FINITE BCI-GROUPS ARE SOLVABLE

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Abstract. Let $S$ be a subset of a finite group $G$. The bi-Cayley graph $BCay(G, S)$ of $G$ with respect to $S$ is an undirected graph with vertex set $G \times \{1, 2\}$ and edge set $\{(x, 1), (sx, 2)\mid x \in G, s \in S\}$. A bi-Cayley graph $BCay(G, S)$ is called a BCI-graph if for any bi-Cayley graph $BCay(G, T)$, whenever $BCay(G, S) \cong BCay(G, T)$ we have $T = gS^\alpha$ for some $g \in G$ and $\alpha \in Aut(G)$. A group $G$ is called a BCI-group if every bi-Cayley graph of $G$ is a BCI-graph. In this paper, we prove that every BCI-group is solvable.

1. Introduction

All graphs considered here are assumed to be undirected, finite and simple unless stated otherwise. For a graph $\Gamma$, we use $V(\Gamma)$, $E(\Gamma)$, and $Aut(\Gamma)$ to denote the vertex set, the edge set and the full automorphism group of $\Gamma$, respectively. For the most part, our notation and terminology is standard and taken from [3] (for permutation group theory) and [2] (for graph theory).

Let $S$ be a subset of a group $G$ not containing the identity element of $G$. Recall that the Cayley graph $\Gamma = Cay(G, S)$ of $G$ with respect to $S$ is the graph with vertex set $G$, where $(x, y)$ is a directed edge if and only if $yx^{-1} \in S$. Also $\Gamma$ is undirected if and only if $S = S^{-1}$. A Cayley graph $Cay(G, S)$ of a group $G$ is called a CI-graph if whenever $T$ is another subset of $G$ such that $Cay(G, S) \cong Cay(G, T)$, there exists an automorphism $\sigma$ of $G$ such that $S^\sigma = T$. For a positive integer $m$, the group $G$ is said to have the $m$-CI property if all Cayley graphs of $G$ of valency $m$ are CI-graphs; further, if $G$ has the $k$-CI property for all $k \leq m$, then $G$ is called an $m$-CI-group, and a $|G|$-CI-group $G$ is called a CI-group. The Cayley isomorphism problem of Cayley graphs, especially determining CI-graphs.

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CI-groups etc., have been an active topic in algebraic graph theory for a long time, see [13] for a survey on this topic.

For a group \( G \), and a subset \( S \) (possibly, containing the identity element) of \( G \), the bi-Cayley graph \( \text{BCay}(G, S) \) of \( G \) with respect to \( S \) is the bipartite graph with vertex set \( G \times \{1, 2\} \) and edge set \( \{(x, 1), (sx, 2)\} \mid x \in G, s \in S \). A bi-Cayley graph \( \text{BCay}(G, S) \) is called a BCI-graph if for any bi-Cayley graph \( \text{BCay}(G, T) \), whenever \( \text{BCay}(G, S) \cong \text{BCay}(G, T) \) we have \( T = gS^\alpha \) for some \( g \in G \) and \( \alpha \in \text{Aut}(G) \). A group \( G \) is called a BCI-group, if all bi-Cayley graphs of \( G \) are BCI-graphs. Also \( G \) is called an \( m \)-BCI-group, if all bi-Cayley graphs of \( G \) of valency at most \( m \) are BCI-graphs (See [7, Definition 1]). Xu, et. al., in [13] found a necessary and sufficient condition for a finite group being a 2-BCI-group.

Recently some authors studied the isomorphisms of bi-Cayley graphs. For example in [17], it is proved that only finite simple non-abelian 3-BCI group is \( A_5 \), the alternating group on five symbols. The Sylow subgroups of 3-BCI-groups are classified in [8] and the nilpotent 3-BCI-groups are determined in [16]. Also in [17], the isomorphisms of connected bi-Cayley graphs of cyclic groups, with valency 4 are discussed. In [3, Corollary 2.7], a Babai’s type theorem for bi-Cayley graphs of finite cyclic groups is proved and the present authors improved it to arbitrary groups in [11].

Li proved that every finite CI-group is solvable, see [12, Theorem 1.2]. In this paper we prove that every finite BCI-group is also solvable.

Throughout the paper we assume that \( G \) is a finite group.

2. Main Results

In [11, Lemma 2.9], the authors proved that if \( G \) is a finite group and \( S \) and \( T \) are two subsets of \( G \) both of which contain the identity, then \( \text{Cay}(G, S \setminus \{1\}) \cong \text{Cay}(G, T \setminus \{1\}) \) implies that \( \text{BCay}(G, S) \cong \text{BCay}(G, T) \). By a similar argument to the proof of [11, Lemma 2.9], one can prove that if \( G \) and \( H \) are two groups, \( S \subseteq G, T \subseteq H, 1_G \in S, 1_H \in T \) and \( \text{Cay}(G, S \setminus \{1_G\}) \cong \text{Cay}(H, T \setminus \{1_H\}) \), then \( \text{BCay}(G, S) \cong \text{BCay}(H, T) \).

In the following lemma we construct some non-BCI graphs. Let us denote by \( C_n, K_n, \Gamma_1 \), and \( \Gamma_1[\Gamma_2] \), a cycle of length \( n \), a complete graph on \( n \) vertices, the complement of a graph \( \Gamma \), and the lexicographic product of graphs \( \Gamma_1 \) and \( \Gamma_2 \), respectively. Recall that the lexicographic product graph of \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) has vertex set \( V_1 \times V_2 \), and \( u = (u_1, u_2) \) is adjacent with \( v = (v_1, v_2) \) whenever \( (u_1, v_1) \in E_1 \) or \( u_1 = v_1 \) and \( (u_2, v_2) \in E_2 \).

**Lemma 2.1.** Assume that \( G \) has two subgroups \( H = \langle a \rangle \cong \mathbb{Z}_{n^2} \) and \( K = \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n \), where \( n \geq 2 \). Then \( G \) has a bi-Cayley graph of valency \( n + 1 \) which is not a BCI-graph.

**Proof.** Firstly we claim that if \( G \) is an arbitrary finite group with a subset \( S \) such that \( \text{BCay}(G, S) \) is a BCI-graph, then for each \( T \subseteq G \) with \( \text{BCay}((SS^{-1}), S) \cong \text{BCay}((TT^{-1}), T) \), we have \( (SS^{-1}) \cong (TT^{-1}) \). To prove the claim we note that since \( \text{BCay}((SS^{-1}), S) \cong \text{BCay}((TT^{-1}), T) \), by Lemma 2.8 of [17], \( \text{BCay}(G, S) \cong \text{BCay}(G, T) \). Thus there exist \( \sigma \in \text{Aut}(G) \) and \( g \in G \) such that \( T = gS^\sigma \).
Therefore we have

\[ \langle SS^{-1} \rangle^\sigma = \langle (SS^{-1})^\sigma \rangle = \langle S^\sigma (S^{-1})^\sigma \rangle = \langle g^{-1}TT^{-1}g \rangle = g^{-1}\langle TT^{-1} \rangle g \]

and thus \( \langle SS^{-1} \rangle \cong \langle SS^{-1} \rangle^\sigma = g^{-1}\langle TT^{-1} \rangle g \cong \langle TT^{-1} \rangle \). This proves the claim.

Now to prove the lemma, we put \( S = a(a^n) \) and \( T = b(c) \). Then \( \text{Cay}(H, S) \cong C_n[K_n] \cong \text{Cay}(K, T) \) and since \( H = \langle S \rangle \) and \( K = \langle T \rangle \), we have \( \text{Cay}(G, S) \cong \text{Cay}(G, T) \) (see [13, Example 2.8 and p. 310]). Let \( S' = S \cup \{1\} \) and \( T' = T \cup \{1\} \). By the discussion preceding the lemma, we have \( \text{BCay}(G, S') \cong \text{BCay}(G, T') \) and \( \text{BCay}(H, S') \cong \text{BCay}(K, T') \). Suppose, for a contradiction, that \( \text{BCay}(G, S') \) is a BCI-graph. Since \( 1 \in S' \cap T' \), \( \langle S' S'^{-1} \rangle = \langle S' \rangle = \langle S \rangle = H \) and \( \langle T' T'^{-1} \rangle = \langle T' \rangle = \langle T \rangle = K \) (see [7, p. 1259]). Hence by the above claim, \( H \cong K \), which is a contradiction. 

In 1976, Babai and Frankl proved that if \( G \) is a CI-group and \( H \) a characteristic subgroup of \( G \), then \( G/H \) is a CI-group (see [13, Lemma 8.2]). Later, in 2013, Dobson and Morris proved that the quotient group of every CI-group is also a CI-group [3]. The following lemma is the BCI-version of the result of Babai and Frankl.

**Lemma 2.2.** Let \( G \) be a BCI-group and \( H \) a characteristic subgroup of \( G \). Then \( G/H \) is a BCI-group.

**Proof.** Let \( \text{BCay}(G/H, S) \cong \text{BCay}(G/H, T) \), where \( S, T \subseteq G/H \). Let \( \pi : G \to G/H \) be the natural projection, and set \( S_1 := \{ g \in G \mid g^\sigma \in S \} \), \( T_1 := \{ g \in G \mid g^\sigma \in T \} \). We claim that \( \text{BCay}(G, S_1) \cong \text{BCay}(G/H, S)[K_{|H|}] \). Let \( R \) be a right transversal of \( H \) in \( G \). Then for any \( g \in G \) there exists a unique \( r \in R \) such that \( Hg = Hr \). Define

\[
\varphi : G \times \{1, 2\} \to (G/H \times \{1, 2\}) \times H \\
(g, i) \mapsto ((g^\sigma, i), gr^{-1}),
\]

where \( Hg = Hr \) (\( r \in R \)). Clearly \( \varphi \) is well-defined. Let \( (g_1, i_1)^\varphi = (g_2, i_2)^\varphi \). Then \( ((g_1^\sigma, i_1), g_1r_1^{-1}) = ((g_2^\sigma, i_2), g_2r_2^{-1}) \), where \( Hg_1 = Hr_1 \) and \( Hg_2 = Hr_2 \), for some \( r_1, r_2 \in R \). So \( i_1 = i_2 \), \( g_1^\sigma = g_2^\sigma \) and \( g_1r_1^{-1} = g_2r_2^{-1} \), which imply that \( r_1 = r_2 \). Hence \( g_1 = g_2 \) and \( i_1 = i_2 \), i.e., \( \varphi \) is 1-1. Now let \( ((Hx, i), h) \in (G/H \times \{0, 1\}) \times H \). Then there exists \( r \in R \) such that \( Hx = Hr \). So \( (hr, i)^\varphi = ((Hhr, i), hrr^{-1}) = ((Hx, i), h) \). Hence \( \varphi \) is onto. Now we show that \( \varphi \) preserves the adjacency.

Let \( \Gamma = \text{BCay}(G, S_1) \) and \( \Sigma = \text{BCay}(G/H, S) \). Let \( (x, 1), (y, 2) \in V(\Gamma) \). Then \( Hx = Hr_1 \) and \( Hy = Hr_2 \), for some \( r_1, r_2 \in R \). We have

\[
\{ (x, 1), (y, 2) \} \in E(\Gamma) \quad \Rightarrow \quad yx^{-1} \in S_1
\]

\[
\Rightarrow \quad (yx^{-1})^\pi \in S
\]

\[
\Rightarrow \quad y^\pi(x^\pi)^{-1} \in S
\]

\[
\Rightarrow \quad \{ (x^\pi, 1), (y^\pi, 2) \} \in E(\Sigma)
\]

\[
\Rightarrow \quad \{ (x, 1)^\varphi, (y, 2)^\varphi \} \in E(\Sigma[K_{|H|}]),
\]

and

\[
\{ (x, 1)^\varphi, (y, 2)^\varphi \} \in E(\Sigma[K_{|H|}]) \quad \Rightarrow \quad \{ ((x^\pi, 1), x^{-1}r_1^{-1}), ((y^\pi, 2), y^{-1}r_2^{-1}) \} \in E(\Sigma[K_{|H|}])
\]
This shows that $\varphi$ is an isomorphism. Similarly we can show that $BCay(G/H, T)[H] \cong BCay(G, T_1)$. Now we have

\[ BCay(G, S_1) \cong BCay(G/H, S)[H] \cong BCay(G/H, T)[H] \cong BCay(G, T_1). \]

Since $G$ is a BCI-group, there exist $\alpha \in Aut(G)$ and $g \in G$ such that $T_1 = gS_1^\alpha$. Also $H$ is a characteristic subgroup of $G$ and so $\overline{\alpha} : G/H \to G/H$, defined by $(Hx)^{\overline{\alpha}} := Hx^\alpha$ is an automorphism. Now $T = T_1^n = g^n(S_1^n)^rn = g^nS^n$. This completes the proof. \( \square \)

**Lemma 2.3.** $A_5$ is not 6-BCI group.

**Proof.** It is shown in [2] that $Cay(A_5, S \setminus \{id\}) \cong Cay(A_5, T \setminus \{id\})$, where

- $S := \{(12435), (15342), (14)(25), (14)(35), (25)(34), \text{id}\}$,
- $T := \{(12435), (15342), (12)(45), (13)(24), (15)(34), \text{id}\}$,

and $S, T$ are not conjugate in $Aut(A_5)$. Also by [3, Lemma 2.9], $BCay(G, S) \cong BCay(G, T)$. Suppose, for a contradiction, that there exist $g \in A_5$ and $\sigma \in Aut(A_5)$ such that $S^\sigma = g^{-1}T$. Since $S$ and $T$ are not conjugate in $Aut(A_5)$, $g \neq \text{id}$. Also $\text{id} \in S$ implies that $g \in T$. We distinguish three cases:

**Case I.** $g = (12435)$ or $g = (15342)$ or $g = (15)(34)$. In the first case $(254) \in S^\sigma$, in the second case $(123) \in S^\sigma$ and in the third case $(245) \in S^\sigma$. So in all cases $S$ must have an element of order 3, which is a contradiction.

**Case II.** $g = (12)(45)$. Then $S^\sigma = \{(14)(35), (25)(34), \text{id}, (14523), (12534), (12)(45)\}$. Therefore $\{(12435)^\sigma, (15342)^\sigma\} = \{(14523), (12534)\}$. This implies that $\text{id} = (12435)^\sigma (15342)^\sigma = (243)$ or $(135)$, which is a contradiction.

**Case III.** $g = (13)(24)$. Then $S^\sigma = \{(15)(23), (14)(35), (13254), \text{id}, (14235), (13)(24)\}$. This implies that $\text{id} = (12435)^\sigma (15342)^\sigma = (152)$ or $(345)$, which is a contradiction.

Hence in all cases we obtain a contradiction. This completes the proof. \( \square \)

Recall that two elements $a$ and $b$ of a group $G$ are said to be fused or inverse-fused if there exists $\sigma \in Aut(G)$ such that $a = b^\sigma$ or $a = (b^{-1})^\sigma$, respectively. $G$ is an FIF-group if every pair of elements of the same order is fused or inverse-fused, see [4]. Now we are ready to prove that finite BCI-groups are solvable.

**Theorem 2.4.** Every finite BCI-group is solvable.
Proof. Let $G$ be an insolvable BCI-group. Then $G$ is a 2-BCI-group and so by [14] (see also [7], Lemma 2.4(2)) is a FIF-group. Now by [11], Corollary 1.3], $G = A \times B$, where $A$ and $B$ have coprime orders, $A$ is solvable and $B$ is one of the groups $PSL_2(q)$, $q = 5, 7, 8, 9$, $PSL_3(4)$, $Sz(8)$, $M_{11}$, $M_{23}$, $SL_2(q)$, $q = 5, 7, 9$ or $(C \times Sz(8)) \rtimes \mathbb{Z}_{3^m}$, where $s, m \geq 1$ and $C$ is an abelian group. Since $(|A|, |B|) = 1$, $A$ and $B$ are characteristic subgroups of $G$. Hence by Lemma 2.3, $B \cong G/A$ is a BCI-group. Since $|B| > 3$, $B$ is a 3-BCI-group. If $B$ is one of the simple non-abelian groups $PSL_2(q)$, $q = 5, 7, 8, 9$, $PSL_3(4)$, $Sz(8)$, $M_{11}$, $M_{23}$, then by [11], Theorem 1.1, $B = PSL_2(5) \cong A_5$, which is not a BCI-group, by Lemma 2.3. Now suppose that $B = SL_2(q)$, where $q = 5, 7, 9$. Since $PSL_2(q) = SL_2(q)/Z(SL_2(q))$ and $Z(SL_2(q))$ is a characteristic subgroup of $SL_2(q)$, so by Lemma 2.3, $PSL_2(q)$ must be a 3-BCI-group. Thus again by [11], Theorem 1.1, $q = 5$ which implies that $B = SL_2(5)$ is a BCI-group. So $A_5 \cong PSL_2(5)$ is a BCI-group, which is a contradiction, by Lemma 2.3. The remaining case is $B = (C \times Sz(8)) \rtimes \mathbb{Z}_{3^m}$. By [7], $Sz(8)$ (and so $B$) has two subgroups isomorphic to $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence Lemma 2.4 implies that $B$ is not a BCI-group. This completes the proof. □

By [11], Corollary 1.3(3), [13] (or [7], Lemma 2.4(2)) and Theorem 2.3, every finite BCI-group $G$ is a semidirect product of a group $A$ by a group $B$ such that $(|A|, |B|) = 1$, $A$ is a nilpotent FIF-group and every Sylow subgroup of $B$ is cyclic or $Q_8$. Note that by [7], Lemma 2.4(2)), $A$ is a 2-BCI-group and the structure of all Sylow subgroups of $A$ are given in [11], Corollary 1.3 (2)).

REFERENCES

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