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THE CHANGES OF DISTANCE SIGNLESS LAPLACIAN ENERGY OF COMPLETE TRIPARTITE GRAPHS DUE TO EDGE DELETIONS

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Abstract. The distance signless Laplacian energy $E^Q(G)$ of a simple connected graph G of order n having distance signless Laplacian eigenvalues λ_i $(i = 1, 2, \ldots, n)$ is defined as $E^Q(G) = \sum_{i=1}^{n} |\lambda_i - \frac{2W}{n}|$ where W is the Wiener index of G. In this paper, we study the distance signless Laplacian energy change caused by an edge deletion in complete tripartite graphs K_{p_1,p_2,p_3} with $p_i \geq 2$.

1. Introduction

In this paper, all graphs considered are finite, connected, undirected and simple. Let $G(V, E)$ be such a graph with finite vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E \subseteq \binom{V}{2}$. Let $k > 1$. Denote by $K_{n_1,n_2,\ldots,n_k}$ the complete k-partite graph on $n = n_1 + n_2 + \cdots + n_k$ vertices. For convenience, we may assume that $n_1 \geq n_2 \geq \cdots \geq n_k > 0$. One particular type of complete k-partite graph is the Turán graph $T(n, k) \cong K_{[\frac{n}{k}],\ldots,[\frac{n}{k}]}$ [13].

Now let us introduce the distance signless Laplacian matrix of a graph. Let G be a graph of order n. The distance matrix $D(G) = (d_{uv})_{u,v \in V(G)}$ is an $n \times n$ matrix where the distance between vertices u and v is denoted by $d_{uv} = d_G(u, v)$ representing the length of the shortest path between these vertices. The sum of the distances from v to all other vertices in G is called the transmission of v and is denoted by

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$Tr_G(v)$, so $Tr_G(v) = \sum_{u \in V(G)} d_{uv}$. The Wiener index of a graph G is equal to the sum of the distances among all vertices in G and is denoted by $W(G)$. We will use shortly W instead of $W(G)$. The distance signless Laplacian matrix $D_Q(G)$ of G is the matrix $D_Q(G) = D(G) + Tr(G)$ where $Tr(G)$ is the diagonal matrix such that $Tr(G) = diag\{Tr_G(v_1), Tr_G(v_2), \ldots, Tr_G(v_n)\}$. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the distance signless Laplacian matrix $D_Q(G)$ are called distance signless Laplacian eigenvalues of G. The distance signless Laplacian spectrum $Sp(D_Q(G))$ of G is a multiset consisting of its distinct distance signless Laplacian eigenvalues together with their multiplicities. Since $D_Q(G)$ is a real and symmetric matrix, all of its eigenvalues are real.

In [3], Alhevaz et al. defined the distance signless Laplacian energy of G as

$$E^Q(G) = \sum_{i=1}^{n} |\lambda_i - \frac{2W}{n}|$$

and found the bounds for the distance signless Laplacian energy of graphs with n vertices. For more details about background and development on distance signless Laplacian matrix and distance signless Laplacian energy, see [15, 6, 7, 2, 16, 10, 3].

In recent years, the impact of edge changes on graph energy has attracted great interests. In [8, 9, 1, 14, 11, 12], the neighborhood energy, distance energy and Seidel energy changes of the graphs due to edge changes are studied.

The distance energy of a simple undirected graph G, denoted by $E^D(G)$, is the sum of the absolute values of the eigenvalues of the distance matrix $D(G)$ of G. In 2020, Tian et al. [11] obtained the following results.

Theorem 1.1. *Assume that we have a complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$ with $p_i \geq 2$ and $r \geq 3$. If $p_1 = p_2 = \cdots = p_r = p$, then, for any edge e, $E^D(K_{p,p,\ldots,p}) < E^D(K_{p,p,\ldots,p} - e)$.*

Theorem 1.2. *If $G = K_{p,p,p+1}$ with $p \geq 2$, then, for any edge e of G, $E^D(G) < E^D(G - e)$.*

Theorem 1.3. *If $G = K_{p,p,p-1}$ with $p \geq 3$, then, for any edge e of G, $E^D(G) < E^D(G - e)$.*

Theorem 1.4. *If $G = T(n, 3)$ with $n \geq 6$, then, for any edge e of G, $E^D(G) < E^D(G - e)$.*

In this paper, we investigate the distance signless Laplacian energy change for Turán graphs by an edge deletion.

We will proceed as follows. In Section 2 we present some important results required for the main results of this paper. In Sections 3, 4 and 5 we study the effects of edge deletion on distance signless Laplacian energy of k-partite Turán graphs $K_{p,p,\ldots,p}$, tripartite Turán graphs $K_{p,p,p+1}$ and $K_{p,p,p-1}$.

2. Preliminaries

In this section, we first give some results that will be used in other sections.

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Definition 2.1. [4] *For a connected graph G, a partition π of $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ is called the distance signless Laplacian equivalent partition if $d(v, V_j) = (q_{ij})$ is a constant for each $v \in V_i$, where $d(v, V_j) = \sum_{u \in V_j} d(v, u)$ and $1 \leq i, j \leq k$. Then, the matrix $Q_\pi = q_{ij}$ is called the distance signless Laplacian quotient matrix.*

Lemma 2.2. [4] *Let $\pi : V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ be a distance signless Laplacian equivalent partition of a connected graph G, let Q_π and $D_Q(G)$ be the distance signless Laplacian quotient matrix and distance signless Laplacian matrix of G, respectively. If α is an eigenvalue of Q_π, then, α is also an eigenvalue of $D_Q(G)$.*

The join $G \vee H$ of two graphs G and H is the graph obtained by joining each vertex of G to each vertex of H. Let $N_G(u)$ denote the set of neighbors of a vertex u in G.

From the [5, Proposition 11 and Theorem 32], we have the following conclusions.

Lemma 2.3. [5] *Suppose that G is a connected graph of order n and S is a subset of the vertex set $V(G)$ such that $N_G(x) = N_G(y)$ for any $x, y \in S$, where $|S| = t$. If S is an independent set, then, $Tr(v)$ is a constant h for each $v \in S$ and $D^Q(G)$ has $h - 2$ as an eigenvalue of with multiplicity at least $t - 1$.*

Lemma 2.4. [5] *Let G be an r_1-regular graph of order n_1 with the adjacency spectrum $\{r_1, \lambda_2, \ldots, \lambda_{n_1}\}$. Also let H be an r_2-regular graph of order n_2 with the adjacency spectrum $\{r_2, \delta_2, \ldots, \delta_{n_2}\}$. Then, the distance signless Laplacian spectrum of $G \vee H$ consists of:*

i) $(2n_1 - r_1 + n_2) - \lambda_i - 4$, *for* $i = 2, 3, \ldots, n_1$;

ii) $(2n_2 - r_2 + n_1) - \delta_i - 4$, *for* $i = 2, 3, \ldots, n_2$;

iii) $\frac{5}{4}(n_1 + n_2) - r_1 - r_2 - 4 \pm$
$\sqrt{\frac{9}{4}{n_1}^2 + (4(r_2 - r_1 - n_2) + (r_1 - r_2 + \frac{1}{2}n_2))n_1 + \frac{9}{4}{n_2}^2 + (4(r_1 - r_2) - (r_1 + r_2))n_2 + {r_1}^2 + 2r_1r_2 + {r_2}^2}$.

3. Distance signless Laplacian energy change for $K_{p,\ldots,p}$

In this section we deal with the k-partite Turán graphs with equal number of vertices in each partite set, that is, $K_{p,p,\ldots,p}$.

Lemma 3.1. *For $p \geq 2$, the distance signless Laplacian spectrum of the graph $K_{p,\ldots,p}$ on $n = kp$ vertices is*

$$\Big\{(kp + p - 4)^{kp-k}, (kp + 2p - 4)^{k-1}, (2kp + 2p - 4)\Big\}.$$

Proof. Let $G = K_{p,\ldots,p}$ with $(k-1)$-partite and $H = pK_1$. Then, complete k-partite graph $K_{p,\ldots,p} = K_{p,\ldots,p} \vee pK_1$. Notice that G is $(k-2)p$-regular with the adjacency spectrum $\{(k-2)p, 0, \ldots, 0, -p, \ldots, -p\}$ including 0 with multiplicity $(k-1)(p-1)$, $-p$ with multiplicity $k-2$ and $(k-2)p$ with unit multiplicity and H is 0-regular with the adjacency spectrum $\{0, 0, \ldots, 0\}$. From Lemma 2.4, we have the desired result. □

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Theorem 3.2. *Let $K_{p,p,\ldots,p}$ be the complete k-partite graph with $p \geq 2$ and e be any edge of $K_{p,\ldots,p}$. Then, the distance signless Laplacian eigenvalues of $K_{p,p,\ldots,p} - e$ are $(k+1)p-4$ with multiplicity $kp-k-2$, $(k+2)p-4$ with multiplicity $k-3$ and the roots $x_i (i = 1,2,3,4,5)$ of the following polynomial*

$$f_1(x) = [x-(k+1)p+4][x-(k+2)p+4][-x^3+(4kp+5p-10)x^2-(5k^2p^2+13kp^2+8p^2-26kp-32p+26)x$$
$$+(2k^3+8k^2+10k+4)p^3-(16k^2+40k+24)p^2+(32k+40)p-8].$$

Proof. Let $V_1, V_2, \ldots, V_k$ be the k-partite sets of $K_{p,p,\ldots,p}$. Without loss of generality, we may assume that $u_p \in V_1$ and $v_1 \in V_2$ are the end vertices of an edge e of $K_{p,p,\ldots,p}$. Now we give the vertex set $V(K_{p,\ldots,p}-e)$ with a partition $\pi_1 : V(K_{p,\ldots,p}-e) = V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5'$, where $V_1' = \{u_1, u_2, \ldots, u_{p-1}\}$, $V_2' = \{u_p\}$, $V_3' = \{v_1\}$, $V_4' = \{v_2, v_3, \ldots, v_p\}$ and the rest of vertices are in V_5'. It is easy to see that each of $V_i' (i = 1,2,3,4,5)$ satisfies the condition of Lemma 2.3. Then, $K_{p,\ldots,p} - e$ has the distance signless Laplacian eigenvalue $(k+1)p-4$ with multiplicity at least $kp-k-2$.

Let $\mathbf{1}$ be a vector with p components whose all elements are 1 and $\mathbf{0}$ be a vector with p components whose all elements are 0. Let e_i be a vector with k components whose i-th entry is $\mathbf{1}$ and other entries are $\mathbf{0}$. Let $e = u_t v_j$ be any edge where $u_t \in V_t, v_j \in V_j$ $(t, j = 1,2,\ldots,k)$. Let $X_i = e_{i+1} - e_s$, where $i \neq t, j$, $i = 1,2,\ldots,k-1$ and $s = min_{\ell \neq t,j}\{\ell\}$. Then,

$$D_Q(K_{p,\ldots,p} - e)X_i = ((k+2)p-4)X_i.$$

It can be seen that X_i's are linearly independent. Thus $(k+2)p-4$ is an eigenvalue of $D_Q(K_{p,\ldots,p}-e)$ with multiplicity $k-3$.

On the other hand, since π_1 is a distance signless Laplacian equivalent partition of $K_{p,\ldots,p}-e$, thus the distance signless Laplacian quotient matrix Q_{π_1} with respect to the partition π_1 may be written in the following form:

$$Q_{\pi_1} = \begin{bmatrix} (k+3)p-6 & 2 & 1 & p-1 & (k-2)p \\ 2p-2 & (k+1)p-1 & 2 & p-1 & (k-2)p \\ p-1 & 2 & (k+1)p-1 & 2p-2 & (k-2)p \\ p-1 & 1 & 2 & (k+3)p-6 & (k-2)p \\ p-1 & 1 & 1 & p-1 & 2kp-4 \end{bmatrix}.$$

By simple calculations, we obtain the characteristic polynomial of Q_{π_1} as follows

$$f_1(x) = [x-(k+1)p+4][x-(k+2)p+4][-x^3+(4kp+5p-10)x^2-(5k^2p^2+13kp^2+8p^2-26kp-32p+26)x$$
$$+(2k^3+8k^2+10k+4)p^3-(16k^2+40k+24)p^2+(32k+40)p-8].$$

By Lemma 2.2, the roots $x_i (i = 1,2,3,4,5)$ of $f_1(x)$ are also the eigenvalues of $D_Q(K_{p,\ldots,p}-e)$. Consequently, the distance signless Laplacian eigenvalues of $K_{p,\ldots,p}-e$ are $(k+1)p-4$ with multiplicity $kp-k-2$, $(k+2)p-4$ with multiplicity $k-3$ and the roots $x_i (i = 1,2,3,4,5)$ of the polynomial $f_1(x)$. □

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For the sake of simplicity, we may write the polynomial $f_1(x)$ in Theorem 3.2 as $f_1(x) = g_1(x)t_1(x)h_1(x)$, where

$$\begin{aligned} g_1(x) &= x - kp - p + 4, \\ t_1(x) &= x - kp - 2p + 4, \\ h_1(x) &= -x^3 + (4kp + 5p - 10)x^2 - (5k^2p^2 + 13kp^2 + 8p^2 - 26kp - 32p + 26)x \\ &+ (2k^3 + 8k^2 + 10k + 4)p^3 - (16k^2 + 40k + 24)p^2 + (32k + 40)p - 8. \end{aligned}$$

Lemma 3.3. *Let x_1 be the root of $g_1(x)$ and x_2 be the root of $t_1(x)$. Then,*
i) for $p = 2$ and $k \geq 3$, $x_1 < x_2 < \frac{2W}{n}$.
ii) for $p \geq 3$ and $k \geq 3$, $x_1 < \frac{2W}{n} < x_2$.

Proof. Since $\frac{2W}{n} = \frac{(k^2+k)p^2-2kp+2}{kp}$, the results follow immediately by simple calculations. □

Lemma 3.4. *If x_3, x_4 and x_5 are three roots of $h_1(x)$ for $p \geq 2$ and $k \geq 3$, then, the polynomial $h_1(x)$ has one root smaller than $\frac{2W}{n}$ and two roots greater than $\frac{2W}{n}$.*

Proof. Without loss of generality, we may assume that $x_3 < x_4 < x_5$ are the roots of the polynomial $h_1(x)$. Then, by the relations between roots and coefficients, we get

$$\begin{aligned} \sum_{i=3}^{5} x_i &= 4kp + 5p - 10, \\ x_3x_4 + x_3x_5 + x_4x_5 &= (5k^2 + 13k + 8)p^2 - (26k + 32)p + 26, \\ \prod_{i=3}^{5} x_i &= (2k^3 + 8k^2 + 10k + 4)p^3 - (16k^2 + 40k + 24)p^2 + (32k + 40)p - 8. \end{aligned}$$

Let $\delta_1 = x_3 - \frac{2W}{n}$, $\delta_2 = x_4 - \frac{2W}{n}$ and $\delta_3 = x_5 - \frac{2W}{n}$. Then, we have,

$$\delta_1 + \delta_2 + \delta_3 = \frac{(k^2 + 2k)p^2 - 4kp - 6}{kp}$$

and

$$\delta_1\delta_2\delta_3 = \frac{-(12k^4 + 6k^3)p^5 - (2k^4 + 4k^3 + 2k^2)p^4 + (16k^3 + 8k^2)p^3 + (8k^2 + 8k)p^2 - 16kp - 8}{k^3p^3}.$$

Clearly, for $p \geq 2$ and $k \geq 3$, we have that, $\delta_1\delta_2\delta_3 < 0$, which implies that, one or all of δ_i's is negative. Since $\delta_1 + \delta_2 + \delta_3 > 0$, all δ_i's can not be negative. Thus the desired result follows. □

Lemma 3.5. *If $p = 2$ and $k \geq 3$, then, $x_3 < \frac{2k^2-2}{k}$.*

Proof. From Lemma 3.4, we know that $x_3 < \frac{2W}{n}$ and it is easy to see that $\frac{2k^2-2}{k} < \frac{2W}{n}$ for $p = 2$ and $k \geq 3$. For $p = 2$ and $k \geq 3$, we have the following exercise.

$$h_1\left(\frac{2k^2 - 2}{k}\right) = \frac{-4(k^4 - 2k^3 + k^2 - 2)}{k^3} < 0.$$

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Therefore, by the relations between roots and coefficients, we have

$$x_3 < \frac{2k^2-2}{k}.$$

□

Lemma 3.6. *If $p \geq 3$ and $k \geq 3$, then, $x_3 < \frac{(k+1)p^2-2p-2}{p}$.*

Proof. Similar to the proof of Lemma 3.5, we have that $\frac{(k+1)p^2-2p-2}{p} < \frac{2W}{n}$ and

$$h_1\left(\frac{kp^2+p^2-2p-2}{p}\right) = \frac{4p^2(-kp+p+k+1)-8(2p-1)}{p^3} < 0,$$

for $p \geq 3$ and $k \geq 3$, so the desired result is arrived. □

Theorem 3.7. *Let $K_{p,\ldots,p}$ be the complete k-partite graph with $p \geq 2$ and $k \geq 3$. Then, for any edge e of $K_{p,p,\ldots,p}$, we have $E^Q(K_{p,\ldots,p}-e) > E^Q(K_{p,\ldots,p})$.*

Proof. We will prove the distance signless Laplacian energy change in two cases.

Cases 1: For $p = 2$ and $k \geq 3$, by the definition of distance signless Laplacian energy and Lemma 3.1, we have $E^Q(K_{p,\ldots,p}) = 4k$. From Theorem 3.2, Lemma 3.3 and Lemma 3.4, we have that

$$\begin{aligned}
E^Q(K_{p,\ldots,p,}-e) &= (k-2)\left|2(k+1)-4-\frac{2W}{n}\right| + (k-3)\left|2(k+2)-4-\frac{2W}{n}\right| + \sum_{i=1}^{5}\left|x_i-\frac{2W}{n}\right| \\
&= (k-2)\left(\frac{2W}{n}-(2k-2)\right)+(k-3)\left(2k-\frac{2W}{n}\right)+\left(\frac{2W}{n}-(2k-2)\right) \\
&+ \left(2k-\frac{2W}{n}\right)+\left(\frac{2W}{n}-x_3\right)+\left(x_4-\frac{2W}{n}\right)+\left(x_5-\frac{2W}{n}\right) \\
&= \frac{16k^2-8}{2k}-2x_3.
\end{aligned}$$

Then, by Lemma 3.5,

$$E^Q(K_{p,\ldots,p}-e)-E^Q(K_{p,\ldots,p}) = \frac{8k^2-8}{2k}-2x_3 > 0.$$

Cases 2: For $p \geq 3$ and $k \geq 3$, by the definition of distance signless Laplacian energy and Lemma 3.1, we have $E^Q(K_{p,\ldots,p}) = 4kp-4k$. From Theorem 3.2, Lemma 3.3 and Lemma 3.4, we have

https://dx.doi.org/10.22108/toc.2026.146794.2326

that

$$\begin{aligned} E^Q(K_{p,\dots,p,}-e) &= (kp-k-2)\left|(k+1)p-4-\frac{2W}{n}\right|+(k-3)\left|(k+2)p-4-\frac{2W}{n}\right| \\ &+ \sum_{i=1}^{5}\left|x_i-\frac{2W}{n}\right| \\ &= (kp-k-2)\left(\frac{2W}{n}-[(k+1)p-4]\right)+(k-3)\left((k+2)p-4-\frac{2W}{n}\right) \\ &+ \left(\frac{2W}{n}-[(k+1)p-4]\right)+\left((k+2)p-4-\frac{2W}{n}\right) \\ &+ \left(\frac{2W}{n}-x_3\right)+\left(x_4-\frac{2W}{n}\right)+\left(x_5-\frac{2W}{n}\right) \\ &= \frac{(6k^2+2k)p^2-(4k^2-4k)p-4k}{kp}-2x_3. \end{aligned}$$

Then, by Lemma 3.6,

$$E^Q(K_{p,\dots,p}-e)-E^Q(K_{p,\dots,p})=\frac{(2k^2+2k)p^2-4kp-4k}{kp}-2x_3>0.$$

This finishes the proof.

□

4. Distance signless Laplacian energy change for $K_{p,p,p+1}$

In this section, we investigate the effects on distance signless Laplacian energy of $K_{p,p,p+1}$ due to an edge deletion. For this we give the following lemma.

Lemma 4.1. *For $p \geq 2$, the distance signless Laplacian spectrum of the graph $K_{p,p,p+1}$ on $n = 3p+1$ vertices is*

$$\left\{(4p-3)^{[2p-2]},(4p-2)^{[p]},(5p-3)^{[1]},\frac{13(p-1)\pm\sqrt{9p^2+2p+9}}{2}\right\}.$$

Proof. Let $G = K_{p,p}$ and $H = (p+1)K_1$. Then, $K_{p,p,p+1} = K_{p,p} \vee (p+1)K_1$. We note that G is p-regular with the adjacency spectrum $\{p,0,\dots,0,-p\}$ and H is 0-regular with the adjacency spectrum $\{0,0,\dots,0\}$. From Lemma 2.4, the result is obtained. □

For $K_{p,p,p+1}$ we face two types edges, therefore we discuss it in two cases. Let us assume that the deleted edge e belong to two vertex subsets of same sizes.

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Theorem 4.2. *For $p \geq 2$, let $G \cong K_{p,p,p+1}$ with $V(G) = V_1 \cup V_2 \cup V_3$. Then, for any $e = u_i v_j (i, j = 1, 2, \ldots, p)$, the distance signless Laplacian eigenvalues of $K_{p,p,p+1} - e$ are $4p-3$ with multiplicity $2p-4$, $4p-2$ with multiplicity p and the roots $x_i (i = 1, 2, 3, 4, 5)$ of the following polynomial*

$$f_2(x) = [x - 4p + 3][x - 5p + 3][x^3 + (4 - 17p)x^2 + (92p^2 - 45p - 3)x - 160p^3 + 120p^2 + 12p - 4].$$

Proof. Without loss of generality, we assume that $e = u_p v_1$. Similar to the proof of Theorem 3.2, we obtain that $4p-3$ is an eigenvalue of $D_Q(K_{p,p,p+1} - e)$ with multiplicity $2p-4$ and $4p-2$ is an eigenvalue of $D_Q(K_{p,p,p+1} - e)$ with multiplicity p.

Now, we give the vertex set $V(K_{p,p,p+1} - e)$ with a partition $\pi_2 : V(K_{p,p,p+1} - e) = V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5'$, where $V_1' = \{u_1, u_2, \ldots, u_{p-1}\}$, $V_2' = \{u_p\}$, $V_3' = \{v_1\}$, $V_4' = \{v_2, v_3, \ldots, v_p\}$, $V_5' = \{w_1, w_2, \ldots, w_{p+1}\}$. So, the quotient matrix Q_{π_2} with respect to the partition π_2 may be written in the form:

$$Q_{\pi_2} = \begin{bmatrix} 6p-5 & 2 & 1 & p-1 & p+1 \\ 2p-2 & 4p & 2 & p-1 & p+1 \\ p-1 & 2 & 4p & 2p-2 & p+1 \\ p-1 & 1 & 2 & 6p-5 & p+1 \\ p-1 & 1 & 1 & p-1 & 6p \end{bmatrix}.$$

By a simple calculation, we get the characteristic polynomial of Q_{π_2} as follows

$$f_2(x) = [x - 4p + 3][x - 5p + 3][x^3 + (4 - 17p)x^2 + (92p^2 - 45p - 3)x - 160p^3 + 120p^2 + 12p - 4].$$

This completes the proof. □

For the sake of simplicity, we may write $f_2(x) = g_2(x)t_2(x)h_2(x)$, where

$$\begin{aligned} g_2(x) &= x - 4p + 3, \\ t_2(x) &= x - 5p + 3, \\ h_2(x) &= x^3 + (4 - 17p)x^2 + (92p^2 - 45p - 3)x - 160p^3 + 120p^2 + 12p - 4. \end{aligned}$$

Lemma 4.3. *If x_1 be the root of $g_2(x)$ and x_2 be the root of $t_2(x)$, then,*
i) For $p = 2$, $x_1 < x_2 < \frac{2W}{n}$.
ii) For $p \geq 3$, $x_1 < \frac{2W}{n} < x_2$.

Proof. Since $\frac{2W}{n} = \frac{12p^2+2p+2}{3p+1}$, the results follow by simple elementary operations. □

Lemma 4.4. *If x_3, x_4 and x_5 are three roots of $h_2(x)$ for $p \geq 2$, then, the polynomial $h_2(x)$ has one root smaller than $\frac{2W}{n}$ and two roots greater than $\frac{2W}{n}$.*

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Proof. Without loss of generality, we may assume that $x_3 < x_4 < x_5$ are the roots of the polynomial $h_2(x)$. Then, by the relation of root and coefficient, we have

$$\begin{aligned} \sum_{i=3}^{5} x_i &= 17p-4, \\ x_3x_4+x_3x_5+x_4x_5 &= 92p^2-45p-3, \\ \prod_{i=3}^{5} x_i &= 160p^3-120p^2-12p+4. \end{aligned}$$

Let $\delta_1 = x_3 - \frac{2W}{n}$, $\delta_2 = x_4 - \frac{2W}{n}$ and $\delta_3 = x_5 - \frac{2W}{n}$. Then, we have

$$\delta_1+\delta_2+\delta_3 = \frac{15p^2-p-10}{3p+1}$$

and

$$\delta_1\delta_2\delta_3 = \frac{-36p^5-306p^4-84p^3+320p^2+120p-14}{(3p+1)^3}.$$

For $p \geq 2$, we see that, $\delta_1\delta_2\delta_3 < 0$, which means that, one or all of δ_i's is negative. Since $\delta_1+\delta_2+\delta_3 > 0$, all δ_i's can not be negative. The result is obvious. □

Lemma 4.5. *If $p \geq 3$, then, $x_3 < \frac{12p^2+p-5}{3p+1}$.*

Proof. From Lemma 4.3, we have $x_3 < \frac{2W}{n}$ and it is easy to see that $\frac{12p^2+p-5}{3p+1} < \frac{2W}{n}$. As a result of a simple calculation, we have,

$$h_2\left(\frac{12p^2+p-5}{3p+1}\right) = \frac{72p^4+282p^3+230p^2+198p-14}{(3p+1)^3} > 0.$$

Therefore

$$x_3 < \frac{12p^2+p-5}{3p+1}.$$

□

Theorem 4.6. *If $p \geq 2$, then,*

$$E^Q(K_{p,p,p+1}) < E^Q(K_{p,p,p+1}-e).$$

Proof. By the definition of distance signless Laplacian energy and Lemma 4.1, for $p \geq 2$ we have, $E^Q(K_{p,p,p+1}) = \frac{36p^2-12p-12}{3p+1}$. We examine the energy change in two cases:

Cases 1: If $p = 2$, then, $E^Q(K_{2,2,3}) \approx 16,287$. By a simple calculation $E^Q(K_{2,2,3}-e) \approx 17,606$. Then, $E^Q(K_{2,2,3}-e) - E^Q(K_{2,2,3}) \approx 1,319 > 0$. Therefore $E^Q(K_{2,2,3}) < E^Q(K_{2,2,3}-e)$.

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Cases 2: Now, by the definition of distance signless Laplacian energy, Theorem 4.2, Lemma 4.3 and Lemma 4.4, for $p \geq 3$ we have,

$$\begin{aligned} E^Q(K_{p,p,p+1}-e) &= (2p-4)\left|4p-3-\frac{2W}{n}\right|+p\left|4p-2-\frac{2W}{n}\right|+\sum_{i=1}^{5}\left|x_i-\frac{2W}{n}\right| \\ &= (2p-4)\left(\frac{2W}{n}-(4p-3)\right)+p\left(\frac{2W}{n}-(4p-2)\right)+\left(\frac{2W}{n}-(4p-3)\right) \\ &+ \left(5p-4-\frac{2W}{n}\right)+\left(\frac{2W}{n}-x_3\right)+\left(x_4-\frac{2W}{n}\right)+\left(x_5-\frac{2W}{n}\right) \\ &= \frac{60p^2-10p-22}{3p+1}-2x_3. \end{aligned}$$

By the definition of distance signless Laplacian energy and Lemma 4.1, for $p \geq 2$ we have, $E^Q(K_{p,p,p+1}) = \frac{36p^2-12p-12}{3p+1}$. Consequently, from Lemma 4.5, we obtain

$$E^Q(K_{p,p,p+1}-e)-E^Q(K_{p,p,p+1}) = \frac{24p^2+2p-10}{3p+1}-2x_3 > 0,$$

which completes the proof.

□

In the following, we consider that the two end vertices of the deleted edge e are belongs to two vertex subsets of different sizes.

Theorem 4.7. *Let $G \cong K_{p,p,p+1}$ be the complete tripartite graph with $p \geq 2$ and with vertex set $V(G) = V_1 \cup V_2 \cup V_3$. Suppose that $e = v_iw_j$ is any edge of $K_{p,p,p+1}$, where $1 \leq i \leq p$ and $1 \leq j \leq p+1$. Then, the distance signless Laplacian eigenvalues of $K_{p,p,p+1}-e$ are $4p-3$ with multiplicity $2p-3$, $4p-2$ with multiplicity $p-1$ and the roots $x_i (i = 1,2,3,4,5)$ of the following polynomial*

$$\begin{aligned} &f_3(x) = x^5+(9-26p)x^4+(265p^2-188p+22)x^3+(-1328p^3+1439p^2-358p-12)x^2 \\ &+(3280p^4-4804p^3+1869p^2+67p-104)x-3200p^5+5920p^4-3160p^3-76p^2+480p-96. \end{aligned}$$

Proof. Without loss of generality, we assume that $e = v_pw_1$. Similar to the proof of Theorem 3.2, one can show that $4p-3$ is an eigenvalue of $D_Q(K_{p,p,p+1}-e)$ with multiplicity $2p-3$, and $4p-2$ is an eigenvalue of $D_Q(K_{p,p,p+1}-e)$ with multiplicity $p-1$.
Now we give the vertex set $V(K_{p,p,p+1}-e)$ with a partition $\pi_3 : V(K_{p,p,p+1}-e) = V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5'$, where $V_1' = \{u_1,u_2,\ldots,u_p\}$, $V_2' = \{v_1,v_2,\ldots,v_{p-1}\}$, $V_3' = \{v_p\}$, $V_4' = \{w_1\}$, $V_5' = \{w_2,w_3,\ldots,w_{p+1}\}$.

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Then, the quotient matrix Q_{π_3} takes the following form:

$$Q_{\pi_3} = \begin{bmatrix} 6p-3 & p-1 & 1 & 1 & p \\ p & 6p-5 & 2 & 1 & p \\ p & 2p-2 & 4p & 2 & p \\ p & p-1 & 2 & 4p+1 & 2p \\ p & p-1 & 1 & 2 & 6p-2 \end{bmatrix}.$$

By straightforward calculation, we get the characteristic polynomial of Q_{π_3} in the following form

$$f_3(x) = x^5 + (9-26p)x^4 + (265p^2 - 188p + 22)x^3 + (-1328p^3 + 1439p^2 - 358p - 12)x^2$$
$$+(3280p^4 - 4804p^3 + 1869p^2 + 67p - 104)x - 3200p^5 + 5920p^4 - 3160p^3 - 76p^2 + 480p - 96.$$

This completes the proof. □

Lemma 4.8. *If x_1,x_2,x_3,x_4 and x_5 are five roots of $f_3(x)$, then,*
i) $x_1 < x_2 < x_3 < \frac{2W}{n} < x_4 < x_5$, for $p = 2$.
ii) $x_1 < x_2 < \frac{2W}{n} < x_3 < x_4 < x_5$, for $3 \leq p \leq 9$.
iii) $x_1 < \frac{2W}{n} < x_2 < x_3 < x_4 < x_5$, for $p \geq 10$.

Proof. i) and *ii)* If $2 \leq p \leq 9$, see Table 1.

TABLE 1. Roots for $2 \leq p \leq 9$

p	x_1	x_2	x_3	x_4	x_5	$\frac{2W}{n}$
2	5.240	6.526	7.552	8	15.682	7.714
3	9.307	10.897	12.320	13.110	23.366	11.6
4	13.333	15.095	17.207	18.169	31.195	15.538
5	17.346	19.209	22.150	23.205	39.090	19.5
6	21.354	23.282	27.116	28.228	47.020	23.474
7	25.359	27.333	32.095	33.244	54.969	27.455
8	29.363	31.370	37.080	38.256	62.931	31.44
9	33.366	35.399	42.069	43.265	70.901	35.429

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iii) Without loss of generality, we may assume that $x_1 < x_2 < x_3 < x_4 < x_5$ are the roots of $f_3(x)$. Using the relations between roots and coefficients, we get

$$\begin{aligned}\sum_{i=1}^{5} x_i &= 26p-9,\\ \sum_{i<j} x_i x_j &= 265p^2-188p+22,\\ \prod_{i=1}^{5} x_i &= -3200p^5+5920p^4-3160p^3-76p^2+480p-96.\end{aligned}$$

Let $\delta_i = x_i - \frac{2W}{n}, i = 1,2,3,4,5$. Then, we have

$$\delta_1+\delta_2+\delta_3+\delta_4+\delta_5 = \frac{18p^2-11p-19}{3p+1}$$

and

$$\delta_1\delta_2\delta_3\delta_4\delta_5 = \frac{-540p^8+5994p^7-4032p^6-23298p^5+468p^4+31270p^3+22344p^2+4658p}{(3p+1)^5}.$$

Now for $p \geq 10$, it is easy to see that $\delta_1\delta_2\delta_3\delta_4\delta_5 < 0$, which implies that one, three or all of δ_i's is negative. However, $\delta_1+\delta_2+\delta_3+\delta_4+\delta_5 > 0$, so all δ_i's can not be negative. Since $f_3(\frac{2W}{n}) > 0$ and $f_3(\frac{2W}{n}+1) < 0$, then, by the intermediate value theorem, we have $\frac{2W}{n} < x_2 < \frac{2W}{n}+1$. Similarly $\frac{2W}{n}+1 < x_3 < \frac{2W}{n}+p-2$ and $\frac{2W}{n}+p-2 < x_4 < \frac{2W}{n}+p$, because $f_3(\frac{2W}{n}+1) < 0$ and $f_3(\frac{2W}{n}+p-2) > 0$, $f_3(\frac{2W}{n}+p-2) > 0$ and $f_3(\frac{2W}{n}+p) < 0$. Since $\delta_2 > 0, \delta_3 > 0, \delta_4 > 0$, three of δ_i's can not be negative. Thus one of δ_i's is negative. □

Lemma 4.9. *If $p \geq 10$, $x_1 < \frac{12p^2-3p-11}{3p+1}$.*

Proof. By a simple computation, we have

$$\begin{aligned}f_3\left(\frac{12p^2-3p-11}{3p+1}\right) &= 540p^8-3969p^7-5868p^6-29921p^5-44452p^4\\ &- 77203p^3-97636p^2-55483p-58968 > 0.\end{aligned}$$

Therefore $x_1 < \frac{12p^2-3p-11}{3p+1}$. □

Theorem 4.10. *If $p \geq 2$, then,*

$$E^Q(K_{p,p,p+1}) < E^Q(K_{p,p,p+1}-e).$$

Proof. Since for $p \geq 2$, we know that $E^Q(K_{p,p,p+1}) = \frac{36p^2-12p-12}{3p+1}$. We investigate the distance signless Laplacian energy change in two cases.

Cases 1: Let $X = E^Q(K_{p,p,p+1})$ and $Y = E^Q(K_{p,p,p+1}-e)$. For $2 \leq p \leq 9$, see Table 2.

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TABLE 2. Distance signless Laplacian energy change for $2 \leq p \leq 9$

p	X	$SpD_Q(K_{p,p,p+1}-e)$	$\frac{2W}{n}$	Y	$Y-X$
2	15.428	{5,6,5.240,6.526,7.552,8,15.682}	7.714	16.506	1.078
3	27.6	$\{[9]^3,[10]^2,9.307,10.897,12.320,13.110,23.366\}$	11.6	27.992	0.392
4	39.692	$\{[13]^5,[14]^3,13.333,15.095,17.207,18.169,31.195\}$	15.538	39.913	0.221
5	51.75	$\{[17]^7,[18]^4,17.346,19.209,22.150,23.205,39.090\}$	19.5	51.89	0.14
6	63.789	$\{[21]^9,[22]^5,21.354,23.282,27.116,28.228,47.020\}$	23.474	63.886	0.097
7	75.818	$\{[25]^{11},[26]^6,25.359,27.333,32.095,33.244,54.969\}$	27.455	75.889	0.071
8	87.84	$\{[29]^{13},[30]^7,29.363,31.370,37.080,38.256,62.931\}$	31.44	87.894	0.054
9	99.857	$\{[33]^{15},[34]^8,33.366,35.399,42.069,43.265,70.901\}$	35.429	99.899	0.042

Cases 2: If $p \geq 10$, then, $E^Q(K_{p,p,p+1}) = \frac{36p^2-12p-12}{3p+1}$. From Theorem 4.7 and Lemma 4.8, we can obtain

$$\begin{aligned} E^Q(K_{p,p,p+1}-e) &= (2p-3)\left|4p-3-\frac{2W}{n}\right| + (p-1)\left|4p-2-\frac{2W}{n}\right| + \sum_{i=1}^{5}\left|x_i-\frac{2W}{n}\right| \\ &= (2p-3)\left(\frac{2W}{n}-(4p-3)\right)+(p-1)\left(\frac{2W}{n}-(4p-2)\right)+\left(\frac{2W}{n}-x_1\right) \\ &+\left(x_2-\frac{2W}{n}\right)+\left(x_3-\frac{2W}{n}\right)+\left(x_4-\frac{2W}{n}\right)+\left(x_5-\frac{2W}{n}\right) \\ &= \frac{60p^2-18p-34}{3p+1}-2x_1. \end{aligned}$$

Because of $E^Q(K_{p,p,p+1}) = \frac{36p^2-12p-12}{3p+1}$ and Lemma 4.9,

$$E^Q(K_{p,p,p+1}-e) - E^Q(K_{p,p,p+1}) = \frac{24p^2-6p-22}{3p+1} - 2x_1 > 0.$$

This is the required proof.

□

Combining Theorem 4.6 with Theorem 4.10, we arrive at:

Theorem 4.11. *Let $G = K_{p,p,p+1}$ with $p \geq 2$. Then, for any edge e of $K_{p,p,p+1}$, we have*

$$E^Q(K_{p,p,p+1}) < E^Q(K_{p,p,p+1}-e).$$

5. Distance signless Laplacian energy change for $K_{p,p,p-1}$

In this section, first of all we examine the distance signless Laplacian spectrum of $K_{p,p,p-1}$ of order $n = kp-1$.

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Lemma 5.1. *For $p \geq 3$, the distance signless Laplacian spectrum of the tripartite Turán graph $K_{p,p,p-1}$ of order $n = 3p-1$ is*

$$\left\{(4p-6)^{[p-2]}, (4p-5)^{[2p-2]}, (5p-5)^{[1]}, \frac{13(p-1) \pm \sqrt{9p^2-26p+9}}{2}\right\}.$$

Proof. The proof is similar to the proof of Lemma 4.1. □

There are two possibilities, when we delete edge e by the edge set of $K_{p,p,p-1}$. First of all we consider that the two end vertices of the deleted edge e belong to two vertex subsets with cardinalities p.

Theorem 5.2. *Assume $G \cong K_{p,p,p-1}$ with $p \geq 3$ and $V(G) = V_1 \cup V_2 \cup V_3$. Then, for any $e = u_i v_j$ where $i, j = 1, 2, \ldots, p$, the distance signless Laplacian eigenvalues of $K_{p,p,p-1} - e$ are $4p-5$ with multiplicity $2p-4$, $4p-6$ with multiplicity $p-2$ and the roots $x_i (i = 1, 2, 3, 4, 5)$ of the following polynomial*

$$f_4(x) = [x-4p+5][x-5p+5][x^3 + (16-17p)x^2 + (92p^2-175p+73)x - 160p^3 + 456p^2 - 380p + 76].$$

Proof. Without loss of generality, we assume that $e = u_p v_1$. Similar to the proof of Theorem 3.2, $4p-5$ and $4p-6$ are eigenvalues of $D_Q(K_{p,p,p-1} - e)$ with multiplicity $2p-4$ and $p-2$, respectively.

In the following, we give the vertex set $V(K_{p,p,p-1} - e)$ with a partition $\pi_4 : V(K_{p,p,p-1} - e) = V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5'$, where $V_1' = \{u_1, u_2, \ldots, u_{p-1}\}$, $V_2' = \{u_p\}$, $V_3' = \{v_1\}$, $V_4' = \{v_2, v_3, \ldots, v_p\}$, $V_5' = \{w_1, w_2, \ldots, w_{p-1}\}$. Then, the quotient matrix Q_{π_4} of $K_{p,p,p-1} - e$ takes the following form:

$$Q_{\pi_4} = \begin{bmatrix} 6p-7 & 2 & 1 & p-1 & p-1 \\ 2p-2 & 4p-2 & 2 & p-1 & p-1 \\ p-1 & 2 & 4p-2 & 2p-2 & p-1 \\ p-1 & 1 & 2 & 6p-7 & p-1 \\ p-1 & 1 & 1 & p-1 & 6p-8 \end{bmatrix}.$$

By a simple calculation, we get the characteristic polynomial of Q_{π_4} as

$$f_4(x) = [x-4p+5][x-5p+5][x^3 + (16-17p)x^2 + (92p^2-175p+73)x - 160p^3 + 456p^2 - 380p + 76].$$

This finishes the proof. □

For simplicity we may assume, that $f_4(x) = g_4(x)t_4(x)h_4(x)$, where

$$\begin{aligned} g_4(x) &= x - 4p + 5, \\ t_4(x) &= x - 5p + 5, \\ h_4(x) &= x^3 + (16-17p)x^2 + (92p^2-175p+73)x - 160p^3 + 456p^2 - 380p + 76. \end{aligned}$$

Lemma 5.3. *Let x_1 be the root of $g_4(x)$ and x_2 be the root of $t_4(x)$. Then, for $p \geq 3$, $x_1 < \frac{2W}{n} < x_2$.*

Proof. The result can be seen immediately by simple elementary operations. □

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Lemma 5.4. *Assume that x_3, x_4 and x_5 are three roots of $h_4(x)$. Then,*
i) For $4 \leq p \leq 9$, $x_3 < \frac{2W}{n} < x_4 \leq x_5$.
ii) For $p \geq 10$, $\frac{2W}{n} < x_3 < x_4 < x_5$.

Proof. *i)* For $4 \leq p \leq 9$, see Table 3.

TABLE 3. Roots of $h_4(x)$ for $4 \leq p \leq 9$

p	x_3	x_4	x_5	$\frac{2W}{n}$
4	12.337	13.698	25.964	12.909
5	16.503	18.676	33.821	16.857
6	20.602	23.668	41.730	20.824
7	24.668	28.665	49.667	24.8
8	28.715	33.664	57.621	28.783
9	32.751	38.663	65.586	32.769

ii) If $x_3 < x_4 < x_5$ are the roots of the polynomial $h_4(x)$, then, we have,

$$\begin{aligned} \sum_{i=3}^{5} x_i &= 17p - 16, \\ x_3x_4 + x_3x_5 + x_4x_5 &= 92p^2 - 175p + 73, \\ \prod_{i=3}^{5} x_i &= 160p^3 - 456p^2 + 380p - 76. \end{aligned}$$

Let $\delta_1 = x_3 - \frac{2W}{n}$, $\delta_2 = x_4 - \frac{2W}{n}$ and $\delta_3 = x_5 - \frac{2W}{n}$. Then, we have

$$\delta_1\delta_2\delta_3 = \frac{36p^5 - 486p^4 + 1476p^3 - 952p^2 + 120p - 2}{(3p-1)^3}.$$

Now $\delta_1\delta_2\delta_3 > 0$, for $p \geq 10$. This implies that, one or all of δ_i's is positive. Since $h_4(\frac{2W}{n}) < 0$ and $h_4(\frac{2W}{n} + 1) > 0$, then, by the intermediate value theorem, $\frac{2W}{n} < x_3 < \frac{2W}{n} + 1$. Similarly because of $h_4(\frac{2W}{n} + p - 4) > 0$ and $h_4(\frac{2W}{n} + p - 3) < 0$, we have $\frac{2W}{n} + p - 4 < x_4 < \frac{2W}{n} + p - 3$. Hence two of δ_i's are positive. That is, it is impossible that one of δ_i's is positive. Consequently all of δ_i's is positive.

□

Theorem 5.5. *If $p \geq 3$, then,*

$$E^Q(K_{p,p,p-1}) < E^Q(K_{p,p,p-1} - e).$$

Proof. By the definition of distance signless Laplacian energy and Lemma 5.1, we observe $E^Q(K_{p,p,p-1}) = \frac{36p^2-60p+12}{3p-1}$. Let's examine the energy change in two cases.

https://dx.doi.org/10.22108/toc.2026.146794.2326

Cases 1: Let $X = E^Q(K_{p,p,p-1})$ and $Y = E^Q(K_{p,p,p-1} - e)$. For $3 \leq p \leq 9$, see Table 4.

TABLE 4. Distance signless Laplacian energy change for $3 \leq p \leq 9$

p	X	$SpD_Q(K_{p,p,p-1} - e)$	$\frac{2W}{n}$	Y	$Y - X$
3	20.166	$\{6, [7]^3, 8, 10, 8.783, 18.\}$	9	20.217	0.051
4	31.636	$\{[10]^2, [11]^5, 15, 12.337, 13.698, 25.964\}$	12.909	31.87	0.234
5	43.714	$\{[14]^3, [15]^7, 20, 16.503, 18.676, 33.821\}$	16.857	43.85	0.136
6	55.765	$\{[18]^4, [19]^9, 25, 20.602, 23.668, 41.730\}$	20.824	55.855	0.09
7	67.8	$\{[22]^5, [23]^{11}, 30, 24.668, 28.665, 49.667\}$	24.8	67.864	0.064
8	79.826	$\{[26]^6, [27]^{13}, 35, 28.715, 33.664, 57.621\}$	28.783	79.874	0.048
9	91.846	$\{[30]^7, [31]^{15}, 40, 32.751, 38.663, 65.586\}$	32.769	91.883	0.037

Cases 2: If $p \geq 10$, then, $E^Q(K_{p,p,p-1}) = \frac{36p^2-60p+12}{3p-1}$. By Theorem 5.2, Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned} E^Q(K_{p,p,p-1} - e) &= (2p-4)\left|4p-5-\frac{2W}{n}\right| + (p-2)\left|4p-6-\frac{2W}{n}\right| + \sum_{i=1}^{5}\left|x_i - \frac{2W}{n}\right| \\ &= (2p-4)\left(\frac{2W}{n} - (4p-5)\right) + (p-2)\left(\frac{2W}{n} - (4p-6)\right) + \left(\frac{2W}{n} - (4p-5)\right) \\ &+ \left(5p-5-\frac{2W}{n}\right) + \left(x_3 - \frac{2W}{n}\right) + \left(x_4 - \frac{2W}{n}\right) + \left(x_5 - \frac{2W}{n}\right) \\ &= \frac{36p^2-58p+6}{3p-1}. \end{aligned}$$

Clearly, $E^Q(K_{p,p,p-1}) = \frac{36p^2-60p+12}{3p-1}$. Then,

$$E^Q(K_{p,p,p-1} - e) - E^Q(K_{p,p,p-1}) = \frac{2p-6}{3p-1} > 0.$$

This finishes the proof.

□

Now we consider, that the two end vertices of the deleted edge e belong to two vertex subsets with different cardinalities. That is the one end vertex belongs to the set with cardinality p and the second end vertex belongs to the set with cardinality $p-1$.

Theorem 5.6. *Assume $G \cong K_{p,p,p-1}$ with $p \geq 3$ and $V(G) = V_1 \cup V_2 \cup V_3$. Then, for any edge $e = v_i w_j$ where $1 \leq i \leq p$, $1 \leq j \leq p-1$, the distance signless Laplacian eigenvalues of $K_{p,p,p-1} - e$ are $4p-5$*

https://dx.doi.org/10.22108/toc.2026.146794.2326

with multiplicity $2p-3$, $4p-6$ *with multiplicity* $p-3$ *and the roots* $x_i (i=1,2,3,4,5)$ *of the following polynomial*

$$f_5(x) = x^5 + (27-26p)x^4 + (265p^2 - 556p + 280)x^3 + (-1328p^3 + 4213p^2 - 4290p + 1388)x^2$$
$$+(3280p^4 - 13964p^3 + 21511p^2 - 14079p + 3250)x - 3200p^5 + 17120p^4 - 35400p^3 + 35052p^2$$
$$-16368p + 2800.$$

Proof. To show that $4p-5$ and $4p-6$ are eigenvalues of $D_Q(K_{p,p,p-1}-e)$ with multiplicities $2p-3$ and $p-3$, respectively, see the proof of Theorem 3.2.

In the following, we give the vertex set $V(K_{p,p,p-1}-e)$ with a partition $\pi_5 : V(K_{p,p,p-1}-e) = V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5'$, where $V_1' = \{u_1, u_2, \ldots, u_p\}$, $V_2' = \{v_1, v_2, \ldots, v_{p-1}\}$, $V_3' = \{v_p\}$, $V_4' = \{w_1\}$ and $V_5' = \{w_2, \ldots, w_{p-1}\}$. So, the quotient matrix Q_{π_5} with respect to the partition π_5 can be written by

$$Q_{\pi_5} = \begin{bmatrix} 6p-5 & p-1 & 1 & 1 & p-2 \\ p & 6p-7 & 2 & 1 & p-2 \\ p & 2p-2 & 4p-2 & 2 & p-2 \\ p & p-1 & 2 & 4p-3 & 2p-4 \\ p & p-1 & 1 & 2 & 6p-10 \end{bmatrix}.$$

By a simple calculation, we get the characteristic polynomial of Q_{π_5} as

$$f_5(x) = x^5 + (27-26p)x^4 + (265p^2 - 556p + 280)x^3 + (-1328p^3 + 4213p^2 - 4290p + 1388)x^2$$
$$+(3280p^4 - 13964p^3 + 21511p^2 - 14079p + 3250)x - 3200p^5 + 17120p^4 - 35400p^3 + 35052p^2$$
$$-16368p + 2800.$$

This finishes the proof. □

Theorem 5.7. *If* $3 \leq p \leq 76$, *then,* $E^Q(K_{p,p,p-1}-e) > E^Q(K_{p,p,p-1})$.

Proof. See Appendix A. □

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Appendix A. In the following table the second column shows the distance signless Laplacian energies of $K_{p,p,p-1}$, note that, we obtain these energies by using the equation $E^Q(K_{p,p,p-1}) = \frac{36p^2-60p+12}{3p-1}$, see, Theorem 5.5. We say $X = E^Q(K_{p,p,p-1})$ and $Y = E^Q(K_{p,p,p-1} - e)$.

p	X	$Sp(D_Q(K_{p,p,p-1} - e))$	$\frac{2W}{n}$	Y	$Y - X$
3	20.166	$\{[7]^3, 6.242, 7.615, 8.773, 10.270, 18.100\}$	9	20.74	0.574
4	31.6364	$\{[10]^1, [11]^5, 10.307, 11.936, 13.667, 15.191, 25.898\}$	12.9090	32.0595	0.4231
5	43.7143	$\{[14]^2, [15]^7, 14.333, 16.111, 18.632, 20.144, 33.779\}$	16.8571	43.9691	0.2548
6	55.7647	$\{[18]^3, [19]^9, 18.346, 20.217, 23.621, 25.115, 41.701\}$	20.8235	55.9328	0.1681
7	67.8	$\{[22]^4, [23]^{11}, 22.354, 24.286, 28.619, 30.094, 49.646\}$	24.8	67.92	0.12
8	79.8261	$\{[26]^5, [27]^{13}, 26.360, 28.335, 33.619, 35.080, 57.606\}$	28.7826	79.9143	0.0882
9	91.8462	$\{[30]^6, [31]^{15}, 30.363, 32.372, 38.621, 40.069, 65.574\}$	32.7692	91.9146	0.0684
10	103.8621	$\{[34]^7, [35]^{17}, 34.366, 36.400, 43.624, 45.061, 73.549\}$	36.7586	103.9163	0.0542
11	115.875	$\{[38]^8, [39]^{19}, 38.368, 40.423, 48.626, 50.055, 81.529\}$	40.75	115.918	0.043
12	127.8857	$\{[42]^9, [43]^{21}, 42.369, 44.441, 53.628, 55.049, 89.512\}$	44.7429	127.9229	0.0372
13	139.8947	$\{[46]^{10}, [47]^{23}, 46.371, 48.456, 58.630, 60.045, 97.498\}$	48.7368	139.9249	0.0302
14	151.9024	$\{[50]^{11}, [51]^{25}, 50.372, 52.469, 63.632, 65.041, 105.486\}$	52.7317	151.9278	0.0254
15	163.9091	$\{[54]^{12}, [55]^{27}, 54.373, 56.481, 68.914, 70.047, 113.184\}$	56.7272	163.9284	0.0193
16	175.9149	$\{[58]^{13}, [59]^{29}, 58.373, 60.489, 73.636, 75.036, 121.466\}$	60.7234	175.9356	0.0207
17	187.92	$\{[62]^{14}, [63]^{31}, 62.374, 64.497, 78.637, 80.033, 129.458\}$	64.72	187.938	0.018
18	199.9245	$\{[66]^{15}, [67]^{33}, 66.375, 68.504, 83.639, 85.031, 137.451\}$	68.7170	199.9401	0.0156
19	211.9286	$\{[70]^{16}, [71]^{35}, 70.375, 72.511, 88.640, 90.029, 145.445\}$	72.7143	211.9423	0.0137
20	223.9322	$\{[74]^{17}, [75]^{37}, 74.375, 76.517, 93.641, 95.028, 153.439\}$	76.7119	223.9448	0.0126
21	235.9355	$\{[78]^{18}, [79]^{39}, 78.376, 80.522, 98.642, 100.026, 161.434\}$	80.7097	235.9459	0.0104
22	247.9385	$\{[82]^{19}, [83]^{41}, 82.376, 84.526, 103.643, 105.025, 169.429\}$	84.7077	247.9498	0.0113
23	259.9412	$\{[86]^{20}, [87]^{43}, 86.376, 88.531, 108.644, 110.024, 177.425\}$	88.7059	259.9507	0.0095
24	271.9437	$\{[90]^{21}, [91]^{45}, 90.377, 92.534, 113.645, 115.023, 185.421\}$	92.7042	271.9526	0.0089
25	283.9459	$\{[94]^{22}, [95]^{47}, 94.377, 96.538, 118.646, 120.022, 193.418\}$	96.7027	283.9538	0.0079
26	295.9481	$\{[98]^{23}, [99]^{49}, 98.377, 100.541, 123.646, 125.021, 201.414\}$	100.7013	295.9562	0.0082
27	307.95	$\{[102]^{24}, [103]^{51}, 102.377, 104.544, 128.647, 130.020, 209.411\}$	104.7	307.9580	0.0008
28	319.9518	$\{[106]^{25}, [107]^{53}, 106.378, 108.547, 133.648, 135.019, 217.409\}$	108.6988	319.9572	0.0054
29	331.9535	$\{[110]^{26}, [111]^{55}, 110.378, 112.550, 138.648, 140.019, 225.406\}$	112.6977	331.9580	0.0045
30	343.9551	$\{[114]^{27}, [115]^{57}, 114.378, 116.552, 143.649, 145.018, 233.403\}$	116.6966	343.9602	0.0051
31	355.9565	$\{[118]^{28}, [119]^{59}, 118.378, 120.554, 148.649, 150.017, 241.401\}$	120.6957	355.9621	0.0056
32	367.9579	$\{[122]^{29}, [123]^{61}, 122.378, 124.556, 153.650, 155.017, 249.399\}$	124.6947	367.9636	0.0057
33	379.9592	$\{[126]^{30}, [127]^{63}, 126.378, 128.558, 158.650, 160.016, 257.397\}$	128.6939	379.9647	0.0055
34	391.9604	$\{[130]^{31}, [131]^{65}, 130.378, 132.560, 163.651, 165.016, 265.395\}$	132.6931	391.9656	0.0052
35	403.9615	$\{[134]^{32}, [135]^{67}, 134.379, 136.562, 168.651, 170.015, 273.393\}$	136.6923	403.9642	0.0027
36	415.9626	$\{[138]^{33}, [139]^{69}, 138.379, 140.563, 173.652, 175.015, 281.392\}$	140.6916	415.9665	0.0039
37	427.9636	$\{[142]^{34}, [143]^{71}, 142.379, 144.565, 178.652, 180.014, 289.390\}$	144.6909	427.9665	0.0029
38	439.9646	$\{[146]^{35}, [147]^{73}, 146.379, 148.566, 183.652, 185.014, 297.389\}$	148.6903	439.9684	0.0038
39	451.9655	$\{[150]^{36}, [151]^{75}, 150.379, 152.568, 188.653, 190.014, 305.387\}$	152.6897	451.9681	0.0026

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p	X	$Sp(D_Q(K_{p,p,p-1}-e))$	$\frac{2W}{n}$	Y	$Y-X$
40	463.9664	$\{[154]^{37},[155]^{77},154.379,156.569,193.653,195.013,313.386\}$	156.6891	463.9695	0.0031
41	475.9672	$\{[158]^{38},[159]^{79},158.379,160.570,198.653,200.013,321.384\}$	160.6885	475.9709	0.0037
42	487.968	$\{[162]^{39},[163]^{81},162.379,164.571,203.654,205.013,329.383\}$	164.688	487.972	0.004
43	499.9688	$\{[166]^{40},[167]^{83},166.379,168.573,208.654,210.012,337.382\}$	168.6875	499.971	0.0022
44	511.9695	$\{[170]^{41},[171]^{85},170.379,172.574,213.654,215.012,345.381\}$	172.6870	511.9719	0.0024
45	523.9701	$\{[174]^{42},[175]^{87},174.379,176.575,218.654,220.012,353.380\}$	176.6866	523.9726	0.0025
46	535.9708	$\{[178]^{43},[179]^{89},178.379,180.576,223.655,225.011,361.379\}$	180.6861	535.9732	0.0024
47	547.9714	$\{[182]^{44},[183]^{91},182.379,184.577,228.655,230.011,369.378\}$	184.6857	547.9737	0.0023
48	559.9720	$\{[186]^{45},[187]^{93},186.380,188.577,233.655,235.011,377.377\}$	188.6853	559.9741	0.0021
49	571.9726	$\{[190]^{46},[191]^{95},190.380,192.578,238.655,240.011,385.376\}$	192.6849	571.9744	0.0018
50	583.9732	$\{[194]^{47},[195]^{97},194.380,196.579,243.656,245.010,393.375\}$	196.6846	583.9746	0.0014
51	595.9737	$\{[198]^{48},[199]^{99},198.380,200.580,248.656,250.010,401.374\}$	200.6842	595.9747	0.001
52	607.9742	$\{[202]^{49},[203]^{101},202.380,204.581,253.656,255.010,409.374\}$	204.6839	607.9748	0.0006
53	619.9747	$\{[206]^{50},[207]^{103},206.380,208.581,258.656,260.010,417.373\}$	208.6835	619.9767	0.002
54	631.9752	$\{[210]^{51},[211]^{105},210.380,212.582,263.656,265.010,425.372\}$	212.6832	631.9766	0.0014
55	643.9756	$\{[214]^{52},[215]^{107},214.380,216.583,268.657,270.009,433.371\}$	216.6829	643.9764	0.0008
56	655.9760	$\{[218]^{53},[219]^{109},218.380,220.583,273.657,275.009,441.371\}$	220.6826	655.9782	0.0022
57	667.9765	$\{[222]^{54},[223]^{111},222.380,224.584,278.657,280.009,449.370\}$	224.6824	667.9779	0.0014
58	679.9769	$\{[226]^{55},[227]^{113},226.380,228.585,283.657,285.009,457.369\}$	228.6821	679.9775	0.0006
59	691.9773	$\{[230]^{56},[231]^{115},230.380,232.585,288.657,290.009,465.369\}$	232.6818	691.9791	0.0018
60	703.9777	$\{[234]^{57},[235]^{117},234.380,236.586,293.657,295.009,473.368\}$	236.6816	703.9786	0.0009
61	715.9780	$\{[238]^{58},[239]^{119},238.380,240.586,298.658,300.008,481.368\}$	240.6813	715.9801	0.0021
62	727.9784	$\{[242]^{59},[243]^{121},242.380,244.587,303.658,305.008,489.367\}$	244.6811	727.9795	0.0011
63	739.9787	$\{[246]^{60},[247]^{123},246.380,248.587,308.658,310.008,497.366\}$	248.6809	739.9809	0.0022
64	751.9791	$\{[250]^{61},[251]^{125},250.380,252.588,313.658,315.008,505.366\}$	252.6806	751.9802	0.0011
65	763.9794	$\{[254]^{62},[255]^{127},254.380,256.588,318.658,320.008,513.365\}$	256.6804	763.9815	0.0021
66	775.9797	$\{[258]^{63},[259]^{129},258.380,260.589,323.658,325.008,521.365\}$	260.6802	775.9808	0.0011
67	787.98	$\{[262]^{64},[263]^{131},262.380,264.589,328.658,330.008,529.364\}$	264.68	787.9820	0.002
68	799.9803	$\{[266]^{65},[267]^{133},266.380,268.590,333.658,335.008,537.364\}$	268.6798	799.9812	0.0009
69	811.9806	$\{[270]^{66},[271]^{135},270.380,272.590,338.659,340.007,545.364\}$	272.6796	811.9823	0.0017
70	823.9809	$\{[274]^{67},[275]^{137},274.380,276.590,343.659,345.007,553.363\}$	276.6794	823.9834	0.0025
71	835.9811	$\{[278]^{68},[279]^{139},278.380,280.591,348.659,350.007,561.363\}$	280.6792	835.9825	0.0014
72	847.9814	$\{[282]^{69},[283]^{141},282.380,284.591,353.659,355.007,569.362\}$	284.6791	847.9836	0.0022
73	859.9817	$\{[286]^{70},[287]^{143},286.380,288.592,358.659,360.007,577.362\}$	288.6789	859.9826	0.0009
74	871.9819	$\{[290]^{71},[291]^{145},290.380,292.592,363.659,365.007,585.362\}$	292.6787	871.9836	0.0017
75	883.9821	$\{[294]^{72},[295]^{147},294.380,296.592,368.659,370.007,593.361\}$	296.6786	883.9846	0.0025
76	895.9824	$\{[298]^{73},[299]^{149},298.380,300.593,373.659,375.007,601.361\}$	300.6784	895.9835	0.0011

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