DIRECTIONALLY \( n \)-SIGNED GRAPHS-III: THE NOTION OF SYMMETRIC BALANCE

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Abstract. Let \( G = (V, E) \) be a graph. By directional labeling (or \( d \)-labeling) of an edge \( x = uv \) of \( G \) by an ordered \( n \)-tuple \( (a_1, a_2, \ldots, a_n) \), we mean a labeling of the edge \( x \) such that we consider the label on \( uv \) as \( (a_1, a_2, \ldots, a_n) \) in the direction from \( u \) to \( v \), and the label on \( x \) as \( (a_n, a_{n-1}, \ldots, a_1) \) in the direction from \( v \) to \( u \). In this paper, we study graphs, called \( (n, d) \)-sigraphs, in which every edge is \( d \)-labeled by an \( n \)-tuple \( (a_1, a_2, \ldots, a_n) \), where \( a_k \in \{+,-\} \), for \( 1 \leq k \leq n \). In this paper, we give different notion of balance: symmetric balance in a \( (n, d) \)-sigraph and obtain some characterizations.

1. Introduction

For graph theory terminology and notation in this paper we follow the book [3]. All graphs considered here are finite and simple.

There are two ways of labeling the edges of a graph by an ordered \( n \)-tuple \( (a_1, a_2, \ldots, a_n) \) (See [12]).

1. Undirected labeling or labeling. This is a labeling of each edge \( uv \) of \( G \) by an ordered \( n \)-tuple \( (a_1, a_2, \ldots, a_n) \) such that we consider the label on \( uv \) as \( (a_1, a_2, \ldots, a_n) \) irrespective of the direction from \( u \) to \( v \) or \( v \) to \( u \).

2. Directional labeling or \( d \)-labeling. This is a labeling of each edge \( uv \) of \( G \) by an ordered \( n \)-tuple \( (a_1, a_2, \ldots, a_n) \) such that we consider the label on \( uv \) as \( (a_1, a_2, \ldots, a_n) \) in the direction from \( u \) to \( v \),
and \((a_n, a_{n-1}, \ldots, a_1)\) in the direction from \(v\) to \(u\).

Note that the \(d\)-labeling of edges of \(G\) by ordered \(n\)-tuples is equivalent to labeling the symmetric digraph \(\overrightarrow{G} = (V, \overrightarrow{E})\), where \(uv\) is a symmetric arc in \(\overrightarrow{G}\) if, and only if, \(uw\) is an edge in \(G\), so that if \((a_1, a_2, \ldots, a_n)\) is the \(d\)-label on \(uv\) in \(G\), then the labels on the arcs \(\overrightarrow{uv}\) and \(\overrightarrow{vu}\) are \((a_1, a_2, \ldots, a_n)\) and \((a_n, a_{n-1}, \ldots, a_1)\) respectively.

Let \(H_n\) be the \(n\)-fold sign group,

\[
H_n = \{+, -\}^n = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n \in \{+, -\}\}
\]

with co-ordinate-wise multiplication. Thus, writing \(a = (a_1, a_2, \ldots, a_n)\) and \(t = (t_1, t_2, \ldots, t_n)\) then \(at := (a_1t_1, a_2t_2, \ldots, a_nt_n)\). For any \(t \in H_n\), the action of \(t\) on \(H_n\) is \(a^t = at\), the co-ordinate-wise product.

Let \(n \geq 1\) be a positive integer. An \(n\)-signed graph (\(n\)-signed digraph) is a graph \(G = (V, E)\) in which each edge (arc) is labeled by an ordered \(n\)-tuple of signs, i.e., an element of \(H_n\). A signed graph \(G = (V, E)\) is a graph in which each edge is labeled by + or -. Thus a 1-signed graph is a signed graph. Signed graphs are well studied in literature (See for example [1, 5, 6, 7, 9, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32]).

In this paper, we study graphs in which each edge is labeled by an ordered \(n\)-tuple \(a = (a_1, a_2, \ldots, a_n)\) of signs (i.e., an element of \(H_n\)) in one direction but in the other direction its label is the reverse: \(a^r = (a_n, a_{n-1}, \ldots, a_1)\), called directionally labeled \(n\)-signed graphs (or \((n,d)\)-signed graphs).

Note that an \(n\)-signed graph \(G = (V, E)\) can be considered as a symmetric digraph \(\overrightarrow{G} = (V, \overrightarrow{E})\), where both \(\overrightarrow{uv}\) and \(\overrightarrow{vu}\) are arcs if, and only if, \(uv\) is an edge in \(G\). Further, if an edge \(uv\) in \(G\) is labeled by the \(n\)-tuple \((a_1, a_2, \ldots, a_n)\), then in \(\overrightarrow{G}\) both the arcs \(\overrightarrow{uv}\) and \(\overrightarrow{vu}\) are labeled by the \(n\)-tuple \((a_1, a_2, \ldots, a_n)\).

In [1], the authors study voltage graph defined as follows: A voltage graph is an ordered triple \(\overrightarrow{G} = (V, \overrightarrow{E}, M)\), where \(V\) and \(\overrightarrow{E}\) are the vertex set and arc set respectively and \(M\) is a group. Further, each arc is labeled by an element of the group \(M\) so that if an arc \(\overrightarrow{uv}\) is labeled by an element \(a \in M\), then the arc \(\overrightarrow{vu}\) is labeled by its inverse, \(a^{-1}\).

Since each \(n\)-tuple \((a_1, a_2, \ldots, a_n)\) is its own inverse in the group \(H_n\), we can regard an \(n\)-signed graph \(G = (V, E)\) as a voltage graph \(\overrightarrow{G} = (V, \overrightarrow{E}, H_n)\) as defined above. Note that the \(d\)-labeling of edges in an \((n,d)\)-signed graph considering the edges as symmetric directed arcs is different from the above labeling. For example, consider a \((4,d)\)-signed graph in Figure 1. As mentioned above, this
can also be represented by a symmetric 4-signed digraph. Note that this is not a voltage graph as defined in [1], since for example; the label on $\overrightarrow{v_2v_1}$ is not the (group) inverse of the label on $\overrightarrow{v_1v_2}$.

Figure 1.

In [10, 11], the authors initiated a study of $(3,d)$ and $(4,d)$-Signed graphs. Also, discussed some applications of $(3,d)$ and $(4,d)$-Signed graphs in real life situations.

In [12], the authors introduced the notion of complementation and generalize the notion of balance in signed graphs to the directionally $n$-signed graphs. In this context, the authors look upon two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge. Also given some motivation to study $(n,d)$-signed graphs in connection with relations among human beings in society.

In [12], the authors defined complementation and isomorphism for $(n,d)$-signed graphs as follows: For any $t \in H_n$, the $t$-complement of $a = (a_1, a_2, \ldots, a_n)$ is: $a^t = at$. The reversal of $a = (a_1, a_2, \ldots, a_n)$ is: $a^r = (a_n, a_{n-1}, \ldots, a_1)$. For any $T \subseteq H_n$, and $t \in H_n$, the $t$-complement of $T$ is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the $t$-complement of an $(n,d)$-signed graph $G = (V,E)$, written $G^t$, is the same graph but with each edge label $a = (a_1, a_2, \ldots, a_n)$ replaced by $a^t$. The reversal $G^r$ is the same graph but with each edge label $a = (a_1, a_2, \ldots, a_n)$ replaced by $a^r$.

Let $G = (V,E)$ and $G' = (V',E')$ be two $(n,d)$-signed graphs. Then $G$ is said to be isomorphic to $G'$ and we write $G \cong G'$, if there exists a bijection $\phi : V \rightarrow V'$ such that if $uv$ is an edge in $G$ which is $d$-labeled by $a = (a_1, a_2, \ldots, a_n)$, then $\phi(u)\phi(v)$ is an edge in $G'$ which is $d$-labeled by $a$, and conversely.

For each $t \in H_n$, an $(n,d)$-signed graph $G = (V,E)$ is t-self complementary, if $G \cong G^t$. Further, $G$ is self reverse, if $G \cong G^r$.

Proposition 1.1. (E. Sampathkumar et al. [12]) For all $t \in H_n$, an $(n,d)$-signed graph $G = (V,E)$ is t-self complementary if, and only if, $G^a$ is t-self complementary, for any $a \in H_n$.

Let $v_1, v_2, \ldots, v_n$ be a cycle $C$ in $G$ and $(a_{k1}, a_{k2}, \ldots, a_{kn})$ be the $n$-tuple on the edge $v_kv_{k+1}, 1 \leq k \leq m - 1$, and $(a_{m1}, a_{m2}, \ldots, a_{mn})$ be the $n$-tuple on the edge $v_mv_1$. 

For any cycle $C$ in $G$, let $P(C)$ denotes the product of the $n$-tuples on $C$ given by:

$(a_{11},a_{12},...,a_{1n})(a_{21},a_{22},...,a_{2n})\cdots(a_{m1},a_{m2},...,a_{mn})$ and $P(C) = (a_{mn},a_{m(n-1)},...,a_{m1})(a_{(m-1)n},a_{(m-1)(n-1)},...,a_{(m-1)1})\cdots(a_{1n},a_{1(n-1)},...,a_{11})$.

An $n$-tuple $(a_1,a_2,...,a_n)$ is identity $n$-tuple, if each $a_k = +$, for $1 \leq k \leq n$, otherwise it is a non-identity $n$-tuple. Further an $n$-tuple $a = (a_1,a_2,...,a_n)$ is symmetric, if $a' = a$, otherwise it is a non-symmetric $n$-tuple. In $(n,d)$-sigraph $G = (V,E)$ an edge labeled with the identity $n$-tuple is called an identity edge, otherwise it is a non-identity edge.

Note that the above products $P(C)$ as well as $P(C)$ are $n$-tuples. In general, these two products need not be equal. However, the following holds.

**Proposition 1.2.** (E. Sampathkumar et al. [12])

For any cycle $C$ of an $(n,d)$-sigraph $G = (V,E)$, $P(C) = P(C)'$.

**Corollary 1.3.** (E. Sampathkumar et al. [12])

For any cycle $C$, $P(C) = P(C)$ if, and only if, $P(C)$ is a symmetric $n$-tuple. Furthermore, $P(C)$ is the identity $n$-tuple if, and only if, $P(C)$ is.

2. Balance in an $(n,d)$-Signed Graph

In [12], the authors defined two notions of balance in an $(n,d)$-signed graph $G = (V,E)$ as follows:

**Definition .** Let $G = (V,E)$ be an $(n,d)$-sigraph. Then,

(i) $G$ is identity balanced (or $i$-balanced), if $P(C)$ on each cycle of $G$ is the identity $n$-tuple, and

(ii) $G$ is balanced, if every cycle contains an even number of non-identity edges.

**Note:** An $i$-balanced $(n,d)$-sigraph need not be balanced and conversely. For example, consider the $(4,d)$-sigraphs in Figure 2. In Figure 2(a) $G$ is an $i$-balanced but not balanced, and in Figure 2(b) $G$ is balanced but not $i$-balanced.

![Figure 2](image-url)
An \((n, d)\)-signed graph \(G = (V, E)\) is \(i\)-balanced if each non-identity \(n\)-tuple appears an even number of times in \(P(C)\) on any cycle of \(G\).

However, the converse is not true. For example see Figure 3(a). In Figure 3(b), the number of non-identity 4-tuples is even and hence it is balanced. But it is not \(i\)-balanced, since the 4-tuple \((+ + - -)\) (as well as \((- - + +)\)) does not appear an even number of times in \(P(C)\) of 4-tuples.

![Figure 3](image)

In \[12\], the authors obtained some characterizations of balanced and \(i\)-balanced \((n, d)\)-signed graphs.

In \[13\], E. Sampathkumar et al. defined the path balance in an \((n, d)\)-signed graphs as follows:

Let \(G = (V, E)\) be an \((n, d)\)-signed graph. Then \(G\) is

1. **Path \(i\)-balanced**, if any two vertices \(u\) and \(v\) satisfy the property that for any \(u - v\) paths \(P_1\) and \(P_2\) from \(u\) to \(v\), \(P(P_1) = P(P_2)\).

2. **Path balanced** if any two vertices \(u\) and \(v\) satisfy the property that for any \(u - v\) paths \(P_1\) and \(P_2\) from \(u\) to \(v\) have same number of non identity \(n\)-tuples.

Clearly, the notion of path balance and balance coincides. That is an \((n, d)\)-signed graph is balanced if, and only if, \(G\) is path balanced. If an \((n, d)\) signed graph \(G\) is \(i\)-balanced then \(G\) need not be path \(i\)-balanced and conversely. In \[13\], the authors obtained the characterization path \(i\)-balanced \((n, d)\)-signed graphs as follows:

**Theorem 2.1.** (Characterization of Path \(i\)-balanced \((n, d)\)-Signed Graphs)

An \((n, d)\)-signed graph is path \(i\)-balanced if, and only if, any two vertices \(u\) and \(v\) satisfy the property that for any two vertex disjoint \(u - v\) paths \(P_1\) and \(P_2\) from \(u\) to \(v\), \(P(P_1) = P(P_2)\).

3. **Symmetric Balance in an \((n, d)\)-Signed Graph**

Let \(n \geq 1\) be an integer. An \(n\)-tuple \((a_1, a_2, \ldots, a_n)\) is symmetric, if \(a_k = a_{n-k+1}, 1 \leq k \leq n\). Let

\(H_n = \{(a_1, a_2, \ldots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}\)

be the set of all symmetric \(n\)-tuples. Note that \(H_n\) is a group under coordinate wise multiplication, and the order of \(H_n\) is \(2^m\), where \(m = \lceil n/2 \rceil\).
We now define a new notion of balance in \((n,d)\)-sigraphs as follows:

**Definition.** Let \(G = (V, E)\) be an \((n,d)\)-sigraph. Then \(G\) is *symmetric balanced* or *s-balanced* if \(P(C)\) on each cycle \(C\) of \(G\) is symmetric \(n\)-tuple.

**Note.**
1. If an \((n,d)\)-sigraph \(G = (V, E)\) is \(i\)-balanced then clearly \(G\) is s-balanced. But a s-balanced \((n,d)\)-sigraph need not be \(i\)-balanced. For example, the \((4,d)\)-sigraphs in Figure 4. \(G\) is an s-balanced but not \(i\)-balanced.
2. A s-balanced \((n,d)\)-sigraph need not be balanced and conversely.
3. In view of Corollary 1.3, the notion of s-balance is well defined since if \(P(C)\) is symmetric \(n\)-tuple then \(P(C)\) is also symmetric.

![Figure 4](image)

**4. Criteria for s-Balance**

In this section, we obtain some characterizations for s-balanced \((n,d)\)-sigraphs:

**Theorem 4.1.** An \((n,d)\)-sigraph is s-balanced if, and only if, every cycle of \(G\) contains an even number of non-symmetric \(n\)-tuples.

**Proof.** (Necessary) Suppose that \(G\) is s-balanced. We first note that product any two non-symmetric \(n\)-tuples is symmetric, it follows that product of an even number of non-symmetric \(n\)-tuples is symmetric. Suppose that there exists a cycle \(C\) in \(G\) containing odd number of non-identity \(n\)-tuple. Since product of odd number of non-symmetric \(n\) tuples is non-symmetric, and product of symmetric \(n\)-tuples is symmetric, \(P(C)\) is non-symmetric \(n\)-tuple, a contradiction.

(Sufficiency) Suppose that every cycle \(C\) of \(G\) contains even number of non-symmetric \(n\)-tuples. Then \(P(C)\) is symmetric and hence \(G\) is s-balanced.

The following result gives a necessary and sufficient condition for a balanced \((n,d)\)-sigraph to be s-balanced.

**Theorem 4.2.** A balanced \((n,d)\)-sigraph \(G = (V, E)\) is s-balanced if and only if every cycle of \(G\) contains even number of non-identity symmetric \(n\) tuples.
Proof. Suppose $G$ is balanced and every cycle of $G$ contains even number of non identity symmetric $n$-tuples. Let $C$ be a cycle in $G$. Since $G$ is balanced, $C$ contains an even number of non identity $n$-tuples and so number of non-symmetric $n$ tuples in $C$ is even. Hence $P(C)$ is symmetric $n$ tuple. Hence $G$ is $s$-balanced.

Conversely suppose that $G$ is balanced and $s$-balanced. Then the number of non-identity $n$-tuples as well as the number of non-symmetric $n$-tuples on any cycle $C$ of $G$ is even. Hence the number of every cycle of $G$ contains an even number of non-identity symmetric $n$-tuples. □

The following result is well known (see [4]).

Theorem 4.3. (Harary [4]).
A sigraph $G = (V, E)$ is balanced, if, and only if, its vertex set $V$ can be partitioned into two sets $V_1$ and $V_2$ such that every negative edge joins a vertex in $V_1$ and a vertex in $V_2$, and every positive edge joins two vertices in $V_1$ or in $V_2$.

Let $G = (V, E)$ be an $(n,d)$-sigraph. An edge in $G$ labelled by a symmetric edge is called symmetric edge. Otherwise it is called non-symmetric edge. We now give another characterization of $s$-balanced $(n,d)$-sigraph, which is analogous to the partition criteria for balance in signed graph due to Harary [4].

Theorem 4.4. (Characterization of $s$-balanced $(n,d)$-sigraph)
An $(n,d)$-sigraph $G = (V, E)$ is $s$-balanced if and only if the vertex set $V(G)$ of $G$ can be partitioned into two sets $V_1$ and $V_2$ such that each symmetric edge joins the vertices in the same set and each non-symmetric edge joins a vertex of $V_1$ and a vertex of $V_2$.

Proof. We associate a sigraph $G'$ with $G$ on the same vertex set $V$ and the edge set $E$ of $G$ as follows: an edge $ab$ in $G'$ is labelled $+$ or $-$ according as $ab$ is a symmetric edge or non-symmetric edge in $G$. Clearly, the $(n,d)$-sigraph $G$ is $s$-balanced if, and only if, the sigraph $G'$ is balanced, and the result follows from Theorem 4.3. □

An $(n,d)$-sigraph is said to be complete if the underlying graph of $G$ is complete. The $s$-balance base with axis $a$ of a complete $(n,d)$-sigraph $G = (V, E)$ consists list of the product of the $n$-tuples on the triangles containing $a$.

Theorem 4.5. A complete $(n,d)$-sigraph is $s$-balanced if, and only if, all the triangles of a base are $s$-balanced.

Proof. Suppose all the triangles a base are $s$-balanced. Indeed, for any triangle $(bed)$ not appearing in the base with axis $a$, we have $P(bed) = P(abc)P(abd)P(acd)=$symmetric $n$-tuple.

Conversely, if the $(n,d)$-sigraph is $s$-balanced, all these triangles are symmetric and particular those of a base. □
5. Locally $s$-Balanced $(n,d)$-Signed Graph

The notion of local balance in signed graph was introduced by F. Harary [5]. A signed graph $G = (V, E)$ is locally at a vertex $v$, or $G$ is balanced at $v$, if all cycles containing $v$ are balanced. A cut point in a connected graph $G$ is a vertex whose removal results in a disconnected graph. The following result due to Harary [5] gives interdependence of local balance and cut vertex of a signed graph.

**Theorem 5.1. (F. Harary [5])**

If a connected signed graph $G = (V, E)$ is balanced at a vertex $u$. Let $v$ be a vertex on a cycle $C$ passing through $u$ which is not a cut point, then $G$ is balanced at $v$.

In [13], the authors extend the notion of local balance in signed graph to $(n,d)$-signed graphs as follows: Let $G = (V, E)$ be a $(n,d)$-signed graph. Then for any vertices $v \in V(G)$, $G$ is locally $i$-balanced at $v$ (locally balanced at $v$) if all cycles in $G$ containing $v$ is $i$-balanced (balanced.)

Analogous to the above result, in [13], the authors obtained the following for an $(n,d)$-signed graphs:

**Theorem 5.2.** If a connected $(n,d)$-signed graph $G = (V, E)$ is locally $i$-balanced (locally balanced) at a vertex $u$ and $v$ be a vertex on a cycle $C$ passing through $u$ which is not a cut point, then $S$ is locally $i$-balanced (locally balanced) at $v$.

By the motivation of the above locally $i$-balanced (locally balanced) in an $(n,d)$-signed graph introduced by E. Sampathkumar et al. [13], in this section, we define locally $s$-balanced for an $(n,d)$-signed graphs:

**Definition.** Let $G = (V, E)$ be a $(n,d)$-sigraph. Then for any vertices $v \in V(G)$, $G$ is locally $s$-balanced at $v$ if all cycles in $G$ containing $v$ is $s$-balanced.

**Theorem 5.3.** If a connected $(n,d)$-signed graph $G = (V, E)$ is locally $s$-balanced at a vertex $u$ and $v$ be a vertex on a cycle $C$ passing through $u$ which is not a cut point, then $S$ is locally $s$-balanced at $v$.

**Proof.** Suppose that $G$ is $s$-balanced at $u$ and $v$ be a vertex on a cycle $C$ passing through $u$ which is not a cut point. Assume that $G$ is not $s$-balanced at $v$. Then there exists a cycle $C_1$ in $G$ which is not $s$-balanced. Since $G$ is $s$-balanced at $u$, the cycle $C$ is $s$-balanced.

With out loss of generality we may assume that $u \notin C$ for if $u$ is in $C$ and $G$ is $s$-balanced at $u$ $C$ is $s$-balanced. Let $e = uv$ be an edge in $C$. Since $v$ is not a cut point there exists a cycle $C_0$ containing $e$ and $v$. Then $C_0$ consists of two paths $P_1$ and $P_2$ joining $u$ and $v$.

Let $v_1$ be the first vertex in $P_1$ and $v_2$ be a vertex in $P_2$ such that $v_1 \neq v_2 \in C$, such points do exist since $v$ is not a cut point and $v \in C$. Since $u, v \in C_0$. Let $P_3$ be the path on $C_0$ from $v_1$ and $v_2$, $P_4$ be a path in $C$ containing $v$ and $P_5$ is the path from $v_1$ to $v_2$. Then $P_3 \cup P_4$ and $P_3 \cup P_5$ are cycles containing $u$ and hence are $s$-balanced, since they contain $u$. That is $P(P_3)$ and $(P(P_5))$ are either symmetric or non-symmetric so that $C = P_3 \cup P_5$ is $s$-balanced. This completes the proof.  

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