) be a positive integer. The graph \( G_p(n) \) is a graph with vertex set \([n] = \{1, 2, \ldots, n\}\), in which there is an arc from \( u \) to \( v \) if and only if \( u \neq v \) and \( p \nmid u + v \). In this paper it is shown that \( G_p(n) \) is a perfect graph. In addition, an explicit formula for the chromatic number of such graph is given.

1. Introduction

Let \( G = (V, E) \) be a graph on the vertex set \( V = [n] = \{1, 2, \ldots, n\} \) and the edge set \( E \). The \textit{clique number} \( \omega(G) \) of \( G \) is the greatest integer \( r \) such that the complete graph on \( r \) vertices is an induced subgraph of \( G \). A \textit{d-coloring} of \( G \) is a map \( c : V \rightarrow \{1, 2, \ldots, d\} \) such that \( c(v) \neq c(u) \) whenever \( vu \) is an edge. The smallest integer \( k \) such that \( G \) has a \( k \)-coloring is the \textit{chromatic number} of \( G \) and denoted by \( \chi(G) \). A graph \( G \) is called \textit{weakly perfect} if \( \chi(G) = \omega(G) \). A graph is \textit{perfect} if every induced subgraph is weakly perfect. Hence, every perfect graph is weakly perfect and there are several classes which show the converse does not hold in general. For instance, let \( C \) be an odd cycle and \( uw \in E_C \) and let \( H \) denote the graph obtained from adding a new vertex \( v \) together with two new edges \( vu \) and \( vw \) to \( C \). It is easy to see that \( \chi(H) = 3 = \omega(H) \) and \( \chi(C) = 3 > 2 = \omega(C) \). So, \( H \) is weakly perfect, but it is not perfect because \( C \) is an induced subgraph of \( H \). In [2] one can find 120 classes of perfect graphs.

Let \( p \) be a prime number and \( n \) be a positive integer. The graph \( G_p(n) \) is a graph with vertex set \([n] = \{1, 2, \ldots, n\}\), in which there is an arc from \( u \) to \( v \) if and only if \( u \neq v \) and \( p \nmid u + v \). In this paper it is shown that \( G_p(n) \) is a perfect graph. In addition, we will give an explicit formula for chromatic numbers.

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2. Main Results

Let \( p \) be a prime number and let \( n \) be an arbitrary positive integer. We define the graph \( G_p(n) \) as follows

\[
V(G_p(n)) = [n], \quad \text{and} \quad E(G_p(n)) = \{uv | u, v \in [n], \ u \neq v, \text{ and } p \nmid u + v\}.
\]

**Remark 2.1.** Let \( n \) be a power of \( p \), say \( n = p^k \) for some positive integer \( k \). Then \( G_p(n) \) is the graph \( G(p^k) \) introduced in [3].

The graph \( G_3(7) \) demonstrate in the following figure.

\[
\chi(G_3(7)) = \omega(G_3(7)) = 3
\]

Let \( m \) be a positive integer. A graph \( G \) is called \( m \)-chordal if it does not have any induced cycle of length more than \( m \).

**Theorem 2.2.** Let \( p \) be a prime number and let \( n \) be an arbitrary positive integer. Then \( G_p(n) \) is a perfect graph.

**Proof.** Let \( H_p(n) = \overline{G_p(n)} \) be the complement of the graph \( G_p(n) \) which is defined as the graph whose vertices are 1, 2, \ldots, \( n \) and two vertices \( i, j \) are adjacent if and only if \( i + j \) is a multiple of \( p \).

For \( p = 2 \) it means that the graph is the disjoint union of two complete graphs, those on the odd numbers and on the even numbers. For \( p > 2 \) the prime number condition implies that \( p \) is odd. Therefore the numbers divisible by \( p \) induce a complete component, and for \( 0 < i < p \) the numbers congruent with \( i \) together with the numbers congruent with \( p - i \) induce a complete bipartite graph component.

It is immediate that complete graphs and complete bipartite graphs are perfect. It is also immediate that the vertex-disjoint union of perfect graphs is perfect. Therefore \( H_p(n) \) and so \( G_p(n) \) are perfect.

In the rest of this paper we give an explicit formula for chromatic number (equivalently clique number) of \( G_p(n) \). First we state some notations. Let \( n \) be a positive integer and let \( p \) be an odd prime number. We denote by \( [n]_p \), the unique remainder of \( n \) modulo \( p \). Let \( [n]'_p \) define as follows

\[
[n]'_p = \begin{cases} 
[n]_p & \text{if } [n]_p < \frac{p-1}{2}, \\
\frac{p-1}{2} & \text{if } [n]_p \geq \frac{p-1}{2}.
\end{cases}
\]
Theorem 2.3. Let \( p \) be a prime number and let \( n \) be a positive integer. Then

\[
\chi(G_p(n)) = \omega(G_p(n)) = \begin{cases} 
2 & \text{if } p = 2, \\
\left\lceil \frac{n}{p} \right\rceil (\frac{p-1}{2}) + [n]_p & \text{if } p > 2.
\end{cases}
\]

Proof. If \( p = 2 \), then there is nothing to prove. Consider that \( p > 2 \) and let \( w = \left\lceil \frac{n}{p} \right\rceil (\frac{p-1}{2}) + [n]_p + 1 \).

Define

\[ W = \{p\} \cup \{\frac{1}{p} \} \cup \{x \in \{1, \ldots, n\} | [x]_p \in \{1, \ldots, \frac{p-1}{2}\} \} \].

It is easy to see that \( W \) is a clique. Now lets compute the cardinality of \( W \). We have the following partition \( W = \bigcup W_i \) where for each \( i \), \( W_i = \{x \in W | [x]_p = i\} \). Hence

\[
|W| = \sum_{i=0}^{(p-1)/2} |W_i|
= 1 + \left\lfloor \frac{n}{p} \right\rfloor + \ldots + \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor + \ldots + \left\lfloor \frac{n}{p} \right\rfloor = w.
\]

So, we have \( \omega(G_p(n)) \geq w \).

Now we will show the converse. Consider the following map

\[ \gamma : V((G_p(n))) \rightarrow \{0, \ldots, p-1\} \]

such that \( \gamma(x) = [x]_p \) and assume that \( W \) is a clique in \( G_p(n) \). For all \( x \) and \( y \) in \( W \) we have \( \gamma(x) \neq \gamma(y) \), so the pigeonhole principle yields that

\[ |\gamma(W) \cap \{1, \ldots, p-1\}| \leq \left( \frac{p-1}{2} \right) \]

on the other hand we have

\[
\gamma^{-1}(i) = \begin{cases} 
\lceil \frac{n}{p} \rceil & \text{if } [n]_p \geq i, \\
\left\lfloor \frac{n}{p} \right\rfloor & \text{if } [n]_p < i.
\end{cases}
\]

therefore

\[
|W| \leq |\gamma^{-1}(\gamma(W))| 
\leq 1 + \left\lfloor \frac{n}{p} \right\rfloor + \ldots + \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor + \ldots + \left\lfloor \frac{n}{p} \right\rfloor = w
\]

as desired. \( \square \)

Corollary 2.4. \[3, \text{ Theorem 2.1.}\] Let \( p \) be a prime number and let \( k \) be a positive integer. Then

\[
\chi(G_p(p^k)) = \omega(G_p(p^k)) = \begin{cases} 
2 & \text{if } p = 2, \\
p^{k-1}(\frac{p-1}{2}) + 1 & \text{if } p > 2.
\end{cases}
\]

Proof. It is easy to see that \( \lfloor \frac{p^k}{p} \rfloor = p^{k-1} \) and \( [p^k]_p = 0 \). \( \square \)

References


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