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A BOUND FOR THE LOCATING CHROMATIC NUMBER OF TREES

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Abstract. Let f be a proper k-coloring of a connected graph G and $\Pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple $c_\Pi(v) = (d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$, where $d(v, V_i) = \min\{d(v, x) : x \in V_i\}, 1 \le i \le k$. If distinct vertices have distinct color codes, then f is called a locating coloring. The minimum number of colors needed in a locating coloring of G is the locating chromatic number of G, denoted by $\chi_{_L}(G)$. In this paper, we study the locating chromatic numbers of trees. We provide a counter example to a theorem of Gary Chartrand et al. [G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, P. Zhang, The locating-chromatic number of a graph, Bull. Inst. Combin. Appl. 36 (2002) 89-101] about the locating chromatic number of trees. Also, we offer a new bound for the locating chromatic number of trees. Then, by constructing a special family of trees, we show that this bound is best possible.

1. Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper k-coloring of G, $k \in \mathbb{N}$, is a function f defined from $V(G)$ onto a set of colors $[k] = \{1, 2, \ldots, k\}$ such that every two adjacent vertices have different colors. In fact, for every i, $1 \le i \le k$, the set $f^{-1}(i)$ is a nonempty independent set of vertices which is called the color class i. When $S \subseteq V(G)$, then $f(S) = \{f(u) : u \in S\}$. The minimum cardinality k for which G has a proper k-coloring is the chromatic number of G, denoted by $\chi(G)$. For a connected graph G, the distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them, and for a subset S of

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$V(G)$, the distance between u and S is given by $d(u,S) = \min\{d(u,x) : x \in S\}$. The diameter of G is $\max\{d(u,v) : u,v \in V(G)\}$. When u is a vertex of G, then the neighbor of u in G is the set $N_G(u) = \{v : v \in V(G), d(u,v) = 1\}$. The degree of u and the maximum degree of vertices of G are given by $\deg(u) = |N_G(u)|$ and $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$, respectively.

Definition 1.1. [4] *Let f be a proper k-coloring of a connected graph G and $\Pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the* **color code** *of v with respect to Π is defined to be the ordered k-tuple*

$$c_\Pi(v) = (d(v,V_1), d(v,V_2), \ldots, d(v,V_k)).$$

If distinct vertices of G have distinct color codes, then f is called a **locating coloring** *of G. The* **locating chromatic number**, *denoted by $\chi_L(G)$, is the minimum number of colors in a locating coloring of G.*

The concept of locating coloring was first introduced and studied by Chartrand et al. in [4]. They established some bounds for the locating chromatic number of a connected graph. They also proved that for a connected graph G with $n \geq 3$ vertices, we have $\chi_L(G) = n$ if and only if G is a complete multipartite graph. Hence, the locating chromatic number of the complete graph K_n is n. Also for paths and cycles of order $n \geq 3$ it is proved in [4] that $\chi_L(P_n) = 3$, $\chi_L(C_n) = 3$ when n is odd, and $\chi_L(C_n) = 4$ when n is even.

The locating chromatic numbers of trees, Kneser graphs, Cartesian product of graphs, and the amalgamation of stars are studied in [4], [3], [2], and [1], respectively. For more results in the subject and related subjects, see [5] to [10].

Obviously, $\chi(G) \leq \chi_L(G)$. Note that the i-th coordinate of the color code of each vertex in the color class V_i is zero and its other coordinates are non zero. Hence, a proper coloring is a locating coloring whenever the color codes of vertices in each color class are different.

In this paper, we investigate the relation between the locating chromatic number of a tree and its maximum degree. We provide a counter example to a theorem established in [4] about the locating chromatic numbers of trees. Then, we offer a new bound which is tight and best possible.

2. Maximum degree and locating chromatic number

For investigating the relation between the locating chromatic number of a tree and its maximum degree, the following theorem is obtained by Chartrand et al. [4].

Theorem 2.1. [4] *Let $k \geq 3$. If T is a tree for which $\Delta(T) > (k-1)2^{k-2}$, then $\chi_L(T) > k$.*

They claimed that the bound in Theorem 2.1 is tight and cannot be improved. For this reason, in [4] they constructed a tree (for the special case $k = 4$) with maximum degree $(4-1)2^{4-2} = 12$ whose locating chromatic number is 4.

Actually, Theorem 2.1 is incorrect and its bound is not tight when k is an integer larger than 4. For instance, Theorem 2.1 implies that the locating chromatic number of each tree with maximum degree larger than $(5-1)2^{5-2}=32$ is at least 6. But Example 2.2 provides a tree with maximum degree 36 whose locating chromatic number is (at most) 5.

Example 2.2. *Consider the labeled tree T_5 which is illustrated in Figure 1. A proper vertex 5-coloring of T_5, call it f_5, is illustrated in Figure 2. For each i, $1 \le i \le 5$, let V_i be the set of vertices of T_5 with color i, and let $\Pi=(V_1,V_2,\dots,V_5)$. Color classes and corresponding color codes of the vertices of T_5 are illustrated in Table 1. Since distinct vertices have distinct color codes, f_5 is a locating 5-coloring of T_5 and hence, $\chi_{_L}(T_5)\le 5$.*

Fore more details about the tree T_5 and its coloring f_5, one can see the arguments before Theorem 2.5. Moreover, since

$$\Delta(T_5)=\deg(x)=36>4\times 3^{4-3},$$

Theorem 2.4 implies that $\chi_{_L}(T_5)>4$. Therefore, $\chi_{_L}(T_5)=5$.

Lemma 2.3. *Let $k\ge 3$ be an integer. Then,*

$$\max\{p2^{p-1}3^q:\ 1+p+q=k,\ p\in\mathbb{N},\ q\in\mathbb{N}\cup\{0\}\}=4\times 3^{k-3}.$$

Moreover, the maximum value happens just for $p\in\{2,3\}$.

Proof. It can be easily checked that the statements of the theorem hold for $k\in\{3,4\}$. Hereafter suppose that $k\ge 5$. Since $1+p+q=k$, we have $q=k-1-p$. Therefore, if $p=2$, then $q=k-3$ and $p2^{p-1}3^q=4\times 3^{k-3}$. Also, if $p=3$, then $q=k-4$ and $p2^{p-1}3^q=4\times 3^{k-3}$. Now for each $p>3$ we have

$$\begin{aligned}(p-1)2^{p-2}3^{q+1}-p2^{p-1}3^q &= 2^{p-2}3^q(3(p-1)-2p)\\ &= 2^{p-2}3^q(p-3)>0.\end{aligned}$$

This means $(p-1)2^{p-2}3^{q+1}>p2^{p-1}3^q$, which completes the proof. □

Theorem 2.4. *Let $k\ge 3$ be an integer. If T is a tree with $\chi_{_L}(T)=k$, then $\Delta(T)\le 4\times 3^{k-3}$.*

Proof. Assume that T is a tree with locating chromatic number k and $f:V(T)\to[k]$ is a locating k-coloring of T. Let x be a vertex of maximum degree in T. Without loss of generality and by renaming the symbols $1,2,\dots,k$, if it is necessary, we can assume that $f(x)=1$ and $f(N_T(x))=\{2,3,\dots,p+1\}$ for some integer p, $1\le p<k$. For each i, $1\le i\le k$, $V_i=\{v:\ v\in V(T),\ f(v)=i\}$ is the i-th color class of T and $\Pi=(V_1,V_2,\dots,V_k)$ is an ordered partition of $V(T)$ into the resulting color classes. The color code of x is $c_{\Pi}(x)=(d_1,d_2,\dots,d_k)$, where for each $j\in\{1,2,\dots,k\}$, $d_j=d(x,V_j)$. Note that $d_1=0$, $d_2=d_3=\cdots=d_{p+1}=1$, and $d_j\ge 2$ for each $j>p+1$. Let $i\in\{2,3,\dots,p+1\}$ be an arbitrary integer and $y\in V_i\cap N_T(x)$. The color code of y is $c_{\Pi}(y)=(c_1,c_2,\dots,c_k)$ where $c_1=1$ and $c_i=0$. Also, for each $r\in\{2,3,\dots,p+1\}\setminus\{i\}$, $c_r\in\{1,2\}$, and for each $j\in\{p+2,p+3,\dots,k\}$, $c_j\in\{d_j-1,d_j,d_j+1\}$. Note that

$$|\{2,3,\dots,p+1\}\setminus\{i\}|=p-1,\ \ |\{p+2,p+3,\dots,k\}|=k-1-p.$$

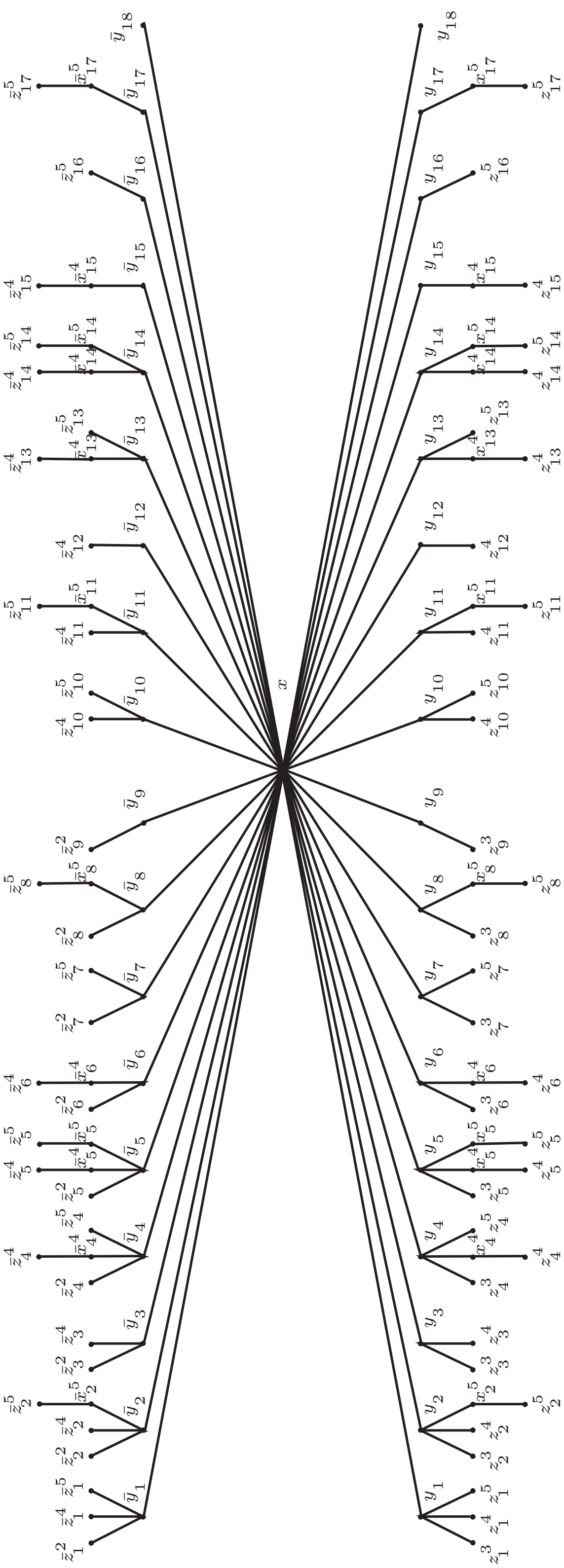

Figure 1. The tree T_5 with maximum degree 36.

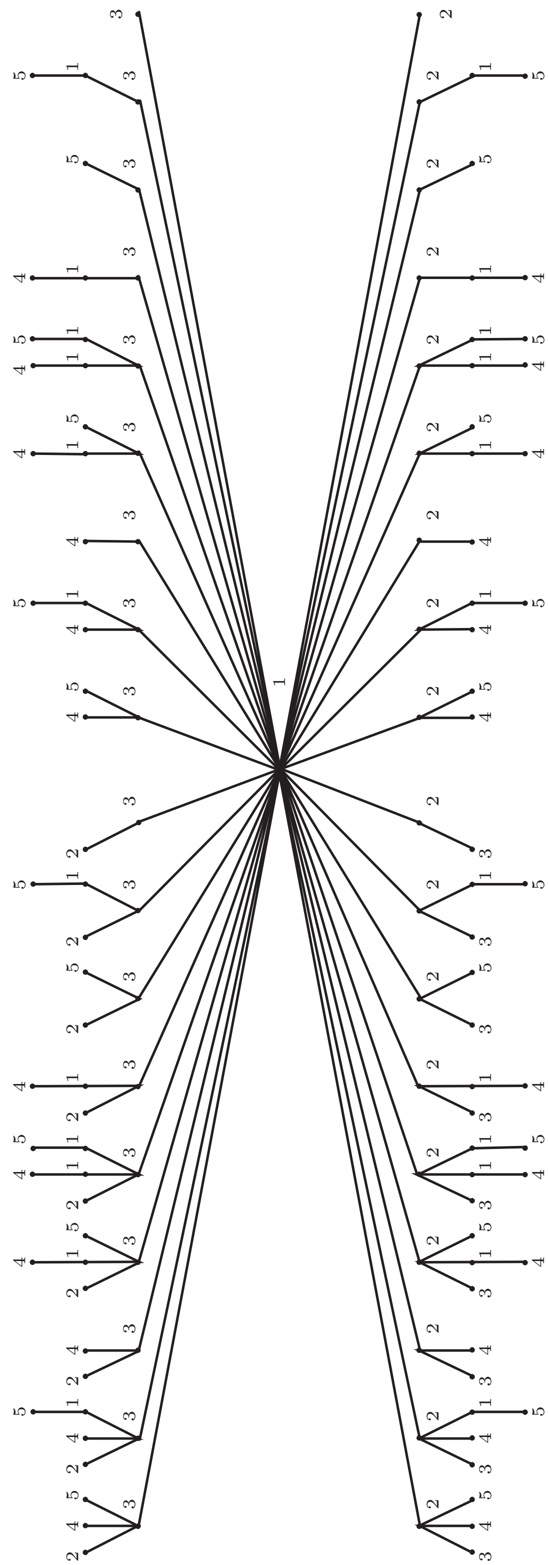

FIGURE 2. The locating 5-coloring f_5 of the tree T_5.

TABLE 1. The color classes and corresponding color codes of T_5 with respect to the locating 5-coloring f_5.

V_1	V_2	V_3	V_4	V_5
x $(0,1,1,2,2)$	y_1 $(1,0,1,1,1)$	$\bar{y}_1$ $(1,1,0,1,1)$	z_1^4 $(2,1,2,0,2)$	z_1^5 $(2,1,2,2,0)$
x_2^5 $(0,1,2,2,1)$	y_2 $(1,0,1,1,2)$	$\bar{y}_2$ $(1,1,0,1,2)$	z_2^4 $(2,1,2,0,3)$	z_2^5 $(1,2,3,3,0)$
x_4^4 $(0,1,2,1,2)$	y_3 $(1,0,1,1,3)$	$\bar{y}_3$ $(1,1,0,1,3)$	z_3^4 $(2,1,2,0,4)$	z_4^5 $(2,1,2,3,0)$
x_5^4 $(0,1,2,1,3)$	y_4 $(1,0,1,2,1)$	$\bar{y}_4$ $(1,1,0,2,1)$	z_4^4 $(1,2,3,0,3)$	z_5^5 $(1,2,3,4,0)$
x_5^5 $(0,1,2,3,1)$	y_5 $(1,0,1,2,2)$	$\bar{y}_5$ $(1,1,0,2,2)$	z_5^4 $(1,2,3,0,4)$	z_7^5 $(2,1,2,4,0)$
x_6^4 $(0,1,2,1,4)$	y_6 $(1,0,1,2,3)$	$\bar{y}_6$ $(1,1,0,2,3)$	z_6^4 $(1,2,3,0,5)$	z_8^5 $(1,2,3,5,0)$
x_8^5 $(0,1,2,4,1)$	y_7 $(1,0,1,3,1)$	$\bar{y}_7$ $(1,1,0,3,1)$	z_{10}^4 $(2,1,3,0,2)$	z_{10}^5 $(2,1,3,2,0)$
x_{11}^5 $(0,1,3,2,1)$	y_8 $(1,0,1,3,2)$	$\bar{y}_8$ $(1,1,0,3,2)$	z_{11}^4 $(2,1,3,0,3)$	z_{11}^5 $(1,2,4,3,0)$
x_{13}^4 $(0,1,3,1,2)$	y_9 $(1,0,1,3,3)$	$\bar{y}_9$ $(1,1,0,3,3)$	z_{12}^4 $(2,1,3,0,4)$	z_{13}^5 $(2,1,3,3,0)$
x_{14}^4 $(0,1,3,1,3)$	y_{10} $(1,0,2,1,1)$	$\bar{y}_{10}$ $(1,2,0,1,1)$	z_{13}^4 $(1,2,4,0,3)$	z_{14}^5 $(1,2,4,4,0)$
x_{14}^5 $(0,1,3,3,1)$	y_{11} $(1,0,2,1,2)$	$\bar{y}_{11}$ $(1,2,0,1,2)$	z_{14}^4 $(1,2,4,0,4)$	z_{16}^5 $(2,1,3,4,0)$
x_{15}^4 $(0,1,3,1,4)$	y_{12} $(1,0,2,1,3)$	$\bar{y}_{12}$ $(1,2,0,1,3)$	z_{15}^4 $(1,2,4,0,5)$	z_{17}^5 $(1,2,4,5,0)$
x_{17}^5 $(0,1,3,4,1)$	y_{13} $(1,0,2,2,1)$	$\bar{y}_{13}$ $(1,2,0,2,1)$	$\bar{z}_1^4$ $(2,2,1,0,2)$	$\bar{z}_1^5$ $(2,2,1,2,0)$
$\bar{x}_2^5$ $(0,2,1,2,1)$	y_{14} $(1,0,2,2,2)$	$\bar{y}_{14}$ $(1,2,0,2,2)$	$\bar{z}_2^4$ $(2,2,1,0,3)$	$\bar{z}_2^5$ $(1,3,2,3,0)$
$\bar{x}_4^4$ $(0,2,1,1,2)$	y_{15} $(1,0,2,2,3)$	$\bar{y}_{15}$ $(1,2,0,2,3)$	$\bar{z}_3^4$ $(2,2,1,0,4)$	$\bar{z}_4^5$ $(2,2,1,3,0)$
$\bar{x}_5^4$ $(0,2,1,1,3)$	y_{16} $(1,0,2,3,1)$	$\bar{y}_{16}$ $(1,2,0,3,1)$	$\bar{z}_4^4$ $(1,3,2,0,3)$	$\bar{z}_5^5$ $(1,3,2,4,0)$
$\bar{x}_5^5$ $(0,2,1,3,1)$	y_{17} $(1,0,2,3,2)$	$\bar{y}_{17}$ $(1,2,0,3,2)$	$\bar{z}_5^4$ $(1,3,2,0,4)$	$\bar{z}_7^5$ $(2,2,1,4,0)$
$\bar{x}_6^4$ $(0,2,1,1,4)$	y_{18} $(1,0,2,3,3)$	$\bar{y}_{18}$ $(1,2,0,3,3)$	$\bar{z}_6^4$ $(1,3,2,0,5)$	$\bar{z}_8^5$ $(1,3,2,5,0)$
$\bar{x}_8^5$ $(0,2,1,4,1)$	$\bar{z}_1^2$ $(2,0,1,2,2)$	z_1^3 $(2,1,0,2,2)$	$\bar{z}_{10}^4$ $(2,3,1,0,2)$	$\bar{z}_{10}^5$ $(2,3,1,2,0)$
$\bar{x}_{11}^5$ $(0,3,1,2,1)$	$\bar{z}_2^2$ $(2,0,1,2,3)$	z_2^3 $(2,1,0,2,3)$	$\bar{z}_{11}^4$ $(2,3,1,0,3)$	$\bar{z}_{11}^5$ $(1,4,2,3,0)$
$\bar{x}_{13}^4$ $(0,3,1,1,2)$	$\bar{z}_3^2$ $(2,0,1,2,4)$	z_3^3 $(2,1,0,2,4)$	$\bar{z}_{12}^4$ $(2,3,1,0,4)$	$\bar{z}_{13}^5$ $(2,3,1,3,0)$
$\bar{x}_{14}^4$ $(0,3,1,1,3)$	$\bar{z}_4^2$ $(2,0,1,3,2)$	z_4^3 $(2,1,0,3,2)$	$\bar{z}_{13}^4$ $(1,4,2,0,3)$	$\bar{z}_{14}^5$ $(1,4,2,4,0)$
$\bar{x}_{14}^5$ $(0,3,1,3,1)$	$\bar{z}_5^2$ $(2,0,1,3,3)$	z_5^3 $(2,1,0,3,3)$	$\bar{z}_{14}^4$ $(1,4,2,0,4)$	$\bar{z}_{16}^5$ $(2,3,1,4,0)$
$\bar{x}_{15}^4$ $(0,3,1,1,4)$	$\bar{z}_6^2$ $(2,0,1,3,4)$	z_6^3 $(2,1,0,3,4)$	$\bar{z}_{15}^4$ $(1,4,2,0,5)$	$\bar{z}_{17}^5$ $(1,4,2,5,0)$
$\bar{x}_{17}^5$ $(0,3,1,4,1)$	$\bar{z}_7^2$ $(2,0,1,4,2)$	z_7^3 $(2,1,0,4,2)$		
	$\bar{z}_8^2$ $(2,0,1,4,3)$	z_8^3 $(2,1,0,4,3)$		
	$\bar{z}_9^2$ $(2,0,1,4,4)$	z_9^3 $(2,1,0,4,4)$		

Since f is a locating coloring, distinct vertices with (the same) color i have distinct color codes. Hence,

$$|V_i \cap N_T(x)| = |\{c_{\Pi}(y) : \; y \in V_i \cap N_T(x)\}| \leq 2^{p-1}3^{k-1-p}. \tag{2.1}$$

Inequality 2.1 holds for each $i \in \{2, 3, \dots, p+1\}$. Therefore,

$$\begin{aligned}\Delta(T) &= |N_T(x)| \\ &= |V_2 \cap N_T(x)| + |V_3 \cap N_T(x)| + \cdots + |V_{p+1} \cap N_T(x)| \\ &\leq 2^{p-1}3^{k-1-p} + 2^{p-1}3^{k-1-p} + \cdots + 2^{p-1}3^{k-1-p} \\ &= p2^{p-1}3^{k-1-p}.\end{aligned}$$

Now with the assumption $q = k - 1 - p$, Lemma 2.3 implies that $\Delta(T) \leq 4 \times 3^{k-3}$. □

As Theorem 2.4 indicates, if $\Delta(T) > 4 \times 3^{k-3}$, then $\chi_{_L}(T) > k$. This theorem shows that trees with large maximum degrees have large locating chromatic numbers. Note that

$$(3-1)2^{3-2} = 4 = 4 \times 3^{3-3},\ (4-1)2^{4-2} = 12 = 4 \times 3^{4-1}.$$

Thus, for $k \in \{3, 4\}$ two bounds in Theorems 2.1 and 2.4 coincide. But for each $k \geq 5$ we have $(k-1)2^{k-2} < 4 \times 3^{k-3}$ which means the bound in Theorem 2.4 is better. In fact, Theorem 2.5 shows that the bound provided by Theorem 2.4 is tight and is best possible. In Theorem 2.5 the structure of tree T_5 investigated in Example 2.2 is generalized.

Consider the tree T_5 and its coloring f_5 mentioned in Example 2.2. Using the Cartesian product of sets, let

$$A = \{1\} \times \{0\} \times \{1,2\} \times \{1,2,3\} \times \{1,2,3\} = \{\alpha_{_1}, \alpha_{_2}, \dots, \alpha_{_{18}}\},$$

and

$$\bar{A} = \{1\} \times \{1,2\} \times \{0\} \times \{1,2,3\} \times \{1,2,3\} = \{\bar{\alpha}_{_1}, \bar{\alpha}_{_2}, \dots, \bar{\alpha}_{_{18}}\},$$

where the elements of A and $\bar{A}$ are ordered using the lexicographic ordering. Therefore,

$$\begin{array}{llll}
\alpha_{_1} = (1,0,1,1,1), & \alpha_{_2} = (1,0,1,1,2), & \alpha_{_3} = (1,0,1,1,3), & \alpha_{_4} = (1,0,1,2,1), \\
\alpha_{_5} = (1,0,1,2,2), & \alpha_{_6} = (1,0,1,2,3), & \alpha_{_7} = (1,0,1,3,1), & \alpha_{_8} = (1,0,1,3,2), \\
\alpha_{_9} = (1,0,1,3,3), & \alpha_{_{10}} = (1,0,2,1,1), & \alpha_{_{11}} = (1,0,2,1,2), & \alpha_{_{12}} = (1,0,2,1,3), \\
\alpha_{_{13}} = (1,0,2,2,1), & \alpha_{_{14}} = (1,0,2,2,2), & \alpha_{_{15}} = (1,0,2,2,3), & \alpha_{_{16}} = (1,0,2,3,1), \\
\alpha_{_{17}} = (1,0,2,3,2), & \alpha_{_{18}} = (1,0,2,3,3), & & \\
\bar{\alpha}_{_1} = (1,1,0,1,1), & \bar{\alpha}_{_2} = (1,1,0,1,2), & \bar{\alpha}_{_3} = (1,1,0,1,3), & \bar{\alpha}_{_4} = (1,1,0,2,1), \\
\bar{\alpha}_{_5} = (1,1,0,2,2), & \bar{\alpha}_{_6} = (1,1,0,2,3), & \bar{\alpha}_{_7} = (1,1,0,3,1), & \bar{\alpha}_{_8} = (1,1,0,3,2), \\
\bar{\alpha}_{_9} = (1,1,0,3,3), & \bar{\alpha}_{_{10}} = (1,2,0,1,1), & \bar{\alpha}_{_{11}} = (1,2,0,1,2), & \bar{\alpha}_{_{12}} = (1,2,0,1,3), \\
\bar{\alpha}_{_{13}} = (1,2,0,2,1), & \bar{\alpha}_{_{14}} = (1,2,0,2,2), & \bar{\alpha}_{_{15}} = (1,2,0,2,3), & \bar{\alpha}_{_{16}} = (1,2,0,3,1), \\
\bar{\alpha}_{_{17}} = (1,2,0,3,2), & \bar{\alpha}_{_{18}} = (1,2,0,3,3). & &
\end{array}$$

Using Table 1, it can be easily checked that $c_{_\Pi}(y_{_i}) = \alpha_{_i}$ and $c_{_\Pi}(\bar{y}_{_i}) = \bar{\alpha}_{_i}$ for each $i \in \{1, 2, \dots, 18\}$. Note that x is a vertex of the maximum degree $4 \times 3^{5-3} = 36$ in T_5 and $N_{T_5}(x)$ is divided into two disjoint sets $\{y_{_i} : \ 1 \leq i \leq 18\}$ and $\{\bar{y}_{_i} : \ 1 \leq i \leq 18\}$. In fact, the tree T_5 and its coloring f_5 are constructed using 5-tuples in $A \cup \bar{A}$ in such a way that the color code of each $y_{_i}$ becomes $\alpha_{_i}$ and the color code of each $\bar{y}_i$ becomes $\bar{\alpha}_i$. For instance, the first component of each α_i is 1, and x is a vertex with color 1 which is adjacent to each y_i. The second component of each α_i is 0 and the color of each y_i is 2. Since $\alpha_1 = (1,0,1,1,1)$, the vertex y_1 is adjacent to vertices z_1^3, z_1^4, and z_1^5 with colors 3, 4, and 5, respectively.

Consider the vertex y_{15} and its corresponding 5-tuple $\alpha_{15} = (1,0,2,2,3)$. Third component of α_{15} is 2 and the distance from y_{15} to vertex $\bar{y}_1$ with color three is 2. Fourth component of α_{15} is 2 and the distance from y_{15} to vertex z_{15}^4 with color four is 2. Fifth component of α_{15} is 3 and the distance from y_{15} to vertex z_1^5 with color five is 3. Now consider the vertex z_{15}^4. The color of z_{15}^4 is 4 and this vertex is adjacent to x_{15}^4 with color 1. Hence, $d(z_{15}^4, V_4) = 0$ and $d(z_{15}^4, V_1) = 1$. For each vertex $v \in V(T_5) \setminus \{x_{15}^4, z_{15}^4\}$, the distance between z_{15}^4 and v in T_5 is given by

$$d(z_{15}^4, v) = d(z_{15}^4, y_{15}) + d(y_{15}, v) = 2 + d(y_{15}, v).$$

Thus, $d(z_{15}^4, V_i) = d(y_{15}, V_i) + 2$ for each $i \in \{1, 2, ..., 5\} \setminus \{1, 4\}$. Therefore, using the componentwise additions, subtractions and scalar multiplications, the color code of z_{15}^4 is

$$\begin{aligned}
c_{\Pi}(z_{15}^4) &= (1, d(y_{15}, V_2) + 2, d(y_{15}, V_3) + 2, 0, d(y_{15}, V_5) + 2) \\
&= (d(\bar{y}_{15}, V_1), d(y_{15}, V_2) + 2, d(y_{15}, V_3) + 2, d(y_{15}, V_4) - 2, d(y_{15}, V_5) + 2) \\
&= (d(\bar{y}_{15}, V_1), d(y_{15}, V_2), d(y_{15}, V_3), d(y_{15}, V_4), d(y_{15}, V_5)) + (0, 2, 2, -2, 2) \\
&= c_{\Pi}(y_{15}) + (2, 2, 2, 2, 2) - (2, 0, 0, 4, 0) \\
&= \alpha_{15} + 2(1, 1, 1, 1, 1) - 2(1, 0, 0, 0, 0) - 4(0, 0, 0, 1, 0).
\end{aligned}$$

Similarly, the color of vertex z_4^3 is 3 and

$$\begin{aligned}
c_{\Pi}(z_4^3) &= (d(y_4, V_1) + 1, d(y_4, V_2) + 1, 0, d(y_4, V_4) + 1, d(y_4, V_5) + 1) \\
&= (d(y_4, V_1) + 1, d(y_4, V_2) + 1, d(y_4, V_3) - 1, d(y_4, V_4) + 1, d(y_4, V_5) + 1) \\
&= (d(y_4, V_1), d(y_4, V_2), d(y_4, V_3), d(y_4, V_4), d(y_4, V_5)) + (1, 1, -1, 1, 1) \\
&= \alpha_4 + (1, 1, 1, 1, 1) - 2(0, 0, 1, 0, 0).
\end{aligned}$$

Similar arguments hold for other vertices of T_5.

Theorem 2.5. *For each integer $k \geq 3$, there exists a tree T_k with maximum degree $4 \times 3^{k-3}$ whose locating chromatic number is k.*

Proof. For $k \in \{3, 4\}$ two trees T_3 and T_4 with their optimum locating colorings are shown in Figure 3. Hereafter suppose that $k \geq 5$. The tree T_5 is investigated in Example 2.2. We want to construct T_k using the ideas involved in the structure of T_5. Let

$$\mathcal{A} = \{1\} \times \{0\} \times \{1,2\} \times \{1,2,3\}^{k-3}, \ \bar{\mathcal{A}} = \{1\} \times \{1,2\} \times \{0\} \times \{1,2,3\}^{k-3}.$$

Note that each element of $\mathcal{A}$ and $\bar{\mathcal{A}}$ is a k-tuple and $|\mathcal{A}| = |\bar{\mathcal{A}}| = 2 \times 3^{k-3}$. Also, note that $\mathcal{A}$ and $\bar{\mathcal{A}}$ are two disjoint sets. If $\gamma \in \mathcal{A} \cup \bar{\mathcal{A}}$ and $1 \leq t \leq k$, then by $\gamma(t)$ we mean the t-th coordinate of γ. Assume that $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_{|\mathcal{A}|}\}$ and $\bar{\mathcal{A}} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{|\bar{\mathcal{A}}|}\}$ where the elements in $\mathcal{A}$ and $\bar{\mathcal{A}}$ are

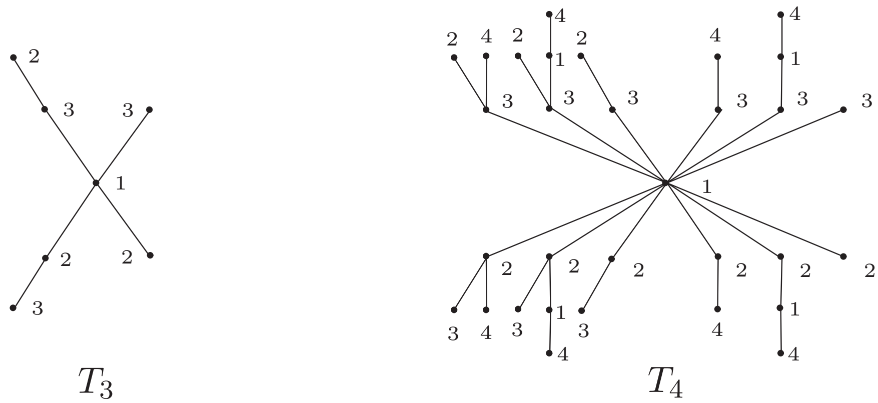

FIGURE 3. Two trees T_3 and T_4 with their optimum locating colorings.

ordered using the lexicographic ordering. Hence, $\alpha_1 = (1,0,1,1,1,\ldots,1)$, $\alpha_{|\mathcal{A}|} = (1,0,2,3,3,\ldots,3)$, $\bar{\alpha}_1 = (1,1,0,1,1,\ldots,1)$, and $\bar{\alpha}_{|\bar{\mathcal{A}}|} = (1,2,0,3,3,\ldots,3)$.
Let T_k be the tree with vertex set $V(T_k)$ and edge set $E(T_k)$ defined as follow.

$$\begin{aligned}
V(T_k) &= \{x\} \\
&\cup \{y_i:\ 1 \le i \le |\mathcal{A}|\} \\
&\cup \{\bar{y}_i:\ 1 \le i \le |\bar{\mathcal{A}}|\} \\
&\cup \{z_i^3:\ 1 \le i \le |\mathcal{A}|,\ \alpha_i(3) = 1\} \\
&\cup \{\bar{z}_i^2:\ 1 \le i \le |\bar{\mathcal{A}}|,\ \bar{\alpha}_i(2) = 1\} \\
&\cup \{z_i^t:\ 1 \le i \le |\mathcal{A}|,\ 4 \le t \le k,\ 1 \le \alpha_i(t) \le 2\} \\
&\cup \{\bar{z}_i^t:\ 1 \le i \le |\bar{\mathcal{A}}|,\ 4 \le t \le k,\ 1 \le \bar{\alpha}_i(t) \le 2\} \\
&\cup \{x_i^t:\ 1 \le i \le |\mathcal{A}|,\ 4 \le t \le k,\ \alpha_i(t) = 2\} \\
&\cup \{\bar{x}_i^t:\ 1 \le i \le |\bar{\mathcal{A}}|,\ 4 \le t \le k,\ \bar{\alpha}_i(t) = 2\},
\end{aligned}$$

and

$$\begin{aligned}
E(T_k) &= \{xy_i:\ 1 \le i \le |\mathcal{A}|\} \\
&\cup \{x\bar{y}_i:\ 1 \le i \le |\bar{\mathcal{A}}|\} \\
&\cup \{y_i z_i^t:\ 1 \le i \le |\mathcal{A}|,\ 2 \le t \le k,\ \alpha_i(t) = 1\} \\
&\cup \{\bar{y}_i \bar{z}_i^t:\ 1 \le i \le |\bar{\mathcal{A}}|,\ 2 \le t \le k,\ \bar{\alpha}_i(t) = 1\} \\
&\cup \{y_i x_i^t:\ 1 \le i \le |\mathcal{A}|,\ 4 \le t \le k,\ \alpha_i(t) = 2\} \\
&\cup \{\bar{y}_i \bar{x}_i^t:\ 1 \le i \le |\bar{\mathcal{A}}|,\ 4 \le t \le k,\ \bar{\alpha}_i(t) = 2\} \\
&\cup \{x_i^t z_i^t:\ 1 \le i \le |\mathcal{A}|,\ 4 \le t \le k,\ \alpha_i(t) = 2\} \\
&\cup \{\bar{x}_i^t \bar{z}_i^t:\ 1 \le i \le |\bar{\mathcal{A}}|,\ 4 \le t \le k,\ \bar{\alpha}_i(t) = 2\}.
\end{aligned}$$

For the case $k=5$, the tree T_5 is illustrated in Figure 1. Note that

$$\Delta(T_k)=\deg(x)=|\mathcal{A}|+|\bar{\mathcal{A}}|=2\times 3^{k-3}+2\times 3^{k-3}=4\times 3^{k-3}>4\times 3^{(k-1)-3}.$$

Thus, Theorem 2.4 implies that the locating chromatic number of T_k is at least k. We show that $\chi_{_L}(T_k)$ is equal to k. For this reason, it is sufficient to provide a locating k-coloring of the vertices of T_k.

Define the vertex k-coloring $f_k : V(T_k)\rightarrow [k]$ as follow.

$$f_k(v)=\begin{cases} 1 & v=x\\ 2 & v=y_i,\ 1\le i\le |\mathcal{A}|\\ 3 & v=\bar{y}_i,\ 1\le i\le |\bar{\mathcal{A}}|\\ 3 & v=z_i^3,\ 1\le i\le |\mathcal{A}|,\ \alpha_i(3)=1\\ 2 & v=\bar{z}_i^2,\ 1\le i\le |\bar{\mathcal{A}}|,\ \bar{\alpha}_i(2)=1\\ 1 & v=x_i^t,\ 1\le i\le |\mathcal{A}|,\ 4\le t\le k,\ \alpha_i(t)=2\\ 1 & v=\bar{x}_i^t,\ 1\le i\le |\bar{\mathcal{A}}|,\ 4\le t\le k,\ \bar{\alpha}_i(t)=2\\ t & v=z_i^t,\ 1\le i\le |\mathcal{A}|,\ 4\le t\le k,\ 1\le \alpha_i(t)\le 2\\ t & v=\bar{z}_i^t,\ 1\le i\le |\bar{\mathcal{A}}|,\ 4\le t\le k,\ 1\le \bar{\alpha}_i(t)\le 2. \end{cases}$$

For the case $k=5$, the coloring f_5 of T_5 is illustrated in Figure 2. Let $\Pi=(V_1,V_2,\ldots,V_k)$ be an ordered partition of $V(T_k)$ into the resulting color classes $V_i=\{v:\ v\in V(T_k),\ f_k(v)=i\}$, $1\le i\le k$. For each i, $1\le i\le k$, let $e_i=(\delta_{1,i},\delta_{2,i},\ldots,\delta_{k,i})$ where $\delta_{j,i}$ is the Kroneker delta. Also, using the componentwise summation, let $e=e_1+e_2+\cdots+e_k$. Therefore,

$$e_1=(1,0,0,\ldots,0,0),\ e_2=(0,1,0,\ldots,0,0),\ \ldots,\ e_k=(0,0,0,\ldots,0,1),\ e=(1,1,1,\ldots,1,1).$$

Using the structure of T_k and by a simple argument similar to what is discussed after Theorem 2.4, we can easily derive the following color codes.

$$c_{_\Pi}(v)=\begin{cases} (0,1,1,2,2,\ldots,2) & v=x\\ \alpha_{_i} & v=y_i,\ 1\le i\le |\mathcal{A}|\\ \alpha_{_i}+e-2e_t & v=z_i^t,\ 1\le i\le |\mathcal{A}|,\ 2\le t\le k,\ \alpha_i(t)=1\\ \alpha_{_i}+e-2e_1-2e_t & v=x_i^t,\ 1\le i\le |\mathcal{A}|,\ 4\le t\le k,\ \alpha_i(t)=2\\ \alpha_{_i}+2e-2e_1-4e_t & v=z_i^t,\ 1\le i\le |\mathcal{A}|,\ 4\le t\le k,\ \alpha_i(t)=2\\ \bar{\alpha}_{_i} & v=\bar{y}_i,\ 1\le i\le |\bar{\mathcal{A}}|\\ \bar{\alpha}_{_i}+e-2e_t & v=\bar{z}_i^t,\ 1\le i\le |\bar{\mathcal{A}}|,\ 2\le t\le k,\ \bar{\alpha}_i(t)=1\\ \bar{\alpha}_{_i}+e-2e_1-2e_t & v=\bar{x}_i^t,\ 1\le i\le |\bar{\mathcal{A}}|,\ 4\le t\le k,\ \bar{\alpha}_i(t)=2\\ \bar{\alpha}_{_i}+2e-2e_1-4e_t & v=\bar{z}_i^t,\ 1\le i\le |\bar{\mathcal{A}}|,\ 4\le t\le k,\ \bar{\alpha}_i(t)=2. \end{cases}$$

Therefore, distinct vertices of T_k have distinct color codes. Hence, f_k is a locating k-coloring of T_k and this completes the proof. □

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