PERFECT STATE TRANSFER IN UNITARY CAYLEY GRAPHS
OVER LOCAL RINGS

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Abstract. In this work, using eigenvalues and eigenvectors of unitary Cayley graphs over finite local rings and elementary linear algebra, we characterize which local rings allow a PST occurring in its unitary Cayley graph. Moreover, we have some developments when $R$ is a product of local rings.

1. Perfect State Transfer and Unitary Cayley Graphs

Let $G$ be an undirected graph whose vertex set $V(G) = \{v_1, \ldots, v_n\}$. The adjacency matrix of $G$, written $A_G$, is the $n \times n$ matrix in which entry $a_{jk}$ is the number of edges in $G$ with endpoint $\{v_j, v_k\}$. Define the matrix-valued function

$$H(t) = \exp(itA_G) \quad \text{for all } t \geq 0.$$ 

We say there is a perfect state transfer (PST) from vertex $v_j$ to vertex $v_k$ if there is a time $t$ such that $|H(t)_{jk}| = 1$. We note that our matrix $H(t)$ determines what is known in graph theory as a continuous quantum walk. For background on quantum walks, we refer the reader to [9] and [10]. A perfect state transfer in continuous-time quantum walk on graphs has received considerable attention in quantum information and computations in Physics (e.g., [2, 4]). An excellent survey of perfect state transfer graphs and related questions are given by Godsil [8]. Observe that $H(t)$ has the following properties:

(i) $H(t)$ is symmetric,
(ii) $\overline{H(t)} = H(t)^{-1}$, where $\overline{\cdot}$ is the complex conjugate,
(iii) $H(t)$ is unitary, i.e., $(H(t))^T = H(t)^{-1}$.

Thus, we have the next proposition.

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Proposition 1.1. If we have a perfect state transfer on $A_G$ from vertex $a$ to vertex $b$ at time $t$, then we have a perfect state transfer from vertex $b$ to vertex $a$ at the same time.

The following theorem is well known in Linear Algebra.

Theorem 1.2. Let $\mathbb{R}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ be an orthogonal decomposition of $\mathbb{R}^n$, where each $W_j$ is spanned by orthogonal basis $\vec{u}_{j1}, \vec{u}_{j2}, \ldots, \vec{u}_{jm_j}$ for some $m_j \in \mathbb{N}$ and for all $j \in \{1, 2, \ldots, k\}$. For each $j \in \{1, 2, \ldots, k\}$, let $E_j$ be the projection of $\mathbb{R}^n$ for $W_j$. Then the $l$th column of the standard matrix of $E_j$ is given by

$$E_j(\vec{e}_l) = \langle \vec{e}_l, \vec{u}_{j1} \rangle \frac{\vec{u}_{j1}}{||\vec{u}_{j1}||^2} + \langle \vec{e}_l, \vec{u}_{j2} \rangle \frac{\vec{u}_{j2}}{||\vec{u}_{j2}||^2} + \cdots + \langle \vec{e}_l, \vec{u}_{jm_j} \rangle \frac{\vec{u}_{jm_j}}{||\vec{u}_{jm_j}||^2}$$

for all $l \in \{1, 2, \ldots, n\}$, where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Recall that $A_G$ is orthogonally diagonalizable. Let $\theta_1, \theta_2, \ldots, \theta_m$ be distinct eigenvalues of $A_G$ and let $E_r$ denote the orthogonal projection on the eigenspace belonging to $\theta_r$ for all $r \in \{1, 2, \ldots, m\}$. Here, we abuse the notation by writing $E_r$ for its standard matrix. It follows from the Spectral Theorem (Theorem 6.25 of [5]) that

(i) $E_jE_k = \delta_{jk}E_j$, for $1 \leq j, k \leq m$,

(ii) $E_1 + E_2 + \cdots + E_r = I_n$,

(iii) $\theta_1E_1 + \theta_2E_2 + \cdots + \theta_mE_m = A_G$.

If $f$ is a differentiable complex-valued function defined on the eigenvalues of $A_G$, then

$$f(A_G) = \sum_{r=1}^{m} f(\theta_r)E_r.$$  

In particular,

$$H(t) = \exp(itA_G) = \sum_{r=1}^{m} \exp(it\theta_r)E_r.$$  

For $j \in \{1, 2, \ldots, n\}$, we write $|\vec{e}_j\rangle = \vec{e}_j$, the $j$th column of the identity matrix $I_n$. The following proposition is Lemma 2.1 of [8]. It will become our main tool, so we record it below.

Proposition 1.3. A perfect state transfer occurs in $G$ from vertex $a$ to vertex $b$ at time $t$ if and only if there is a $\gamma \in \mathbb{C}$ such that $|\gamma| = 1$ and $E_r |b\rangle = \gamma \exp(-it\theta_r)E_r |a\rangle$ for all $r \in \{1, 2, \ldots, m\}$.

Corollary 1.4. If there is a perfect state transfer from vertex $a$ to vertex $b$, then $E_r |a\rangle = \pm E_r |b\rangle$ for all $r \in \{1, 2, \ldots, m\}$.

Proof. It follows from the fact that $E_r |a\rangle$ and $E_r |b\rangle$ are real vectors for all $r \in \{1, 2, \ldots, m\}$. \hfill $\square$

By Proposition 1.3, another main tool for studying perfect state transfers is the spectral decomposition of $A_G$.

Let $R$ be a finite commutative ring with unity $1 \neq 0$ and let $R^\times$ denote the unit group of invertible elements of $R$. The unitary Cayley graph of $R$, $G_R = \text{Cay}(R, R^\times)$ is the Cayley graph whose vertex set is $R$ and edge set is $\{\{a, b\} : a, b \in R$ and $a - b \in R^\times\}$. 


For two graphs $G$ and $H$, their weak product, $G \otimes H$, is the graph defined on $V(G) \times V(H)$ where $(a, b)$ is adjacent to $(a', b')$ if and only if $a$ is adjacent to $a'$ in $G$ and $b$ is adjacent to $b'$ in $H$. The adjacency matrix of $G \times H$ is $A_G \otimes A_H = \begin{bmatrix} a_{11}A_H & \ldots & a_{1n}A_H \\ \vdots & \ddots & \vdots \\ a_{n1}A_H & \ldots & a_{nn}A_H \end{bmatrix}$, where $a_{jk}$ is the entry in $A_G$ for all $j, k \in \{1, 2, \ldots, n\}$.

Recall that a local ring $R$ is a commutative ring with unity 1 which has a unique maximal ideal $M$. Note that, if $R$ is a local ring with unique maximal ideal $M$, then $R^\times = R \setminus M$. Furthermore, every finite commutative ring is a product of local rings. The structure of $G_R$ is presented in the next proposition.

**Proposition 1.5.** [1]

Let $R$ be a finite commutative ring.

(i) $G_R$ is a regular graph of degree $|R^\times|$.

(ii) If $R$ is a local ring with maximal ideal $M$, then $G_R$ is a complete multipartite graph whose partite sets are the cosets of $M$ in $R$. In particular, $G_R$ is a complete graph if and only if $R$ is a field.

(iii) If $R \cong R_1 \times \cdots \times R_s$ is a product of local rings, then $G_R \cong \bigotimes_{i=1}^s G_{R_i}$.

As is standard, if $\theta_1, \ldots, \theta_k$ are eigenvalues of a graph $G$ with multiplicities $m_1, \ldots, m_k$, respectively, we use the notation $\text{Spec} G = \begin{pmatrix} \theta_1 & \ldots & \theta_k \\ m_1 & \ldots & m_k \end{pmatrix}$ to describe the spectrum of $G$. We have the following fact.

**Proposition 1.6.** [1][11]

Let $R$ be a finite local ring with maximal ideal $M$ of size $m$. Then

$$\text{Spec} G_R = \begin{pmatrix} |R^\times| & -m \\ 1 & |R| - 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{|R|}{m} - 1 \frac{|R|}{m} (m - 1) \end{pmatrix}.$$

In particular, if $F$ is a finite field, then

$$\text{Spec} G_F = \begin{pmatrix} |F^\times| & -1 \\ 1 & |F^\times| \end{pmatrix}.$$

When $R = \mathbb{Z}_n$, Bašić et al. [3] have investigated a perfect state transfer on $G_R$. They proved that if $n$ and $n/2$ are not square-free integers, there is a PST in $G_{\mathbb{Z}_n}$. Moreover, they showed that the only unitary Cayley graphs of the ring $\mathbb{Z}_n$ that have a PST are $K_2$ (path of length two) and $C_4$ (4-cycle).

In this work, using Propositions 1.3 and 1.6 we characterize which local rings allowing PST occurring in its unitary Cayley graph in Section 2. Further developments when $R$ is a product of local rings are studied in Section 3.

## 2. PST of $G_R$ when $R$ is Local

Throughout this section, we let $R$ be a finite local ring with unique maximal ideal $M$ of size $m$. For $k, l \in \mathbb{N}$, we write $0_{k \times l}$ and $J_{k \times l}$ for the $k \times l$ matrix whose all entries are 0 and 1, respectively. We
also use \( \bar{0}_k = 0_{k \times 1} \) and \( \bar{1}_k = J_{k \times 1} \). By Proposition 1.5 (ii), we have

\[
A_{GR} = \begin{bmatrix}
0_{m \times m} & J_{m \times m} & J_{m \times m} & \cdots & J_{m \times m} \\
J_{m \times m} & 0_{m \times m} & J_{m \times m} & \cdots & J_{m \times m} \\
J_{m \times m} & J_{m \times m} & 0_{m \times m} & \cdots & J_{m \times m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
J_{m \times m} & J_{m \times m} & J_{m \times m} & \cdots & 0_{m \times m}
\end{bmatrix}.
\]

By Proposition 1.6, \( G_R \) has eigenvalues \( \theta_1 = |R^×|, \theta_2 = -m \) and \( \theta_3 = 0 \) with multiplicities \( \frac{|R|}{m} - 1 \) and \( \frac{|R|}{m} (m - 1) \), respectively, and eigenspace spanned, respectively, by the columns of the following orthogonal matrices:

\[
A_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{|R| \times 1}, \quad A_2 = \begin{bmatrix}
\bar{1}_m & \frac{1}{2} \bar{1}_m & \frac{1}{3} \bar{1}_m & \cdots & \frac{1}{\frac{|R|}{m} - 1} \bar{1}_m \\
-\bar{1}_m & \frac{1}{2} \bar{1}_m & \frac{1}{3} \bar{1}_m & \cdots & \frac{1}{\frac{|R|}{m} - 1} \bar{1}_m \\
\bar{0}_m & -\bar{1}_m & \frac{1}{3} \bar{1}_m & \cdots & \frac{1}{\frac{|R|}{m} - 1} \bar{1}_m \\
\bar{0}_m & \bar{0}_m & -\bar{1}_m & \cdots & \frac{1}{\frac{|R|}{m} - 1} \bar{1}_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{0}_m & \bar{0}_m & \bar{0}_m & \cdots & \frac{1}{\frac{|R|}{m} - 1} \bar{1}_m \end{bmatrix}_{|R| \times \frac{|R|}{m} - 1},
\]

and

\[
A_3 = \begin{bmatrix} W \\ \vdots \\ W \end{bmatrix}_{|R| \times \frac{|R|}{m} (m - 1)}
\]

where

\[
W = \begin{bmatrix} 1 & 1 & 1 \\
\omega & \omega^2 & \omega^{m-1} \\
\omega^2 & \omega^4 & \omega^{2(m-1)} \\
\vdots & \vdots & \vdots \\
\omega^{m-1} & \omega^{2(m-1)} & \omega^{(m-1)(m-1)} \end{bmatrix}_{m \times (m - 1)} \quad \text{and} \quad \omega = \exp(2\pi i/m).
\]

Using Theorem 1.2, the standard matrices of \( E_j, j = 1, 2, 3 \), can be directly computed. We record them in the next theorem.

**Theorem 2.1.** Let \( R \) be a finite local ring with unique maximal ideal \( M \) of size \( m \). For \( j \in \{1, 2, 3\} \), let \( E_j \) be the orthogonal projection on the eigenspace belonging to \( \theta_j \) of \( A_{GR} \). Then

(i) \( E_1 = \frac{1}{|R|} J_{|R| \times |R|} \),

(ii) \( E_2 = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_{|R| - 1} & \bar{u}_{|R|} \end{bmatrix} \), where

\[
\bar{w}_s = \sum_{l=1}^{\frac{|R|}{m} - 1} \frac{\bar{u}_l}{(l+1)m}, \quad \bar{w}_{km+s} = \sum_{l=k}^{\frac{|R|}{m} - 1} \frac{\bar{u}_l}{(l+1)m} - \frac{\bar{u}_k}{(k+1)m}, \quad \bar{w}_{|R| - m + s} = \frac{m - |R|}{|R|m} \bar{u}_{\frac{|R|}{m} - 1}.
\]
for all \( s \in \{1, 2, \ldots, m\}, \ k \in \{1, 2, \ldots, \frac{|R|}{m} - 2\} \) and \( \bar{u}_l \) is the \( l \)th column of \( A_2 \) for all \( l \in \{1, 2, \ldots, \frac{|R|}{m} - 1\} \), and

\[
(iii) \quad E_3 = \frac{1}{m} \begin{bmatrix}
    M & M & \cdots & M
  \end{bmatrix}_{|R| \times |R|}, \quad \text{where } M = \begin{bmatrix}
    m - 1 & -1 & -1 & -1 \\
    -1 & -1 & -1 & m - 1 \\
    -1 & -1 & -1 & -1 \\
    \vdots & \vdots & \vdots & \vdots \\
    -1 & -1 & m - 1 & -1 \\
    -1 & m - 1 & -1 & -1
  \end{bmatrix}_{m \times m}.
\]

The above computations and Corollary 1.4 give the following necessity condition.

**Theorem 2.2.** Let \( R \) be a finite local ring with the maximal ideal \( M \) of size \( m \). If there is a perfect state transfer from vertex \( v_j \) to vertex \( v_k \) in the graph \( G_R \) for some \( 1 \leq j < k \leq |R| \), then \( m \) is 1 or 2.

**Proof.** By Corollary 1.4 we have \( E_3|v_j| = \pm E_3|v_k| \). Thus, the \( j \)th column of \( E_3 \) is equal to \( \pm \) the \( k \)th column of \( E_3 \), which implies that \( m - 1 = -1, 0 \) or 1. Since \( m > 0 \), \( m = 1 \) or 2. \( \square \)

Note that if \( m = 1 \), then \( R \) is a finite field. We obtain a further result on the number of elements of \( R \) in the next theorem.

**Theorem 2.3.** Let \( R \) be the finite field with \( q \) elements. Then there is a perfect state transfer in the graph \( G_R \) if and only if \( q = 2 \).

**Proof.** If \( q = 2 \), then \( A_{GR} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and \( H(t) = e^{itA_{GR}} = \begin{bmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{bmatrix} \) for all \( t \geq 0 \). Thus, \( H(\frac{\pi}{2}) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \) and \( G_R \) has a perfect state transfer at \( t = \frac{\pi}{2} \). Conversely, assume that \( q \geq 3 \). We have

\[
E_2 = [\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_{q-1} \ \vec{w}_q],
\]

where

\[
\vec{w}_1 = \sum_{l=1}^{q-1} \frac{\bar{u}_l}{l+1}, \quad \vec{w}_s = \sum_{l=s}^{q-1} \frac{\bar{u}_l}{l+1} - \frac{\bar{u}_{s-1}}{||\bar{u}_{s-1}||^2} \ (s = 2, 3, \ldots, q-1), \quad \vec{w}_q = \left(1 - \frac{q}{q}ight) \bar{u}_{q-1},
\]

and \( \bar{u}_l \) is the \( l \)th column of \( A_2 \) for all \( l \in \{1, 2, \ldots, q-1\} \). Let \( 1 \leq j < k \leq q \).

**Case 1.** \( j = 1 \) and \( k \in \{2, 3, \ldots, q - 1\} \). Since

\[
\vec{w}_1 - \vec{w}_k = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1/2 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \frac{1}{k} \begin{bmatrix} 1/k-1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \left(\frac{k-1}{k}\right) \begin{bmatrix} 1/k-1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\]
The first entry of $\vec{w}_1 - \vec{w}_k$ is nonzero. Also,

\[
\vec{w}_1 + \vec{w}_k = \frac{1}{2} \begin{bmatrix}
1 \\
-1 \\
0 \\
\vdots \\
0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
1/2 \\
-1 \\
0 \\
\vdots \\
0
\end{bmatrix} + \cdots + \frac{1}{k} \begin{bmatrix}
1/(k-1) \\
\vdots \\
0 \\
\vdots \\
0
\end{bmatrix} + \frac{2}{k+1} \begin{bmatrix}
1/k \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

The last entry of $\vec{w}_1 + \vec{w}_k$ is nonzero. Hence, $\vec{w}_1 \neq \pm \vec{w}_k$.

**Case 2.** $j = 1$ and $k = q$. Since

\[
\vec{w}_1 - \vec{w}_q = \frac{1}{2} \begin{bmatrix}
1 \\
-1 \\
0 \\
\vdots \\
0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
1/q-1 \\
-1 \\
0 \\
\vdots \\
0
\end{bmatrix} + \cdots + \frac{1}{q} \begin{bmatrix}
1/q-1 \\
\vdots \\
0 \\
\vdots \\
0
\end{bmatrix} - \left( \frac{q-1}{q} \right) \begin{bmatrix}
1/q-1 \\
\vdots \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

we have the $q$th entry of $\vec{w}_1 - \vec{w}_q$ is $-1$ and of $\vec{w}_1 + \vec{w}_q$ is $-\frac{2+q}{q} \geq -\frac{2+3}{3} \neq 0$ because $q > 3$. Thus, $\vec{w}_1 \neq \pm \vec{w}_k$. 
Case 3. $j, k \in \{2, \ldots, q-1\}$. Since

$$
\vec{w}_j - \vec{w}_k = \frac{1}{j+1} - 1 + \cdots + \frac{1}{k} \begin{bmatrix}
\frac{1}{j} \\
\vdots \\
\frac{1}{j} \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
- \left(\frac{j-1}{j}\right) \begin{bmatrix}
\frac{1}{k-1} \\
\vdots \\
\frac{1}{k-1} \\
-1 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
+ \left(\frac{k-1}{k}\right) \begin{bmatrix}
-1 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix},
$$

the $k$th entry of $\vec{w}_j - \vec{w}_k$ is $-1$. Moreover,

$$
\vec{w}_j + \vec{w}_k = \frac{1}{j+1} - 1 + \cdots + \frac{1}{k} \begin{bmatrix}
\frac{1}{j} \\
\vdots \\
\frac{1}{j} \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
+ \frac{2}{k+1} \begin{bmatrix}
\frac{1}{k-1} \\
\vdots \\
\frac{1}{k-1} \\
-1 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
+ \cdots + \frac{1}{q} \begin{bmatrix}
\frac{1}{q-1} \\
\vdots \\
\frac{1}{q-1} \\
-1 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix},
$$

$$
- \left(\frac{j-1}{j}\right) \begin{bmatrix}
\frac{1}{j-1} \\
\vdots \\
\frac{1}{j-1} \\
-1 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
- \left(\frac{k-1}{k}\right) \begin{bmatrix}
\frac{1}{k-1} \\
\vdots \\
\frac{1}{k-1} \\
-1 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}.
$$
the \( q \)th row of \( \vec{w}_j + \vec{w}_k \) is not equal to 0.

**Case 4.** \( j, k = 2, \ldots, q - 1, k = q \). Since

\[
\vec{w}_j - \vec{w}_q = \frac{1}{j + 1} + \cdots + \frac{1}{q} - \left( \frac{i - 1}{j} \right) - \left( \frac{1 - q}{q} \right),
\]

and

\[
\vec{w}_j + \vec{w}_q = \frac{1}{j + 1} + \cdots + \frac{1}{q} - \left( \frac{i - 1}{j} \right) + \left( \frac{1 - q}{q} \right),
\]

the \( q \)th entry of \( \vec{w}_j - \vec{w}_q \) is not equal to 0, and of \( \vec{w}_j + \vec{w}_q = \frac{2 + q}{q} \geq \frac{1}{3} > 0 \). Hence, \( \vec{w}_j \neq \pm \vec{w}_k \) for all \( 1 \leq j < k \leq q \). That is, \( E_2 |v_j\rangle \neq \pm E_2 |v_k\rangle \), for all \( 1 \leq j < k \leq q \).

By Proposition 1.4 there is no perfect state transfers in \( G_R \).

For \( m = 2 \), we have \( |R| = 2^k \) for some \( k \geq 2 \). We get the following result.

**Theorem 2.4.** Let \( R \) be a finite local ring with maximal ideal \( M \) of size two. Then the graph \( G_R \) has a perfect state transfer at time \( t = \frac{\pi}{2} \).

**Proof.** Recall that \( G_R \) has three distinct eigenvalues, \( \theta_1 = 2^k - 2, \theta_2 = -2 \) and \( \theta_3 = 0 \). Choose \( \gamma = -1 \) and \( t = \frac{\pi}{2} \). Then \( \exp(-i t \theta_1) = \exp(-i t \theta_2) = -1 \) and \( \exp(-i t \theta_3) = 1 \). Following Proposition 1.3 and Theorem 2.1 we show

\[
E_1 |v_1\rangle = E_1 |v_2\rangle = \gamma \exp(-i t \theta_1) E_1 |v_2\rangle,
\]

\[
E_2 |v_1\rangle = E_2 |v_2\rangle = \gamma \exp(-i t \theta_2) E_2 |v_2\rangle,
\]

and

\[
E_3 |v_1\rangle = -E_3 |v_2\rangle = \gamma \exp(-i t \theta_3) E_3 |v_2\rangle.
\]

Hence, \( G_R \) has a perfect state transfer from vertex \( v_1 \) to vertex \( v_2 \) at time \( t = \frac{\pi}{2} \).
We conclude all discussions in this section in the next theorem.

**Theorem 2.5.** Let $R$ be a finite local ring with maximal ideal $M$ of size $m$. Then $G_R$ has a perfect state transfer if and only if $R = \mathbb{F}_2$ or $m = 2$. In particular, a perfect state occurs at time $t = \frac{\pi}{2}$.

Moreover, if $R$ is a local ring with $m = 2$, it follows from [6] that $|R|$ must be 4. Thus, $R$ is $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ as shown in [12]. Hence, we conclude that:

**Corollary 2.6.** Let $R$ be a finite local ring. Then $G_R$ has a perfect state transfer if and only if $R = \mathbb{F}_2$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

### 3. Further Developments

In this section, we present some results when $R$ is a product of finite local rings. We begin with the following lemma.

**Lemma 3.1.** Let $G$ and $H$ be undirected graphs. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $G$ corresponding to eigenvectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$, respectively, and let $\mu_1, \mu_2, \ldots, \mu_m$ be eigenvalues of $H$ corresponding to eigenvectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$, respectively.

(i) For $k \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, m\}$, we have $\vec{u}_k \otimes \vec{w}_l$ is an eigenvectors of $G \otimes H$ with eigenvalue $\lambda_k \mu_l$.

(ii) Let $g_1, g_2 \in V(G)$ and $h_1, h_2 \in V(H)$. Then

$$
\langle (g_2, h_2) | \exp(itA_{G \otimes H}) | (g_1, h_1) \rangle = \sum_{k=1}^{n} \sum_{l=1}^{m} \langle h_2 | \vec{w}_l \vec{w}_l^T | h_1 \rangle \exp(it\lambda_k \mu_l) \langle \vec{u}_k \vec{u}_k^T | g_1 \rangle.
$$

**Proof.** (i) Let $k \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, m\}$. Then

$$
A_{G \otimes H}(\vec{u}_k \otimes \vec{w}_l) = A_G \vec{u}_k \otimes A_H \vec{w}_l = \lambda_k \vec{u}_k \otimes \mu_l \vec{w}_l = \lambda_k \mu_l (\vec{u}_k \otimes \vec{w}_l).
$$

(ii) Let $g_1, g_2 \in V(G)$ and $h_1, h_2 \in V(H)$. From (i), let

$$
P = \begin{bmatrix}
\vec{u}_1 \otimes \vec{w}_1 & \vec{u}_1 \otimes \vec{w}_2 & \cdots & \vec{u}_1 \otimes \vec{w}_m & \vec{u}_2 \otimes \vec{w}_1 & \vec{u}_2 \otimes \vec{w}_2 & \cdots & \vec{u}_n \otimes \vec{w}_m
\end{bmatrix}
$$

be the $nm \times nm$ matrix such that

$$
\exp(itA_{G \otimes H}) = P \begin{bmatrix}
e^{it\lambda_1 \mu_1} & e^{it\lambda_1 \mu_2} & \cdots & e^{it\lambda_1 \mu_m} 
e^{it\lambda_2 \mu_1} & e^{it\lambda_2 \mu_2} & \cdots & e^{it\lambda_2 \mu_m} 
e^{it\lambda_m \mu_1} & e^{it\lambda_m \mu_2} & \cdots & e^{it\lambda_m \mu_m}
\end{bmatrix} P^T.
$$
Then

\[
\langle (g_2, h_2) \mid \exp(itA_{G \times R}) \mid (g_1, h_1) \rangle = \sum_{k=1}^{n} \sum_{l=1}^{m} \langle g_2 | \vec{u}_k \vec{u}_k^T | g_1 \rangle \langle h_2 | \vec{w}_l \vec{w}_l^T | h_1 \rangle \exp(it \lambda_k \mu_l)
\]

\[
= \sum_{k=1}^{n} \langle g_2 | \left( \sum_{l=1}^{m} \langle h_2 | \vec{w}_l \vec{w}_l^T | h_1 \rangle \exp(it \lambda_k \mu_l) \right) \vec{u}_k \vec{u}_k^T | g_1 \rangle.
\]

Hence, we have the lemma. \( \square \)

If \( R \) is a finite local ring and \( G_R \) has no even eigenvalues, then by Proposition 1.6, \( R \) is the finite field of \( 2^r \) elements for some \( r \in \mathbb{N} \). Thus, \( G_R \) is complete and its adjacency matrix is given by

\[
A_{G_R} = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0
\end{bmatrix}.
\]

It can be shown that for \( j \in \{0, 1, \ldots, |R| - 1\} \), the vector

\[
\vec{w}_j = \frac{1}{|R|} \begin{bmatrix}
1 \\
\omega_j \\
\omega_j^2 \\
\vdots \\
\omega_j^{|R| - 1}
\end{bmatrix}^T,
\]

where \( \omega_j = \exp \left( \frac{2 \pi i j}{|R|} \right) \),

is an eigenvector of \( A_{G_R} \) with eigenvalue \( \mu_j = \sum_{k=1}^{|R|-1} \omega_j^k \). Note that \( \mu_0 = |R| - 1 \) and \( \mu_j = -1 \) if \( j \geq 1 \) (which are the same results found in Proposition 1.6). Moreover, we observe that

\[
(3.1) \quad \langle 0 | \vec{w}_j \vec{w}_j^T | 0 \rangle = \frac{1}{|R|}
\]

for all \( j \in \{0, 1, \ldots, |R| - 1\} \). The following proposition was proved for circulant graphs in [7]. However, the observation above yields the same result.

**Proposition 3.2.** Let \( G \) be a graph on \( n \) vertices with perfect state transfer at time \( t_G \) so that

\[
t_G \text{Spec} G \subseteq \mathbb{Z} \pi := \{a \pi : a \in \mathbb{Z})\).
\]

Then \( G \otimes G_R \) has a perfect state transfer at time \( t_G \) if \( R \) is a finite local ring and \( G_R \) has no even eigenvalues (that is, \( R \) is the finite field of \( 2^r \) elements for some \( r \in \mathbb{N} \)).

**Proof.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be eigenvalues of \( G \) corresponding to eigenvectors \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \), respectively. Assume that \( G \) has a perfect state transfer at time \( t_G \) from vertex \( g_1 \) to vertex \( g_2 \). By Lemma 3.1(ii)
and Eq. (3.1), we have
\[
\langle (g_1, 0) \mid e^{it_G A_G \times H} \mid (g_2, 0) \rangle = \sum_{k=1}^{n} \langle g_1 \mid \sum_{j=0}^{[R]-1} \langle 0 \mid \vec{w}_j \vec{w}_j^T \mid 0 \rangle \exp(it_G \lambda_k \mu_j) \vec{u}_k \vec{u}_k^T \mid g_2 \rangle
\]
\[
= \frac{1}{|R|} \sum_{k=1}^{n} \langle g_1 \mid \exp(it_G \lambda_k \mu_0) + \sum_{j=1}^{[R]-1} \exp(it_G \lambda_k \mu_j) \rangle \vec{u}_k \vec{u}_k^T \mid g_2 \rangle
\]
\[
= \frac{1}{|R|} \sum_{k=1}^{n} \langle g_1 \mid \exp(it_G \lambda_k (|R| - 1)) + \sum_{j=1}^{[R]-1} \exp(it_G \lambda_k (-1)) \rangle \vec{u}_k \vec{u}_k^T \mid g_2 \rangle
\]
\[
= \sum_{k=1}^{n} \langle g_1 \mid \exp(-it_G \lambda_k) \vec{u}_k \vec{u}_k^T \mid g_2 \rangle \quad \text{(because } t_G \lambda_k \in \mathbb{Z} \pi)\]
\[
= \langle g_1 \mid \exp(-it_G A_G) \mid g_2 \rangle.
\]
Since a perfect state transfer occurs from vertex $g_1$ to vertex $g_2$,
\[
|\langle g_1 \mid \exp(-it_G A_G) \mid g_2 \rangle| = |\langle g_1 \mid \exp(it_G A_G) \mid g_2 \rangle| = 1,
\]
so we have a perfect state transfer from vertex $(g_1, 0)$ to vertex $(g_2, 0)$. □

Theorem 2.5 and Proposition 3.2 give the following theorem.

**Theorem 3.3.** Let $\mathbb{F}_{2^r}$ be the finite field with $2^r$ elements and $R$ a finite local ring with $m = 2$. Then $G_R \otimes G_{\mathbb{F}_{2^r}}$ has a perfect state transfer. Moreover, let $m \in \mathbb{N}$ and $\mathbb{F}_{2^{r_1}}, \mathbb{F}_{2^{r_2}}, \ldots, \mathbb{F}_{2^{r_m}}$ be the finite fields with $2^{r_1}, 2^{r_2}, \ldots, 2^{r_m}$ elements, respectively. Then $G_R \otimes G_{\mathbb{F}_{2^{r_1}}} \otimes \cdots \otimes G_{\mathbb{F}_{2^{r_m}}}$ has a perfect state transfer.

**Proof.** Since $R$ is a local ring with $m = 2$, by Theorem 2.5 we have a perfect state transfer at time $t = \frac{\pi}{2}$. It follows from Proposition 1.6 that $t \text{ Spec } G_R \subseteq \mathbb{Z} \pi$. Hence, Proposition 3.2 inductively gives the desired results. □

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