UNICYCLIC GRAPHS WITH STRONG EQUALITY BETWEEN THE 2-RAINBOW DOMINATION AND INDEPENDENT 2-RAINBOW DOMINATION NUMBERS

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Abstract. A 2-rainbow dominating function (2RDF) on a graph $G = (V, E)$ is a function $f$ from the vertex set $V$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled. A 2RDF $f$ is independent (I2RDF) if no two vertices assigned nonempty sets are adjacent. The weight of a 2RDF $f$ is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The 2-rainbow domination number $\gamma_{r2}(G)$ (respectively, the independent 2-rainbow domination number $i_{r2}(G)$) is the minimum weight of a 2RDF (respectively, I2RDF) on $G$. We say that $\gamma_{r2}(G)$ is strongly equal to $i_{r2}(G)$ and denote by $\gamma_{r2}(G) \equiv i_{r2}(G)$, if every 2RDF on $G$ of minimum weight is an I2RDF.

In this paper we characterize all unicyclic graphs $G$ with $\gamma_{r2}(G) \equiv i_{r2}(G)$.

1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [8,13] for terminology and notation in graph theory. Specifically, let $G$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. A unicyclic graph is a connected graph containing exactly one cycle. A vertex of degree one is called a leaf, and its neighbor is called a support vertex. If $v$ is a support vertex, then $L_v$ will denote the set of all leaves adjacent to $v$. A support vertex $v$ is called a strong support vertex if $|L_v| > 1$. For $r, s \geq 1$, a double

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star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to $r$ leaves and the other to $s$ leaves.

For a positive integer $k$, a $k$-rainbow dominating function (kRDF) of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The weight of a kRDF $f$ is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The $k$-rainbow domination number of a graph $G$, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of $G$. A $\gamma_{rk}(G)$-function is a $k$-rainbow dominating function of $G$ with weight $\gamma_{rk}(G)$. Note that 1-rainbow domination number is the classical domination number $\gamma(G)$.

The $k$-rainbow domination was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors (see for example [3, 6, 11, 12]).

A $k$-rainbow dominating function $f$ is called an independent $k$-rainbow dominating function (abbreviated IkRDF) on $G$ if the set $\{v \in V \mid f(v) \neq \emptyset\}$ is independent. The independent $k$-rainbow domination number, denoted by $i_{rk}(G)$, is the minimum weight of an IkRDF on $G$. An independent $k$-rainbow dominating function with weight $i_{rk}(G)$ is called an $i_{rk}(G)$-function. The independent $k$-rainbow domination number is investigated in [1, 3].

Obviously each independent $k$-rainbow dominating function is a $k$-rainbow dominating function, and so $\gamma_{rk}(G) \leq i_{rk}(G)$. If $\gamma_{rk}(G) = i_{rk}(G)$, then every $i_{rk}(G)$-function is also a $\gamma_{rk}(G)$-function. However not every $\gamma_{rk}(G)$-function is an $i_{rk}(G)$-function, even when $\gamma_{rk}(G) = i_{rk}(G)$. We say that $\gamma_{rk}(G)$ and $i_{rk}(G)$ are strongly equal and denote by $\gamma_{rk}(G) \equiv i_{rk}(G)$, if every $\gamma_{rk}(G)$-function is an $i_{rk}(G)$-function.

The strong equality between two parameters was introduced by Haynes and Slater in [11] in the first. Also in [11] and [13], Haynes, Henning and Slater gave constructive characterizations of trees with strong equality between some domination parameters.

In this paper, we characterize all unicyclic graphs $G$ with $\gamma_{r2}(G) \equiv i_{r2}(G)$. For this aim, we use the constructive characterization of trees $T$ with $\gamma_{r2}(T) \equiv i_{r2}(T)$ provided recently by Amjadi et al. [11]. Below we present the procedure given in [11] to built such trees. Let $\mathcal{F}_1$ be the family of trees that can be obtained from $k \geq 1$ disjoint stars $K_{1,2}$ by adding either a new vertex $v$ or a path $uv$ and joining the centers of stars to $v$. Also let $\mathcal{F}_2$ be the family including $P_5$ and all trees obtained from $k \geq 2$ disjoint $P_3$ by adding either a new vertex $v$ or a path $uv$ and joining $v$ to a leaf of each $P_3$. If $T$ belongs to $\mathcal{F}_1 \cup \mathcal{F}_2 - \{P_5\}$ then we call the vertex $v$, the special vertex of $T$ and if $T = P_5$, then its support vertices are special vertices of $T$. We now define recursively a collection $\mathcal{F}$ of trees such that $K_{1,2} \in \mathcal{F}$, and if $T$ is any tree in $\mathcal{F}$, then we put in $\mathcal{F}$ any tree $T'$ that can be obtained from $T$ by any of the following seven operations:

- **Operation $O_1$:** If $z$ is a strong support vertex of $T \in \mathcal{F}$, then $O_1$ adds a new vertex $x$ and an edge $xz$.
- **Operation $O_2$:** If $z$ is a vertex of $T \in \mathcal{F}$, then $O_2$ adds a new tree $T_1 \in \mathcal{F}_1$ with special vertex $x$ and an edge $xz$ provided that if $x$ is a support vertex, then $\gamma_{r2}(T - z) \geq \gamma_{r2}(T)$.
- **Operation $O_3$:** If $z$ is a strong support vertex of $T \in \mathcal{F}$, then $O_3$ adds a path $zxy$. 

• Operation $O_4$: If $z$ is a vertex of $T \in \mathcal{F}$ which is adjacent to a support vertex of degree 2, then $O_4$ adds a path $zxy$.

• Operation $O_5$: If $z$ is a vertex of $T \in \mathcal{F}$ which is adjacent to a strong support vertex, then $O_5$ adds a path $zxyw$.

• Operation $O_6$: If $z$ is a vertex of $T \in \mathcal{F}$, then $O_6$ adds new tree $T_2 \in \mathcal{F}_2$ with special vertex $x$ and an edge $xz$ provided that if $x$ is a support vertex, then $\gamma_{r}(T-z) \geq \gamma_{r}(T)$.

• Operation $O_7$: If $z$ is a vertex of $T \in \mathcal{F}$ such that every $\gamma_{r}(T)$-function assigns $\emptyset$ to $z$, then $O_7$ adds the double star $S(1, 2)$ and an edge $zx$ where $x$ is a leaf of $S(1, 2)$ whose support vertex has degree 3.

**Theorem A** (Amjadi et al. [1]). Let $T$ be a tree. Then $i_{r}(T) \equiv \gamma_{r}(T)$ if and only if $T \in \mathcal{F} \cup \{K_1\}$.

2. Unicyclic graphs $G$ with $\gamma_{r}(G) \equiv i_{r}(G)$

We begin by giving some definitions and results that will be useful in our characterization. A vertex $v \in V(G)$ is called an empty vertex if $f(v) = \emptyset$ for every $\gamma_{r}(G)$-function $f$.

**Observation 2.1.** If $v \in V(G)$ is an empty vertex of $G$, then $\gamma_{r}(G) = \gamma_{r}(G-\{v\})$.

**Proof.** Since $v$ is an empty vertex of $G$, each $\gamma_{r}(G)$-function is clearly a 2RDF of $G-\{v\}$ implying that $\gamma_{r}(G-\{v\}) \leq \gamma_{r}(G)$. If $\gamma_{r}(G-\{v\}) < \gamma_{r}(G)$, then let $f$ be a $\gamma_{r}(G-\{v\})$-function and define the function $h$ by $h(v) = \{1\}$ and $h(x) = f(x)$ for $x \in V(G)-\{v\}$. Clearly $h$ is a 2RDF of $G$ of weight $\gamma_{r}(G)$ that leads to a contradiction because $v$ is an empty vertex of $G$. Thus $\gamma_{r}(G) = \gamma_{r}(G-\{v\})$. □

A pair $(x, y)$ of vertices of a tree $T \in \mathcal{F}$ is called a complementary pair, or c-pair for short, if there exists a $\gamma_{r}(T)$-function $f$ such that $\{f(x), f(y)\} = \{\{1\}, \{2\}\}$. For the pair $(x, y)$, let $G_{T_{x,y}}$ be the graph obtained from $T$ by adding a new vertex $z$ and edges $xz$ and $zy$. Further, if $z$ is an empty vertex of $G_{T_{x,y}}$, then $(x, y)$ is called a c-pair of type 1, and if there is a $\gamma_{r}(G_{T_{x,y}})$-function that assigns $\{1, 2\}$ to $z$, then $(x, y)$ is called a c-pair of type 2. Besides these two cases, $(x, y)$ is called a c-pair of type 3. The following observation is straightforward.

**Observation 2.2.** If $T \in \mathcal{F}$ and $x, y \in V(T)$ are a c-pair, then $\gamma_{r}(T) = \gamma_{r}(G_{T_{x,y}})$.

A vertex $v$ of a tree $T$ is said to be an independent vertex if (i) every $\gamma_{r}(T)$-function $f$ with $|f(v)| = 1$ is independent and there is at least one $\gamma_{r}(T)$-function with this property, (ii) there is no $\gamma_{r}(T)$-function $f$ such that $|f(v)| = 2$. We denote by $\mathcal{H}$ the set of all trees in $\mathcal{F}$ having an independent vertex.

In the next, we provide a constructive characterization of unicyclic graphs $G$ with $\gamma_{r}(G) \equiv i_{r}(G)$. Let $\mathcal{U}$ be the family of graphs $G$ such that $G$ is obtained from some trees in $\mathcal{F}$ by one of the operations $T_1, T_2, T_3, T_4$ or obtained from even number of trees in $\mathcal{H}$ by operation $T_5$.

• Operation $T_1$: Let $x$ and $y$ be two non-adjacent vertices of $T \in \mathcal{F}$. The Operation $T_1$ adds the edge $xy$ if the following two conditions hold:
(1) every $\gamma_{r2}(T)$-function assigns $\emptyset$ to at least one of the $x$ and $y$, and
(2) $T - u$, $(u \in \{x, y\})$ has no 2RDF $f$ of weight $\gamma_{r2}(T)$ such that $f$ is not independent and
\[ \bigcup_{v \in (N_{r2}(x) \cup \{x, y\})} f(v) = \{1, 2\}. \]

- **Operation $T_2$:** Assume $T_0, T_1, \ldots, T_t \in \mathcal{F}$, $x, y \in V(T_0)$ and $v_i \in V(T_i)$ for $i \in \{1, \ldots, t\}$ and let
(1) $(x, y)$ is a c-pair of type 1,
(2) for each $i \in \{1, \ldots, t\}$, $v_i$ is an empty vertex of $T_i$.
Then Operation $T_2$ adds a new vertex $z$ and edges $zx$, $zy$ and $zv_i$ for each $i \in \{1, \ldots, t\}$.

- **Operation $T_3$:** Assume $T_0, T_1, \ldots, T_t \in \mathcal{F}$, $x, y \in V(T_0)$ and $v_i \in V(T_i)$ for $i \in \{1, \ldots, t\}$ and let
(1) $(x, y)$ is a c-pair of type 2,
(2) $T_0 - \{x, y\}$ has no 2RDF $f$ of weight $\gamma_{r2}(T_0) - 1$ that is not independent and $j \in \bigl( \bigcup_{u \in N_{r2}(y) - \{x\}} f(u) \bigr) \cap \bigl( \bigcup_{u \in N_{r2}(x) - \{y\}} f(u) \bigr)$ for some $j \in \{1, 2\}$,
(3) for each $i \in \{1, \ldots, t\}$, $v_i$ is an empty vertex of $T_i$,
(4) $T_i - v_i \in \mathcal{F}$ for each $i \in \{1, \ldots, t\}$.
Then Operation $T_3$ adds a new vertex $z$ and edges $zx$, $zy$ and $zv_i$ for $i \in \{1, \ldots, t\}$.

- **Operation $T_4$:** Assume $T_0, T_1, \ldots, T_t \in \mathcal{F}$, $x, y \in V(T_0)$ and $v_i \in V(T_i)$ for $i \in \{1, \ldots, t\}$ such that
(1) $(x, y)$ is a c-pair of type 3,
(2) $T_0 - \{x, y\}$ has no 2RDF $f$ of weight $\gamma_{r2}(T_0) - 1$ that is not independent and $j \in \bigl( \bigcup_{u \in N_{r2}(y) - \{x\}} f(u) \bigr) \cap \bigl( \bigcup_{u \in N_{r2}(x) - \{y\}} f(u) \bigr)$ for $j \in \{1, 2\}$,
(3) $v_i$ is an empty vertex of $T_i$ for each $i \in \{1, \ldots, t\}$,
(4) for each $i \in \{1, \ldots, t\}$, $T_i - v_i$ has no 2RDF $g$ of weight $\gamma_{r2}(T_i)$ such that $g$ is not independent and $1 \in \bigcup_{u \in N_{r2}(v_i)} g(u)$ or $2 \in \bigcup_{u \in N_{r2}(v_i)} g(u)$.
Then Operation $T_4$ adds a new vertex $z$ and edges $zx$, $zy$ and $zv_i$ for $i \in \{1, \ldots, t\}$.

- **Operation $T_5$:** Let $t \neq 0$ be an even integer, $T_1, T_2, \ldots, T_t \in \mathcal{H}$, and $v_i \in V(T_i)$ be an independent vertex for each $i \in \{1, \ldots, t\}$.
Then Operation $T_5$ adds new vertices $u_1, u_2, \ldots, u_t$ and edges $u_iv_i$, for each $i \in \{1, \ldots, t\}$, $v_iu_{i+1}$ for each $i \in \{1, \ldots, t - 1\}$ and the edge $v_1u_1$.

**Lemma 2.3.** If $T$ is a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and $G$ is a unicyclic graph obtained from $T$ by Operation $T_1$, then $\gamma_{r2}(G) \equiv i_{r2}(G)$.

**Proof.** Let $T \in \mathcal{F}$ and let $x$ and $y$ be non-adjacent vertices of $T$ jointed by Operation $T_1$. Clearly every $\gamma_{r2}(T)$-function is a 2RDF of $G$ and hence $\gamma_{r2}(G) \leq i_{r2}(G) \leq \gamma_{r2}(T) = i_{r2}(T)$. Now we show that $\gamma_{r2}(G) = \gamma_{r2}(T)$. Suppose to the contrary that $\gamma_{r2}(G) < \gamma_{r2}(T)$ and let $f$ be a $\gamma_{r2}(G)$-function.
If $f(x) = f(y) = \emptyset$, or $\emptyset \notin \{f(x), f(y)\}$, then $f$ is a 2RDF of $T$ with weight less than $\gamma_{r2}(T)$, a contradiction. If $f(x) \neq \emptyset$ and $f(y) = \emptyset$ (the case $f(x) = \emptyset$ and $f(y) \neq \emptyset$ is similar), then the function $g$ defined by $g(y) = \{1\}, g(w) = f(w)$ for $w \in V(G) - \{y\}$ is a $\gamma_{r2}(T)$-function which contradicts (i). Hence $\gamma_{r2}(G) = i_{r2}(G) = \gamma_{r2}(T)$. It will now be shown that $\gamma_{r2}(G) \equiv i_{r2}(G)$. Suppose $g$ is a $\gamma_{r2}(G)$-function that is not independent. If $g(x) \neq \emptyset$ and $g(y) \neq \emptyset$ then $g$ is a $\gamma_{r2}(T)$-function, that contradicts (i). If $g(x) = g(y) = \emptyset$, then $g$ is a $\gamma_{r2}(T)$-function that is not independent, a contradiction with $T \in \mathcal{F}$. If $g(x) \neq \emptyset$ and $g(y) = \emptyset$ (the case $g(x) = \emptyset$ and $g(y) \neq \emptyset$ is similar), then $g$ is a 2RDF of
Let $T_0, T_1, \ldots, T_t \in \mathcal{F}$, $x, y \in V(T_0)$ be a c-pair of type 1 and $v_i \in V(T_i)$ is an empty vertex of $T_i$ for every $i \in \{1, \ldots, t\}$. If $G$ is a unicyclic graph obtained from $T_0, T_1, \ldots, T_t$ by Operation $T_2$, then $\gamma_{r_2}(G) \equiv i_{r_2}(G)$.

\textbf{Proof.} Assume $z$ is the new vertex added by Operation $T_2$ for obtaining $G$. First we show that $\gamma_{r_2}(G) = i_{r_2}(G)$. Let $f_i$ be a $\gamma_{r_2}(T_i)$-function for every $i \in \{0, \ldots, t\}$ and let $\{f_0(x), f_0(y)\} = \{\{1\}, \{2\}\}$. Define $h : V(G) \rightarrow \mathcal{P}(\{1, 2\})$ by $h(z) = \emptyset$ and $h(u) = f_i(u)$ for $u \in V(T_i)$ and $0 \leq i \leq t$. Since $T_i \in \mathcal{F}$, we deduce that $h$ is an I2RDF of $G$ with weight $\sum_{i=0}^t \gamma_{r_2}(T_i)$ implying that

\begin{equation}
\gamma_{r_2}(G) \leq i_{r_2}(G) = \sum_{i=0}^t \gamma_{r_2}(T_i) = \sum_{i=0}^t i_{r_2}(T_i).
\end{equation}

Assume $g$ is a $\gamma_{r_2}(G)$-function. We claim that $g(z) = \emptyset$. Suppose to the contrary that $g(z) \neq \emptyset$. If $\omega(g|_{V(T_j)}) < \gamma_{r_2}(T_j)$ for some $1 \leq j \leq t$, then the function $f$ defined by $f(v_j) = \{1\}$ and $f(u) = g(u)$ for $u \in V(T_j) - \{v_j\}$ is a 2RDF of $T_j$ with weight at most $\gamma_{r_2}(T_j)$ which leads to a contradiction because $v_j$ is an empty vertex of $T_j$. Thus $\omega(g|_{V(T_j)}) \geq \gamma_{r_2}(T_j)$ for each $j \in \{1, \ldots, t\}$. It follows from (2.1) that $\omega(g|_{V(T_0)}) \leq \gamma_{r_2}(T_0)$. This implies that $g_{|V(G_{T_{0,xy}})}$ is a $\gamma_{r_2}(G_{T_{0,xy}})$-function with $g(z) \neq \emptyset$, a contradiction. Hence $g(z) = \emptyset$. Thus $g|_{V(T_i)}$ is a 2RDF for each $i \in \{0, \ldots, t\}$. Therefore $\omega(g|_{V(T_j)}) \geq \gamma_{r_2}(T_i) = \gamma_{r_2}(T_i)$ for every $i \in \{0, \ldots, t\}$. Hence $\gamma_{r_2}(G) = \omega(g) \geq \sum_{i=0}^t \gamma_{r_2}(T_i)$. We claim that $\gamma_{r_2}(G) \equiv i_{r_2}(G)$. Assume $h$ is a $\gamma_{r_2}(G)$-function that is not independent. Using an argument similar to that described above, shows that $h(z) = \emptyset$ and the function $h|_{V(T_i)}$ is a $\gamma_{r_2}(T_i)$-function for each $i$. Since $h$ is not independent, we deduce that $h|_{V(T_i)}$ is not independent for some $i$, a contradiction with $T_i \in \mathcal{F}$. This completes the proof. \hfill \Box

\textbf{Lemma 2.5.} Let $T_0, T_1, \ldots, T_t \in \mathcal{F}$, $x, y \in V(T_0)$ and $v_i \in V(T_i)$ for $1 \leq i \leq t$ satisfy in the condition of Operation $T_3$. If $G$ is a unicyclic graph obtained from $T_0, T_1, \ldots, T_t$ by Operation $T_3$, then $\gamma_{r_2}(G) \equiv i_{r_2}(G)$.

\textbf{Proof.} Suppose $z$ is the vertex added by Operation $T_3$ for obtaining $G$. We first show that $\gamma_{r_2}(G) = i_{r_2}(G) = \sum_{i=0}^t \gamma_{r_2}(T_i)$. As in the proof of Lemma 2.4, we can see that

\begin{equation}
\gamma_{r_2}(G) \leq i_{r_2}(G) = \sum_{i=0}^t \gamma_{r_2}(T_i) = \sum_{i=0}^t i_{r_2}(T_i).
\end{equation}

Assume $g$ is a $\gamma_{r_2}(G)$-function. As in the proof of Lemma 2.4, we have $\omega(g|_{V(T_j)}) \geq \gamma_{r_2}(T_j)$ for each $j \in \{1, \ldots, t\}$. On the other hand, the function $g$, restricted to $G_{T_{0,xy}}$ is a 2RDF of $G_{T_{0,xy}}$ implying that $\omega(g|_{G_{T_{0,xy}}}) \geq \gamma_{r_2}(G_{T_{0,xy}}) = \gamma_{r_2}(T_0)$ by Observation 2.2. Thus $\gamma_{r_2}(G) \geq \sum_{i=0}^t \gamma_{r_2}(T_i)$. It follows from (2.2) that $\gamma_{r_2}(G) = \sum_{i=0}^t \gamma_{r_2}(T_i) = \sum_{i=0}^t i_{r_2}(T_i)$. Now we show that $\gamma_{r_2}(G) \equiv i_{r_2}(G)$. Assume $h$ is a $\gamma_{r_2}(G)$-function that is not independent. By above argument, $h|_{V(T_i)}$ is a $\gamma_{r_2}(T_i)$-function for each $i \in \{1, \ldots, t\}$. We consider two cases.
Case 1. $h(z) = \emptyset$.

Since $h$ is not independent, $h \big|_{V(T_i)}$ for some $i$, is a $\gamma_{r_2}(T_i)$-function that is not independent, a contradiction with $T_i \in \mathcal{F}$.

Case 2. $h(z) \neq \emptyset$.

If $h(v_i) \neq \emptyset$ for some $1 \leq i \leq t$, then $h \big|_{V(T_i)}$ is a $\gamma_{r_2}(T_i)$-function and $v_i$ is not an empty vertex of $T_i$, a contradiction. Thus $h(v_i) = \emptyset$ for every $i \in \{1, \ldots, t\}$. This implies that $h \big|_{V(T_i)-\{v_i\}}$ is a $\gamma_{r_2}(T_i-v_i)$-function, by Observation 1. If $h \big|_{V(T_i)-\{v_i\}}$ is not independent for some $i$, we obtain a contradiction with $T_i - v_i \in \mathcal{F}$. Henceforth, we let $h \big|_{V(T_i)-\{v_i\}}$ is independent for each $i$. Thus $h|_{G_{T_0}}$ is a $\gamma_{r_2}$-function that is not independent. Consider two subcases.

Subcase 2.1. $h(z) = \{1, 2\}$.

If $h(x) \neq \emptyset$ (the case $h(y) \neq \emptyset$ is similar), then the function $g$ defined by $g(y) = \{1\}$, $g(u) = h(u)$ for $u \in V(T_0) - \{x\}$ is a 2RDF of $T_0$ of weight less than $\gamma_{r_2}(T_0)$, a contradiction. Therefore, we assume $h(x) = h(y) = \emptyset$. Then the function $g$ defined by $g(x) = g(y) = \{1\}$, $g(u) = h(u)$ for $u \in V(T_0)-\{x, y\}$ is a $\gamma_{r_2}(T_0)$-function that is not independent, a contradiction.

Subcase 2.2. $|h(z)| = 1$.

We may assume without loss of generality that $h(z) = \{1\}$. If $\emptyset \not\subseteq \{h(x), h(y)\}$, then $h \big|_{V(T_0)}$ is a 2RDF of weight less than $\gamma_{r_2}(T_0)$, a contradiction. If $h(x) \neq \emptyset$ and $h(y) = \emptyset$ (the case $h(x) = \emptyset$ and $h(y) \neq \emptyset$ is similar), then $2 \in \bigcup_{u \in \mathcal{N}_{T_0}(y)} h(u) \neq \emptyset$ and the function $g$ defined by $g(y) = \{1\}$ and $g(u) = h(u)$ for $u \in V(T_0) - \{y\}$ is a 2RDF of weight $\gamma_{r_2}(T_0)$ that is not independent, a contradiction with $T_0 \in \mathcal{F}$. Hence, let $h(x) = h(y) = \emptyset$. Then $2 \in \bigcup_{u \in \mathcal{N}_{T_0}(x)} h(u)$ for $v \in \{x, y\}$. If $\bigcup_{u \in \mathcal{N}_{T_0}(x)} h(u) = \{1, 2\}$ (the case $\bigcup_{u \in \mathcal{N}_{T_0}(x)} h(u) = \{1\}$ is similar), then the function $g$ defined by $g(y) = \{1\}$ and $g(u) = h(u)$ for $u \in V(T_0)-\{y\}$ is a 2RDF of weight $\gamma_{r_2}(T_0)$ that is not independent which is a contradiction again. Thus $\bigcup_{u \in \mathcal{N}_{T_0}(x)} h(u) = \bigcup_{u \in \mathcal{N}_{T_0}(y)} h(u) = \{2\}$. But then $h \big|_{V(T_0)-\{x, y\}}$ is a 2RDF on $T_0 - \{x, y\}$ of weight $\gamma_{r_2}(T_0)-1$ that is not independent contradicting the assumption. This completes the proof. 

□

Lemma 2.6. Let $T_0, T_1, \ldots, T_t \in \mathcal{F}$, $x, y \in V(T_0)$ and $v_i \in V(T_i)$ for each $i \in \{1, \ldots, t\}$ satisfy in the conditions of Operation 4. If $G$ is the unicyclic graph obtain from $T_0, T_1, \ldots, T_t$ by Operation $\mathcal{T}_i$, then $\gamma_{r_2}(G) = i_{r_2}(G)$.

Proof. Suppose $z$ is the vertex added by Operation $\mathcal{T}_i$ to obtain $G$. As in the proof of Lemma 2.3, we obtain

$$\gamma_{r_2}(G) \leq i_{r_2}(G) \leq \sum_{i=0}^{t} \gamma_{r_2}(T_i).$$

Assume $g$ is a $\gamma_{r_2}(G)$-function. If $g(z) = \emptyset$ then $g|_{V(T_i)}$ is a 2RDF of $T_i$ for every $i \in \{0, \ldots, t\}$ and hence $\omega(g|_{V(T_i)}) \geq \gamma_{r_2}(T_i)$ for each $i \in \{0, \ldots, t\}$. Therefore $\gamma_{r_2}(G) = \omega(g) \geq \sum_{i=0}^{t} \gamma_{r_2}(T_i)$. Assume $g(z) \neq \emptyset$. Then $g|_{V(T_0) \cup \{z\}}$ is a 2RDF of $G_{T_0, y}$ and for each $i$ either $g|_{V(T_i)}$ is a 2RDF of $T_i$ or $g|_{V(T_i)-\{v_i\}}$ is a 2RDF of $T_i - v_i$. By Observations 2.3 and 2.2, we deduce that $\gamma_{r_2}(G) \geq \sum_{i=1}^{t} \gamma_{r_2}(T_i)$. It follows from (2.3) that $\gamma_{r_2}(G) = i_{r_2}(G) = \sum_{i=0}^{t} \gamma_{r_2}(T_i)$. Now we show that $\gamma_{r_2}(G) = i_{r_2}(G)$. Suppose to the contrary that $g$ is a $\gamma_{r_2}(G)$-function that is not independent. If $g(z) = \emptyset$, then as in the proof of Case
1 in Lemma 2(a) we obtain a contradiction. Let \( g(z) \neq \emptyset \). Since \( x \) and \( y \) are a c-pair of type 3, we have \( |g(z)| = 1 \). Assume, without loss of generality, that \( g(z) = \{1\} \). Since \( g \) is not independent, there are two vertices \( u, v \in V(G) \) such that \( \emptyset \nsubseteq \{f(u), f(v)\} \). Since \( \omega(g|_{V(T_i)}) = \gamma_{2r}(T_i) \) and \( v_i \) is an empty vertex of \( T_i \), we deduce that \( g(v_i) = \emptyset \) for each \( 1 \leq i \leq t \). It follows that \( 2 \notin \bigcup_{w \in N_{T_i}(v_i)} g(w) \) for each \( 1 \leq i \leq t \), and \( u, v \in V(G_{T_0}) \) or \( u, v \in V(T_i) \) for some \( 1 \leq i \leq t \). By condition 3, we deduce that \( u, v \in V(G_{T_0}) \). If \( g(x) = g(y) = \emptyset \), then we must have \( 2 \notin \bigcup_{w \in N_{T_0}(x)} f(w) \cap \bigcup_{w \in N_{T_0}(y)} f(w) \). Let without loss of generality that \( f(x) \neq \emptyset \). Then the function \( h : V(G) \to \mathcal{P}\{1, 2\} \) defined by \( h(y) = \{1\} \) and \( h(w) = g(w) \) for \( w \in V(T_0) - \{y\} \), is a \( \gamma_{2r}(T_0) \)-function that is not independent, a contradiction. Thus \( \gamma_{2r}(G) = i_r(G) \) and the proof is complete.

**Lemma 2.7.** Assume \( t \) is an even integer, \( T_1, T_2, \ldots, T_t \in \mathcal{H} \), and \( v_i \in V(T_i) \) is an independent vertex for each \( i \in \{1, \ldots, t\} \). If \( G \) is the unicyclic graph obtained from \( T_1, \ldots, T_t \) by Operation \( T_5 \), then \( \gamma_{2r}(G) = i_r(G) \).

**Proof.** We use the same notation as defined in Operation \( T_5 \). Suppose \( f_i \) is an independent \( \gamma_{2r}(T_i) \)-function such that \( |f(v_i)| = 1 \) for each \( i \). We may assume that \( f(v_i) = \{1\} \) if \( i \) is odd and \( f(v_i) = \{2\} \) if \( i \) is even. Define the function \( f : V(G) \to \mathcal{P}\{1, 2\} \) by \( f(u) = f_i(u) \) for \( u \in V(T_i) \) and \( f(u_i) = \emptyset \) for each \( i \in \{1, \ldots, t\} \). Clearly \( f \) is an I2RDF of \( G \) of weight \( \sum_{i=1}^{t} \gamma_{2r}(T_i) \) implying that

\[
\gamma_{2r}(G) \leq i_r(G) \leq \sum_{i=1}^{t} \gamma_{2r}(T_i).
\]

It will now be shown that \( \gamma_{2r}(G) = \sum_{i=1}^{t} \gamma_{2r}(T_i) \). Assume to the contrary that \( \gamma_{2r}(G) < \sum_{i=1}^{t} \gamma_{2r}(T_i) \) and let \( f \) be a \( \gamma_{2r}(G) \)-function such that \( \sum_{i=1}^{t} |f(u_i)| \) is as small as possible. We claim that \( f(u_i) \neq \{1, 2\} \) for each \( i \). Suppose to the contrary that \( f(u_i) = \{1, 2\} \) for some \( i \), say \( i = 2 \). Then the function \( g : V(G) \to \mathcal{P}\{1, 2\} \) defined by \( g(u_2) = \emptyset, g(v_1) = f(v_1) \cup \{1\}, g(v_2) = f(v_2) \cup \{2\} \) and \( g(x) = f(x) \) otherwise, is a 2RDF of \( G \) such that \( \sum_{i=1}^{t} |g(u_i)| < \sum_{i=1}^{t} |f(u_i)| \), a contradiction. Hence \( f(u_i) \neq \{1, 2\} \) for each \( i \). Since \( \gamma_{2r}(G) < \sum_{i=1}^{t} \gamma_{2r}(T_i) \), we deduce that \( \omega(f|_{V(T_i) \cup \{u_{i+1}\}}) < \gamma_{2r}(T_i) \) for some \( 1 \leq i \leq t \), say \( i = 1 \). If \( f|_{V(T_1)} \) is a 2RDF of \( T_1 \), then its weight is less than \( \gamma_{2r}(T_1) \) which is a contradiction. Let \( f|_{V(T_1)} \) is not a 2RDF of \( T_1 \). This implies that \( f(v_1) = \emptyset \). If \( f(u_2) \neq \emptyset \), then \( |f(u_2)| = 1 \) and the function \( h : V(T_1) \to \mathcal{P}\{1, 2\} \) defined by \( h(v_1) = f(u_2), h(x) = f(x) \) for \( x \in V(T_1) - \{v_1\} \) is a 2RDF of \( T_1 \) of weight less than \( \gamma_{2r}(T_1) \) which is a contradiction again. Thus \( f(u_2) = \emptyset \). Now to rainbow dominate \( v_1 \), we must have \( f(u_1) = \emptyset \). Since \( f(u_1) \neq \{1, 2\} \), we may assume without loss of generality that \( f(u_1) = \{1\} \). It follows that \( 2 \notin \bigcup_{w \in N_{T_1}(v_1)} f(w) \). But then the function \( h_1 : V(T_1) \to \mathcal{P}\{1, 2\} \) defined by \( h_1(v_1) = \{1\}, h_1(x) = f(x) \) for \( x \in V(T_1) - \{v_1\} \) is a \( \gamma_{2r}(T_1) \)-function such that \( h_1 \) is not independent and \( |h_1(v_1)| = 1 \), a contradiction by \( T_1 \in \mathcal{H} \). Thus \( \omega(f|_{V(T_1) \cup \{u_{i+1}\}}) \geq \gamma_{2r}(T_i) \) for each \( i \in \{1, \ldots, t\} \) and hence \( \gamma_{2r}(G) \geq \sum_{i=1}^{t} \gamma_{2r}(T_i) \). It follows from (2.3) that \( \gamma_{2r}(G) = \sum_{i=1}^{t} \gamma_{2r}(T_i) \) implying that \( \gamma_{2r}(G) = i_r(G) \). Now we show that \( \gamma_{2r}(G) = i_r(G) \). Suppose to the contrary that \( g \) is a \( \gamma_{2r}(G) \)-function that is not independent and let \( u \) and \( v \) be two vertices such that \( uv \in E(G) \) and \( \emptyset \notin \{g(u), g(v)\} \). We consider two cases.
Case 1. \(u, v \in V(T_1)\) for some \(1 \leq i \leq t\).

If \(|g(v_i)| = 1\) then \(g|_{V(T_1)}\) is a \(\gamma_r(T_1)\)-function that is not independent, contradicting (i) of the definition of independent vertex. If \(|g(v_i)| = 2\), then obviously \(g|_{V(T_1)}\) is a 2RDF of \(T_1\). Since \(T_1 \in \mathcal{H}\), it follows from the property (2) that \(g|_{V(T_1)}\) is not a \(\gamma_r(T_1)\)-function which implies \(\omega(g|_{V(T_1) \cup \{u_{i+1}\}}) \geq \omega(g|_{V(T_1)}) > \gamma_r(T_1)\). Since \(\omega(g|_{V(T_1) \cup \{u_{i+1}\}}) \geq \gamma_r(T_1)\) for each \(1 \leq i \leq t\), we obtain \(\gamma_r(G) > \sum_{i=1}^{t} \gamma_r(T_i)\), a contradiction. Henceforth, we assume \(|g(v_i)| = 0\). If \(\omega(g|_{V(T_1)}) < \gamma_r(T_1)\) then we define \(h(v_i) = 1\) and \(h(w) = g(w)\) for \(w \in V(T_1) - \{v_i\}\) to obtain a \(\gamma_r(T_1)\)-function that is not independent, a contradiction with the property (1). Let \(\omega(g|_{V(T_1)}) = \gamma_r(T_1)\). Then we must have \(g(u_{i+1}) = \emptyset\), otherwise \(\omega(g|_{V(T_1) \cup \{u_{i+1}\}}) \geq \omega(g|_{V(T_1)}) > \gamma_r(T_1)\) and we obtain a contradiction as above. It follows from \(g(u_{i+1}) = \emptyset\) and \(g(v_i) = \emptyset\) that \(g(v_{i+1}) = \{1, 2\}\). By the property (2), we have \(\omega(g|_{V(T_{i+1}) \cup \{u_{i+1}\}}) > \gamma_r(T_{i+1})\) which leads to a contradiction as above.

Case 2. \(\{v, u\} = \{v_i, u_{i+1}\}\) for some \(i\). (the case \(\{v, u\} = \{v_i, u_i\}\) is similar).

Assume, without loss of generality, that \(v = v_i\) and \(u = u_{i+1}\). Since \(g|_{V(T_1)}\) is a \(\gamma_r(T_2)\)-function and since \(g(u_{i+1}) \neq \emptyset\), we deduce that \(\omega(g|_{V(T_1) \cup \{u_i\}}) > \gamma_r(T_1)\). Since \(\omega(g|_{V(T_j) \cup \{u_{j+1}\}}) \geq \gamma_r(T_j)\) for each \(1 \leq j \leq t\), we obtain \(\gamma_r(G) > \sum_{i=1}^{t} \gamma_r(T_i)\), a contradiction. Thus \(g\) is independent and hence \(\gamma_r(G) \equiv i_r(G)\). This completes the proof.

In the next, we show that if \(G\) is a unicyclic graph with \(i_r(G) \equiv \gamma_r(G)\), then \(G \in \mathcal{U}\).

Theorem 2.8. If \(G\) is a unicyclic graph of order \(n \geq 3\), with \(i_r(G) \equiv \gamma_r(G)\), then \(G \in \mathcal{U}\).

Proof. Let \(C = (u_1, u_2, \ldots, u_k)\) be the unique cycle of \(G\) and let

\[
C_0 = \{u \in V(C) \mid \text{there is a } \gamma_r(G) - \text{function that assigns } \emptyset \text{ to } u\}.
\]

Clearly \(C_0 \neq \emptyset\) because \(\gamma_r(G) \equiv i_r(G)\). Also let \(C_1 = \{u_i \in C_0 \mid \text{there is a } \gamma_r(G) \text{ - function } f \text{ with } f(u_i) = \emptyset \text{ such that } f(u_{i-1}) \subseteq f(u_{i+1}) \text{ or } f(u_{i+1}) \subseteq f(u_{i-1})\}\). We consider two cases.

Case 1. \(C_1 \neq \emptyset\).

Assume, without loss of generality, that \(u_2 \in C_1\). By assumption, \(G\) has an \(i_r(G)\)-function \(f\) such that \(f(u_2) = \emptyset\) and \(f(u_1) \subseteq f(u_3)\) or \(f(u_3) \subseteq f(u_1)\). Suppose, without loss of generality, that \(f(u_1) \subseteq f(u_3)\). Let \(T = G - u_1u_2\). Clearly \(T\) is a tree. We show that \(T \in \mathcal{F}\) and \(G\) can be obtained from \(T\) by Operation \(\mathcal{T}_1\). Obviously \(f\) is a 2RDF of \(T\) that yields \(\gamma_r(T) \leq i_r(T) \leq \gamma_r(G)\). On the other hand, every \(\gamma_r(T)\)-function is a 2RDF of \(G\) implying that \(\gamma_r(T) \geq \gamma_r(G)\). Therefore \(\gamma_r(T) = i_r(T) = \gamma_r(G)\). If \(T\) has a non independent \(\gamma_r(T)\)-function \(f\), then \(f\) is \(\gamma_r(G)\)-function that is not independent, a contradiction with \(i_r(G) \equiv \gamma_r(G)\). This yields \(i_r(T) \equiv \gamma_r(T)\). If \(T\) has a \(\gamma_r(T)\)-function \(g\) such that \(\emptyset \notin \{g(u_1), g(u_2)\}\), then \(g\) is a \(\gamma_r(G)\)-function that is not independent, a contradiction again. Hence \(T\) satisfies (i) of Operation \(\mathcal{T}_1\). Suppose \(u \in \{u_1, u_2\}\). If \(T - u\) has a 2RDF \(g\) of weight \(\gamma_r(T)\) such that \(g\) is not independent and \(\cup_{v \in N_T(u) \cup \{u_1, u_2\} - \{u\}} g(v) = \{1, 2\}\), then the function \(h : V(G) \rightarrow \mathcal{P}\{\{1, 2\}\}\) defined by \(h(u) = \emptyset\) and \(h(w) = g(w)\) for \(w \in V(G) - \{u\}\) is a 2RDF of \(G\) of weight \(\gamma_r(G)\) that is not independent, a contradiction with \(i_r(G) \equiv \gamma_r(G)\). Therefore \(T\) satisfies (ii) of Operation \(\mathcal{T}_1\). Thus \(G\) can be obtained from \(T\) by applying Operation \(\mathcal{T}_1\).
Case 2. $C_1 = \emptyset$.

Assume, without loss of generality, that $u_2 \in C_0$ and $\deg(u_2) \geq \deg(x)$ for every $x \in C_0$. Since $C_1 = \emptyset$, for every $\gamma_2(G)$-function $g$ with $g(u_2) = 0$, we have $g(u_1) \notin g(u_3)$ and $g(u_3) \notin g(u_1)$. This implies \{g(u_1), g(u_3)\} = \{\{1\}, \{2\}\}. Let $f$ be a $\gamma_2(G)$-function with $f(u_2) = 0$, $f(u_1) = \{1\}$ and $f(u_3) = \{2\}$. We consider the following subcases.

Subcase 2.1. $\deg(u_2) \geq 3$.

Let $N(u_2) = \{u_1, u_3\} = \{v_1, v_2, \ldots, v_t\}$ and let $T_0, T_1, \ldots, T_t$ be the component of $G - u_2$ such that $u_1, u_3 \in V(T_0)$ and $v_i \in V(T_i)$ for each $i \in \{1, \ldots, t\}$. If $f(v_j) \neq \emptyset$ for some $1 \leq j \leq t$, then $f(u_1) \subseteq f(v_j)$ or $f(u_3) \subseteq f(v_j)$, say $f(u_1) \subseteq f(v_j)$, and an argument similar to that described in Case 1, shows that $G$ can be obtained from $T = G - u_1 u_2$ by Operation $T_1$. Henceforth, we assume that $f(v_i) = \emptyset$ for each $i \in \{1, \ldots, t\}$. Therefore we may assume for any $\gamma_2(G)$-function $g$ with $g(u_2) = 0$, $g(v_i) = \emptyset$ for each $i \in \{1, \ldots, t\}$. We claim that $T_i \in \mathcal{F}$, $\omega(f|_{V(T_i)}) = \gamma_2(T_i)$ and $v_i$ is an empty vertex of $T_i$ for each $i \in \{1, \ldots, t\}$. Since $i_2(G) \equiv \gamma_2(G)$ and since $f(u_2) = \emptyset$, the function $f|_{V(T_i)}$ is an I2RDF of $T_i$ and hence $\gamma_2(T_i) \leq i_2(T_i) \leq \omega(f|_{V(T_i)})$ for each $0 \leq i < t$. If $\gamma_2(T_i) < \omega(f|_{V(T_i)})$ for some $i$, then let $g_1$ be a $\gamma_2(T_i)$-function and define $h : V(G) \rightarrow \mathcal{P}(\{1, 2\})$ by $h(u) = f(u)$ for $u \in V(G) - V(T_i)$ and $h(u) = g_1(u)$ for $u \in V(T_i)$. Obviously $h$ is a 2RDF of $G$ with weight less than $\gamma_2(G)$, a contradiction. So $\gamma_2(T_i) = \omega(f|_{V(T_i)})$ for each $i \in \{0, \ldots, t\}$. If $T_i \notin \mathcal{F}$ for some $i$, then let $g_2$ be a $\gamma_2(T_i)$-function that is not independent and define $h_1 : V(G) \rightarrow \mathcal{P}(\{1, 2\})$ by $h_1(u) = f(u)$ for $u \in V(G) - V(T_i)$ and $h_1(u) = g_2(u)$ for $u \in V(T_i)$. Clearly $h_1$ is a $\gamma_2(G)$-function that is not independent, a contradiction with $\gamma_2(G) \equiv i_2(G)$. Similarly, we can see that $v_i$ is an empty vertex of $T_i$ for each $i \in \{1, \ldots, t\}$. If $u_2$ is an empty vertex of $G$ then $u_1$ and $u_3$ are a $c$-pair of type 1 of $T_0$ and $G$ can be obtained by using Operation $T_2$. Now suppose there is a $\gamma_2(G)$-function $h$ that assigns \{1, 2\} to $u_2$. Then $u_1$ and $u_3$ are a $c$-pair of type 2. If $T_i - v_i$, for some $1 \leq i \leq t$, has a 2RDF $g$ of weight $\gamma_2(T_i)$ that is not independent, then define $h_1(u) = h(u)$ for $u \in V(G) - V(T_i)$ and $h_1(u) = g(u)$ for $u \in V(T_i - v_i)$ and $h_1(v_i) = \emptyset$ to obtain a non independent $\gamma_2(G)$-function, a contradiction with $\gamma_2(G) \equiv i_2(G)$. Thus $T_i - v_i \in \mathcal{F}$ for each $i \in \{1, \ldots, t\}$. If $T_0 - \{u_1, u_3\}$ has a 2RDF $g$ of weight $\gamma_2(T_0) - 1$ such that $g$ is not independent and $j \in (\cup_{u \in N_{T_0}(u_1)} g(u)) \cap (\cup_{u \in N_{T_0}(u_3)} g(u))$ for some $j \in \{1, 2\}$, then define $h_2(u) = g(u)$ for $u \in V(T_0) - \{u_1, u_3\}$, $h_2(u_1) = h_2(u_3) = \emptyset$, $h_2(\{1, 2\}) = \{j\}$ and $h_2(u) = g_i(u)$ for $u \in V(T_i)$, $1 \leq i \leq t$, where $g_i$ is arbitrary $\gamma_2(T_i)$-function to obtain a non independent $\gamma_2(G)$-function, a contradiction again. Thus the conditions of Operation $T_3$ holds and $G$ can be obtain from $T_0, T_1, \ldots, T_t$ by Operation $T_3$. Finally, let there is a $\gamma_2(G)$-function $h$ that assign $\{1\}$ (the case $\{2\}$ is similar) to $u_2$. Then $u_1$ and $u_3$ are a $c$-pair of type 3. Since $\gamma_2(G) \equiv i_2(G)$, $h(v_i) = \emptyset$ for each $1 \leq i \leq t$. It follows that $2 \in N_{T_i}(v_i)$ for each $i \in \{1, \ldots, t\}$. If $\omega(h|_{V(T_i)}) < \gamma_2(T_i)$ for some $i$, then the function $k : V(T_i) \rightarrow \mathcal{P}(\{1, 2\})$ defined by $k(v_i) = \{1\}$ and $k(w) = h(w)$ for $w \in V(T_i) - \{v_i\}$, is a $\gamma_2(T_i)$-function that is not independent, a contradiction. Hence, $\omega(h|_{V(T_i)}) = \gamma_2(T_i)$ for each $i$. We claim that for each $i$, $T_i - v_i$ has no 2RDF $g$ of weight $\gamma_2(T_i)$ that it is not independent and $j \in \cup_{u \in N_{T_i}(v_i)} g(u)$ for some $j \in \{1, 2\}$. Assume to the contrary that $T_i - v_i$ has a 2RDF $g$ of weight $\gamma_2(T_i)$ such that $j \in \cup_{u \in N_{T_i}(v_i)} g(u)$ ($j \in \{1, 2\}$) and $g$ is not independent for some $i$. If $j = 2$, then the
function $h_3$ defined by $h_3(v_i) = \emptyset, h_3(u) = h(u)$ for $u \in V(G) - V(T_i)$ and $h_3(u) = g(u)$ for $u \in V(T_i)$, is a $\gamma_{r2}(G)$-function that is not independent, a contradiction with $\gamma_{r2}(G) \equiv i_{r2}(G)$. If $j = 1$, the defined $h_4$ by $h_4(v_i) = \emptyset, h_4(u) = h(u)$ for $u \in V(G) - V(T_i)$, $h_4(u) = \{1, 2\} \setminus g(u)$ if $u \in V(T_i)$ and $|g(u)| = 1$, and $h_4(u) = g(u)$ when $u \in V(T_i)$ and $|g(v_i)| \neq 1$, to obtain a $\gamma_{r2}(G)$-function that is not independent, a contradiction again. This proves the claim. As above we can see that $T_0 - \{u_1, u_3\}$ has no 2RDF $g$ of weight $\gamma_{r2}(T_0) - 1$ that is not independent and $j \in (\cup_{u \in N_{T_0}(u_1)} f(u)) \cap (\cup_{u \in N_{T_0}(u_2)} f(u))$ for some $j \in \{1, 2\}$. Therefore $G$ can be obtained from $T_0, T_1, \ldots, T_4$ by Operation $T_4$.

**Subcase 2.2.** $\deg(u_2) = 2$.
Assume, without loss of generality, that $f(u_1) = \{1\}$ and $f(u_3) = \{2\}$. Since $\gamma_{r2}(G) \equiv i_{r2}(G)$, $f$ is independent and hence $f(u_4) = \emptyset$. By the choice of $u_2$ and the fact that $C_1 = \emptyset$, we have $\deg(u_4) = 2$ and $f(u_5) = \{1\}$. Repeating this process, we deduce that $f(u_2m) = \emptyset, f(u_{2m-1}) = \{1\}$ and $f(u_{2m+1}) = \{2\}$ for each integer $m \geq 1$. It follows that $k = 4s$ for some $s$ and every vertex of $C$ with even index belongs to $C_0$. By the choice of $u_2$, we deduce that $\deg(u_2) = \cdots = \deg(u_k) = 2$.

First let $C_0 - \{u_2, u_4, \ldots, u_k\} \neq \emptyset$. Assume, without loss of generality, that $u_3 \in C_0 - \{u_2, \ldots, u_k\}$. By the choice of $u_2$, $\deg(u_3) = 2$. Using an argument similar to that described in this subcase, we obtain $\{u_1, u_3, \ldots, u_{k-1}\} \subseteq C_0$ and $\deg(u_1) = \cdots = \deg(u_{k-1}) = 2$. Since $G$ is connected, we deduce that $G = C_4$. Now we can obtain $G$ from 2s single vertices by Operation $T_5$. Now let $C_0 = \{u_2, u_4, \ldots, u_k\}$. Assume $T_1, T_3, \ldots, T_{4s-1}$ are the components of $G - \{u_2, \ldots, u_k\}$ and $u_{2m-1} \in V(T_{2m-1})$ for $1 \leq m \leq 2s$. First we show that $\gamma_{r2}(T_{2m-1}) = \omega(f|_{V(T_{2m-1})})$ for each $m$. Clearly $f|_{V(T_{2m-1})}$ is an independent 2RDF of $T_{2m-1}$ and hence $\gamma_{r2}(T_{2m-1}) \leq \omega(f|_{V(T_{2m-1})})$ for each $m$. Assume to the contrary that $\gamma_{r2}(T_{2m-1}) < \omega(f|_{V(T_{2m-1})})$ for some $m$ and let $g$ be a $\gamma_{r2}(T_{2m-1})$-function. If $|g(v_{2m-1})| \geq 1$, then we can assume that $f(v_{2m-1}) \subseteq g(v_{2m-1})$. Define the function $h$ by $h(u) = g(u)$ if $u \in V(T_{2m-1})$ and $h(u) = f(u)$ otherwise, to obtain a 2RDF of $G$ of weight less than $\gamma_{r2}(G)$, a contradiction. If $|g(v_{2m-1})| = 0$, then $|\cup_{u \in N_{T_{2m-1}}(v_{2m-1})} g(u)| = 2$. Define the function $h$ by $h(u) = f(u)$ for $u \in V(G) - V(T_{2m-1} - v_{2m-1})$ and $h(u) = g(u)$ otherwise to obtain a $\gamma_{r2}(G)$-function that is not independent, a contradiction with $\gamma_{r2}(G) \equiv i_{r2}(G)$. Thus $\gamma_{r2}(T_{2m-1}) = \omega(f|_{V(T_{2m-1})})$. Now we show that $T_1, T_3, \ldots, T_{4s-1} \in \mathcal{H}$. Note that $f|_{V(T_{2m-1})}$ is an $i_{r2}(T_{2m-1})$-function with $|f(v_{2m-1})| = 1$ for $1 \leq m \leq 2s$. Let $g$ be an $i_{r2}(T_{2m-1})$-function such that $|g(v_{2m-1})| = 1$. Let without loss of generality that $g(v_{2m-1}) = f(v_{2m-1})$. Then the function $h$ defined by $h(u) = g(u)$ if $u \in V(T_{2m-1})$ and $h(u) = f(u)$ otherwise, is an $i_{r2}(G)$-function. Since $\gamma_{r2}(G) \equiv i_{r2}(G)$, $h$ is independent and hence $g$ is independent. Thus $T_{2m-1}$ satisfies (i) in the definition of independent vertex. If there is a $\gamma_{r2}(T_{2m-1})$-function $g$ with $|g(v_{2m-1})| = 2$, then $v_{2m-2}, v_{2m} \in C_1$ a contradiction with $C_1 = \emptyset$. Thus $T_{2m-1}$ satisfies (ii) in the definition of independent vertex, for each $m$. Thus $v_{2m-1}$ is an independent vertex of $T_{2m-1}$ and hence $T_1, T_3, \ldots, T_{4s-1} \in \mathcal{H}$. Thus $G$ can be obtain from $T_1, T_3, \ldots, T_{4s-1}$ by Operation $T_5$. This completes the proof. □
Now we are ready to state the main theorem of this paper.

**Theorem 2.9.** Let $G$ be a unicyclic graph. Then $i_{r2}(G) \equiv \gamma_{r2}(G)$ if and only if $G \in \mathcal{U}$.

**REFERENCES**


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