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EDGE COALITIONS IN GRAPHS

NAZLI BESHARATI*, AZAM SADAT EMADI, AND IMAN MASOUMI

Abstract. Coalition theory for vertex sets has been extensively studied, whereas its edge counterpart has received much less attention. In this paper, we introduce edge coalitions, edge coalition partitions, the edge coalition number, and edge coalition graphs. We prove that every graph with at least one edge admits an edge coalition partition and establish bounds on the edge coalition number, including lower bounds for complete and complete bipartite graphs. We characterize trees and unicyclic graphs G satisfying $EC(G) = |E(G)|$. For connected graphs that are neither trees nor unicyclic, we reduce the problem to a line-graph condition; a direct structural characterization in terms of G remains open. We also determine the edge coalition number for several standard graph families, including paths, cycles, stars, and complete graphs of small order. In addition, we establish structural properties of edge coalition graphs and characterize all connected self-edge coalition graphs. These results extend coalition theory from vertices to edges and provide a foundation for further study of edge coalition structures.

1. Introduction

Community issues are often too complex to be addressed by a single agency or organization, making coalitions an effective strategy for achieving common objectives. Such cooperation arises in education, business, government, and coalition voting under proportional representation systems. Motivated by these applications, coalition structures have attracted considerable attention in graph theory. Throughout this paper, we restrict our attention to coalitions formed by two disjoint sets. In [5],

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*Corresponding author.

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Haynes et al. introduced the concept of coalition for vertices and further investigated its properties in [7, 8, 6]. A coalition consists of two disjoint vertex sets, neither of which is a dominating set individually, while their union forms a dominating set. They also introduced vertex coalition partitions and established several fundamental results. More recently, coalition theory has been extended to related settings, including connected coalitions [1] and double coalitions in general graphs [9] and in regular graphs [10].

Edge domination is the edge counterpart of vertex domination and has been widely studied [2, 3]. Coalition concepts, however, have so far been examined only for vertex sets. This raises a natural question: can the notion of coalition be extended from vertices to edges? In this paper we answer this question in the affirmative. To the best of our knowledge, edge coalition partitions and the associated edge coalition graphs have not been studied before.

Although many basic properties of edge coalitions follow directly from the line graph correspondence $EC(G) = C(L(G))$, established as Lemma 3.3 below, we also provide direct combinatorial arguments in the original graph. We introduce the concepts of edge coalition, edge coalition partition, edge coalition number, and edge coalition graph, and investigate their fundamental properties.

Several edge-set partitions are already known in graph theory and computer science, such as proper edge colorings (partitions in to matchings) [4] and edge-domatic partitions [14]. The notion introduced here provides another natural partition, this time arising from edge domination and coalition structures.

The main contributions of this paper are as follows. We introduce the concepts of edge coalition, edge coalition partition, edge coalition number, and edge coalition graph; prove that every graph with at least one edge admits an edge coalition partition; characterize trees and unicyclic graphs G satisfying $EC(G) = m$, and reduce the remaining connected case to a line-graph condition (Proposition 6.3), leaving a direct structural characterization in terms of G as an open problem; determine the edge coalition number for several graph classes and characterize the case $EC(G) \in \{1, 2, 3\}$; and determine the edge coalition graphs associated with edge coalition partitions for several important graph classes.

The remainder of the paper is organized as follows. Section 2 presents the notation and preliminary results. Section 3 develops the basic theory of edge coalitions. Sections 4 and 5 establish bounds and certain characterizations of the edge coalition number. Section 6 characterizes trees and unicyclic graphs satisfying $EC(G) = m$, and reduces the remaining connected case to a line-graph condition. Section 7 investigates edge coalition graphs of several graph classes. Section 8 concludes with open problems and future directions.

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2. Preliminaries

All graphs considered in this paper are finite, simple, and undirected. In this section, we recall the notation and terminology used throughout the paper. For undefined notation and terminology, we refer to [4, 11].

Two edges are adjacent if they share a common vertex. The edge degree of an edge $e = uv$, denoted by $\deg(e)$, is the number of edges adjacent to e. Equivalently, $\deg(e) = \deg(u) + \deg(v) - 2$. An edge e is called isolated if $deg(e) = 0$, i.e., it has no adjacent edges. The open neighborhood of an edge e is $N(e) = \{f \in E(G) : f \text{ is adjacent to } e\}$. Each edge in $N(e)$ is called a neighbor of e; thus $deg(e) = |N(e)|$. The closed neighborhood of an edge e is $N[e] = N(e) \cup \{e\}$.

Let G be a graph of size m. A vertex $v \in V(G)$ is called a *full vertex* if it is adjacent to every other vertex of G, that is, $\deg(v) = n-1$, where $n = |V(G)|$. An edge e is called a *full edge* if $deg(e) = m-1$, that is, if e is adjacent to every other edge of G. For example, every edge of a star is a full edge.

Let $G = (V, E)$ be a graph. A subset $E_i \subseteq E$ is called a *singleton set* if $|E_i| = 1$; otherwise, if $|E_i| \geq 2$, it is called a *non-singleton set*.

A set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) \setminus S$ is adjacent to a vertex of S. Similarly, an edge-dominating set is a subset $D \subseteq E(G)$ such that every edge in $E(G) \setminus D$ is adjacent to an edge of D. The minimum cardinality of an edge-dominating set is called the *edge domination number* of G and is denoted by $\gamma'(G)$. An edge-dominating set D is *minimal* if no proper subset of D is an edge-dominating set.

An edge-domatic partition of a graph G is a partition of $E(G)$ into edge-dominating sets. For further details, see [2, 14, 13].

The line graph of a graph G, denoted by $L(G)$, is the graph whose vertices correspond to the edges of G, where two vertices are adjacent if and only if the corresponding edges are adjacent [11]. It is well known that every edge-dominating set of G corresponds to a dominating set of $L(G)$; hence, $\gamma'(G) = \gamma(L(G))$. It follows immediately that every full edge forms a singleton edge-dominating set.

A unicyclic graph is a connected graph containing exactly one cycle and therefore has the same number of vertices and edges. Examples include paw, pan, and sunlet graphs.

The double star $S(p, q)$, where $p \geq q \geq 0$, is the graph obtained from two stars $K_{1,p}$ and $K_{1,q}$ by joining their centers with an edge.

For a graph G, a *vertex coalition* consists of two disjoint sets of vertices, neither of which is a dominating set, but whose union is a dominating set. A *vertex coalition partition* (or *c-partition*) is a partition of $V(G)$ such that each part is either a singleton dominating set or forms a vertex coalition with another part. The *vertex coalition number* $C(G)$ is the maximum order of a c-partition of G. These notions were introduced in [5].

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3. Edge Coalitions

In this section, we introduce the concept of edge coalitions and investigate their fundamental properties.

Definition 3.1. *Let $G = (V, E)$ be a graph. An edge coalition in G consists of two disjoint nonempty sets of edges E_1 and E_2, where neither E_1 nor E_2 is an edge-dominating set, but their union $E_1 \cup E_2$ forms an edge-dominating set. We say that E_1 and E_2 form an edge coalition and refer to them as edge coalition partners.*

Definition 3.2. *An* edge coalition partition*, henceforth called an* ec-partition*, of a graph G is an edge partition $\pi = \{E_1, \ldots, E_k\}$ such that every set E_i of π is either a singleton edge-dominating set, or is not an edge-dominating set but forms an edge coalition with another set E_j $(j \neq i)$ in π. In particular, no part of an ec-partition can be a non-singleton edge-dominating set. The* edge coalition number*, denoted by $EC(G)$, is the maximum number of parts k of an ec-partition of G. An ec-partition with $EC(G)$ parts is called an $EC(G)$-partition.*

Lemma 3.3. *For any graph G with at least one edge, $EC(G) = C(L(G))$. Moreover, two disjoint sets of edges form an edge coalition in G if and only if the corresponding sets of vertices form a vertex coalition in $L(G)$. Consequently, ec-partitions of G correspond exactly to c-partitions of $L(G)$.*

Proof. For each $A \subseteq E(G)$ let $A^* = \{v_e : e \in A\} \subseteq V(L(G))$. By definition of $L(G)$, A edge-dominates G if and only if A^* dominates $L(G)$. Hence for disjoint $A, B \subseteq E(G)$, the pair (A, B) is an edge coalition in G if and only if (A^*, B^*) is a vertex coalition in $L(G)$. Since $A \mapsto A^*$ is a bijection between partitions of $E(G)$ and partitions of $V(L(G))$ preserving the number of parts, ec-partitions of G correspond to c-partitions of $L(G)$. Therefore $EC(G) = C(L(G))$. □

Note that if G has no full edges, then no singleton E_i in an ec-partition can be an edge-dominating set. Consequently, every subset E_i must form an edge coalition with another subset E_j in the partition.

Definition 3.4. *Let G be a graph of size m with an edge set $E = \{e_1, e_2, \ldots, e_m\}$. The singleton partition, denoted π_0, of G is the partition of E into m singleton sets, that is, $\pi_0 = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$.*

Example 3.5. *Consider the cycle C_5 with edge set $E(C_5) = \{e_1, e_2, e_3, e_4, e_5\}$, where the indices are taken modulo 5. Let $\pi_0 = \big\{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}\big\}$. No singleton edge is edge-dominating. Moreover, for each $i \in \{1, \ldots, 5\}$, the set $\{e_i, e_{i+2}\}$ is edge-dominating. Thus, every part of π_0 has an edge coalition partner, and hence π_0 is an ec-partition of C_5. Since π_0 has five parts and C_5 has five edges, it follows that $EC(C_5) = 5$. Thus, π_0 is an $EC(C_5)$-partition.*

Definition 3.6. *Let $\pi = \{E_1, E_2, \ldots, E_k\}$ be an ec-partition of a graph $G = (V, E)$. The* edge coalition graph*, denoted by $ECG(G, \pi)$, is the graph whose vertices are in one-to-one correspondence with the*

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sets of π. Two vertices of $ECG(G,\pi)$ are adjacent if and only if their corresponding sets E_i and E_j form an edge coalition in G.

Note that, in Definition 3.6, we use the same notation E_i for both a part of π and the corresponding vertex of $ECG(G,\pi)$; the intended meaning is clear from the context. Moreover, for any graph G and any $EC(G)$-partition π, there exists a corresponding edge coalition graph $ECG(G,\pi)$ with $EC(G)$ vertices.

4. Fundamental Results on the Edge Coalition Number

In this section, we establish fundamental results and bounds for the edge coalition number and determine it for several important graph classes.
Every graph has a vertex coalition partition [5]; applying this to $L(G)$ and using the correspondence above yields the following.

Theorem 4.1. *Every graph with at least one edge has an ec-partition.*

Proof. By Lemma 3.3, $EC(G) = C(L(G))$. Every graph admits a vertex coalition partition [5], so $L(G)$ has a c-partition. Its preimage under the bijection of Lemma 3.3 is an ec-partition of G. □

The following values follow immediately from the corresponding results on coalition numbers of line graphs together with the identities $L(K_{1,n}) = K_n$, $L(P_n) = P_{n-1}$, and $L(C_n) = C_n$. Additionally, the line graph $L(S(p,q))$ consists of two cliques K_{p+1} and K_{q+1} sharing one vertex; its vertex coalition number is $p+q+1$. Thus, we have the following observation.

Observation 4.2. *For any star $K_{1,n}$, $EC(K_{1,n}) = n$, and for any double star $S(p,q)$, $EC(S(p,q)) = p+q+1$.*

For any path P_n, Lemma 3.3 and [5, Theorem 7] *imply that:*

$$EC(P_n) = \begin{cases} n-1, & \text{if } n \leq 5, \\ 4, & \text{if } n = 6, \\ 5, & \text{if } 7 \leq n \leq 10, \\ 6, & \text{if } n \geq 11. \end{cases}$$

For any cycle C_n, Lemma 3.3 and [5, Theorem 8] *imply that:*

$$EC(C_n) = \begin{cases} n, & \text{if } n \leq 6, \\ 5, & \text{if } n = 7, \\ 6, & \text{if } n \geq 8. \end{cases}$$

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Example 4.3. *Consider the double star $S(2,1)$ with edge set $E(S(2,1)) = \{e_1, e_2, e_3, e_4\}$, where e_1 is the edge joining the two centers. The singleton partition $\Pi = \big\{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}\big\}$ is an ec-partition of $S(2,1)$. Indeed, $\{e_1\}$ is an edge-dominating set, while neither $\{e_2\}$, $\{e_3\}$, nor $\{e_4\}$ is edge-dominating. Moreover, $\{e_2\}$ and $\{e_4\}$ form an edge coalition, and $\{e_3\}$ and $\{e_4\}$ form an edge coalition. Hence, every part of Π is either edge-dominating or has an edge coalition partner, and therefore $EC(S(2,1)) = 4$.*

Consider the path P_{13} with edges $e_1, e_2, \ldots, e_{12}$, and let $\Pi = \{E_1, E_2, E_3, E_4, E_5, E_6\}$, where $E_1 = \{e_1, e_6, e_9, e_{11}\}, E_2 = \{e_2, e_5, e_{10}, e_{12}\}, E_3 = \{e_3\}, E_4 = \{e_4\}, E_5 = \{e_7\}, E_6 = \{e_8\}$.

Label the edges, in order, by the sequence $(1,2,3,4,2,1,5,6,1,2,1,2)$, where an edge labeled i belongs to E_i. To see that Π is an ec-partition of P_{13}, observe that none of its parts is an edge-dominating set. Moreover, the pairs $(E_1, E_3), (E_1, E_4), (E_2, E_5), (E_2, E_6)$ form edge coalitions. Hence, every part of Π has an edge coalition partner in Π, and therefore Π is an ec-partition of P_{13}.

Proposition 4.4. *Let G be a graph with maximum degree $\Delta(G) \geq 2$, and let π be an $EC(G)$-partition. If $X \in \pi$, then X has at most $2\Delta(G) - 1$ edge coalition partners.*

Proof. If X edge-dominates G then X is a singleton and has no partner. Assume therefore that X is not edge-dominating, and choose an edge $e = uv$ not dominated by X. Any partner of X must dominate e, hence must contain an edge of $N[e]$. Since the parts of π are disjoint, at most $|N[e]|$ such sets exist. Now $|N[e]| = \deg(e) + 1 = \deg(u) + \deg(v) - 1 \leq 2\Delta(G) - 1$. □

The following bounds are immediate from the definition and the fact that any edge partition of an m-edge graph has at most m parts.

Corollary 4.5. *For any connected graph $G = (V, E)$ of order $n \geq 2$ and size m, the edge coalition number satisfies $1 \leq EC(G) \leq m \leq \frac{n(n-1)}{2}$.*

Now, we can characterize the graph G for which $EC(G) = \frac{n(n-1)}{2}$ holds.

Proposition 4.6. *For any connected graph G with $n \geq 2$, $EC(G) = \frac{n(n-1)}{2}$ if and only if G is one of the complete graphs in the set $\{K_2, K_3, K_4, K_5\}$.*

Proof. Let $G \cong K_n$ for some $n \in \{2,3,4,5\}$. For $n = 2, 3$, every edge is a full edge, so the singleton edge partition is an $EC(K_n)$-partition. For $n = 4, 5$, every edge has a disjoint edge f, and the set $\{e, f\}$ is edge-dominating. Hence, the singleton edge partition is an *ec*-partition. Therefore, $EC(K_n) = |E(K_n)| = \frac{n(n-1)}{2}$.

For the converse, assume that $EC(G) = \frac{n(n-1)}{2}$. Since $EC(G) \leq |E(G)| \leq \frac{n(n-1)}{2}$, we have $|E(G)| = \frac{n(n-1)}{2}$, and hence $G \cong K_n$. It remains to determine those n for which the singleton edge partition of K_n is an *ec*-partition. For $n \geq 6$, consider any two edges of K_n. If they are adjacent, they dominate $3n - 6$ edges; if they are disjoint, they dominate $4n - 10$ edges. Thus, for $n \geq 6$, any

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two edges dominate at most $4n - 10 < \frac{n(n-1)}{2}$ edges. Hence no singleton edge has an edge coalition partner, and the singleton edge partition is not an *ec*-partition. Therefore $n \leq 5$. Consequently, $G \cong K_n$ for some $n \in \{2, 3, 4, 5\}$. □

Note that K_2 is the only graph that attains the lower bound in Corollary 4.5, while the complete graphs K_n for $n \in \{2, 3, 4, 5\}$ attain the upper bound.

Theorem 4.7. [2] *For any connected graph G of even order n, the edge domination number satisfies $\gamma'(G) = \frac{n}{2}$ if and only if G is isomorphic to K_n or $K_{\frac{n}{2},\frac{n}{2}}$.*

Proposition 4.8. *For any complete graph K_n of even order $n \geq 4$, $EC(K_n) \geq 2(n-1)$.*

Proof. Since K_n admits a 1-factorization [4], its edge set can be partitioned into $n-1$ pairwise edge-disjoint perfect matchings: $E(K_n) = M_1 \cup M_2 \cup \cdots \cup M_{n-1}$. Each M_i has cardinality $n/2$. By Theorem 4.7, $\gamma'(K_n) = n/2$, so each M_i is a minimum edge-dominating set. Partition each M_i into two nonempty proper subsets $M_{i,1}$ and $M_{i,2}$. Since M_i is minimum, neither $M_{i,1}$ nor $M_{i,2}$ is edge-dominating, while $M_{i,1} \cup M_{i,2} = M_i$ is edge-dominating. Thus $M_{i,1}$ and $M_{i,2}$ form an edge coalition. Therefore $\{M_{1,1}, M_{1,2}, \ldots, M_{n-1,1}, M_{n-1,2}\}$ is an ec-partition of K_n with $2(n-1)$ parts. Hence $EC(K_n) \geq 2(n-1)$. The bound is sharp for K_4. □

Proposition 4.9. *For the complete bipartite graph $K_{r,s}$ with $2 \leq r \leq s$, $EC(K_{r,s}) \geq 2s$.*

Proof. Let the partite sets of $K_{r,s}$ be $X = \{a_1, \ldots, a_r\}$ and $Y = \{b_1, \ldots, b_s\}$. For each $1 \leq i \leq s$, define $X_i = \{a_1b_i, a_2b_i, \ldots, a_rb_i\}$. Each X_i is an edge-dominating set: indeed, any edge not in X_i is of the form a_jb_ℓ with $\ell \neq i$, and it is adjacent to $a_jb_i \in X_i$. Moreover, X_i is minimal. To see this, remove any edge a_jb_i from X_i. Then the edge a_jb_ℓ with $\ell \neq i$ is not adjacent to any remaining edge of X_i, so the remaining set is not edge-dominating. Thus X_i is a minimal edge-dominating set.

The sets $X_1, \ldots, X_s$ are pairwise edge-disjoint. For each $1 \leq i \leq s$, partition X_i into two nonempty subsets $X_{i,1} = \{a_1b_i\}$, $X_{i,2} = \{a_2b_i, \ldots, a_rb_i\}$. Since $r \geq 2$, the set $X_{i,2}$ is nonempty. Because X_i is a minimal edge-dominating set, neither $X_{i,1}$ nor $X_{i,2}$ is edge-dominating, while their union $X_{i,1} \cup X_{i,2} = X_i$ is edge-dominating. Hence $X_{i,1}$ and $X_{i,2}$ form an edge coalition.

Therefore $\pi = \{X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}, \ldots, X_{s,1}, X_{s,2}\}$ is an ec-partition of $K_{r,s}$ with $2s$ parts. Finally, since $r \geq 2$, no singleton edge is edge-dominating: a single edge a_ib_j does not dominate the edge a_kb_ℓ with $k \neq i$ and $\ell \neq j$. Thus $EC(K_{r,s}) \geq 2s$. The bound is sharp for every $K_{2,s}$ with $s \geq 2$, since $EC(K_{2,s}) = 2s$. However, it is not sharp in general; for example, $EC(K_{3,3}) \geq 7 > 6 = 2s$. □

5. Bounds and Small Values of the Edge Coalition Number

In this section, we establish bounds on the edge coalition number of a graph and characterize the graphs G for which $EC(G) \in \{1, 2, 3\}$.

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Observation 5.1. *If G has neither isolated edges nor full edges, then*

$$EC(G) \geq \delta(L(G)) + 2 = \min_{uv \in E(G)} \big(\deg_G(u) + \deg_G(v)\big) \geq 3.$$

Proof. Since G has no isolated edges, $\delta(L(G)) \geq 1$. Moreover, since G has no full edges, $L(G)$ has no full vertices. By [5, Theorem 3], $C(L(G)) \geq \delta(L(G)) + 2$. Hence, by Lemma 3.3, $EC(G) = C(L(G)) \geq \delta(L(G)) + 2$. Finally, for every edge $uv \in E(G)$, $\deg_{L(G)}(uv) = \deg_G(u) + \deg_G(v) - 2$. Consequently, $\delta(L(G)) + 2 = \min_{uv \in E(G)} \big(\deg_G(u) + \deg_G(v)\big) \geq 3$, as required. □

5.1. **Graphs with Small Values of the Edge Coalition Number.** We characterize graphs with edge coalition numbers at most three.

Theorem 5.2. *Let G be a connected graph of order $n \geq 2$. Then*

(1) $EC(G) = 1$ *if and only if* $G \cong K_2$.
(2) $EC(G) = 2$ *if and only if* $G \cong P_3$.
(3) $EC(G) = 3$ *if and only if* $G \in \{C_3, P_4, K_{1,3}\}$.

Proof. (1) If $G \cong K_2$, its only edge is a full edge, so the singleton partition is an ec-partition with one part. Conversely, if $EC(G) = 1$, then an $EC(G)$-partition has one part, namely $\{E(G)\}$. By Definition 3.2, this part must be a singleton edge-dominating set. Hence $|E(G)| = 1$, and since G is connected, $G \cong K_2$.

(2) If $G \cong P_3$, then $L(P_3) \cong K_2$, and by Lemma 3.3 and [5, Theorem 5], $EC(P_3) = C(K_2) = 2$. Conversely, if $EC(G) = 2$, then $C(L(G)) = 2$ by Lemma 3.3. Since $L(G)$ has no isolated vertices, by [5, Theorem 5] we get $L(G) \cong K_2$, so $G \cong P_3$.

(3) By Lemma 3.3, $EC(G) = C(L(G))$. If $G \in \{C_3, P_4, K_{1,3}\}$, then $L(G) \cong K_3$ or $L(G) \cong K_{1,2}$. Indeed, $L(C_3) \cong L(K_{1,3}) \cong K_3$ and $L(P_4) \cong K_{1,2}$. By [5, Theorem 6], $C(L(G)) = 3$, so $EC(G) = 3$.

Conversely, suppose that $EC(G) = 3$. Then, by Lemma 3.3, $C(L(G)) = 3$. Since G is connected and has at least three edges, the graph $L(G)$ has no isolated vertices. Hence, by [5, Theorem 6], $L(G) \cong K_3$ or $L(G) \cong K_{1,r}$ for some $r \geq 2$.

If $L(G) \cong K_3$, then the three edges of G are pairwise adjacent. Thus, either they form a triangle or they have a common endpoint. Hence, $G \cong C_3$ or $G \cong K_{1,3}$.

Suppose that $L(G) \cong K_{1,r}$. Let $e = uv$ be the edge of G corresponding to the center of $K_{1,r}$. The remaining r edges of G are all adjacent to e and are pairwise nonadjacent. Hence, at most one of them can be incident with u, and at most one can be incident with v. Hence $r \leq 2$. Since $r \geq 2$, we have $r = 2$, and the three edges of G form a path of length 3. Thus, $G \cong P_4$. □

6. Graphs with Maximum Edge Coalition Number

In this section, we study graphs G for which the edge coalition number equals the size of G.

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6.1. **Unicyclic Graphs.** Let G be a connected unicyclic graph of order n and size m. Recall that every connected unicyclic graph of order n has size $m = n$. In this subsection, we characterize all connected unicyclic graphs satisfying $EC(G) = m$.

Let Θ denote the family of unicyclic graphs defined as follows: $\Theta = \Theta_a \cup \Theta_b \cup \Theta_c \cup \Theta_d \cup \Theta_e$, where
(a) $\Theta_a = \{C_n : 3 \leq n \leq 6\}$.
(b) Θ_b is the set of graphs obtained from C_3 by attaching stars $K_{1,r_1}, \ldots, K_{1,r_t}$, where $r_i \geq 1$ and $1 \leq t \leq 3$, at t distinct vertices of C_3, one star at each vertex, with the center of each attached star identified with its corresponding vertex of C_3.
(c) Θ_c is the set of graphs obtained from C_3 by choosing a vertex v_0 of C_3, attaching a path v_0v_1 to v_0, and attaching $p \geq 1$ pendant edges at v_1.
(d) Θ_d is the set of graphs obtained from C_4 by attaching either one star $K_{1,r}$ or two stars $K_{1,r}$ and $K_{1,s}$, where $r, s \geq 1$, at distinct vertices of C_4, with the center of each attached star identified with its corresponding vertex of C_4.
(e) Θ_e is the set of graphs obtained from C_5 by attaching a star $K_{1,r}$, where $r \geq 1$, at one vertex of C_5, with the center of the attached star identified with that vertex.

Representative graphs illustrating the five types of graphs in the family Θ are shown in Figure 1.

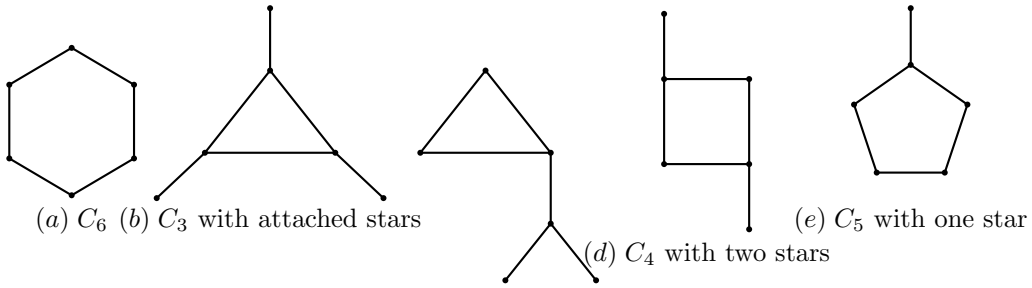


FIGURE 1. Representative graphs illustrating the five types in the family Θ.

Theorem 6.1. *For any connected unicyclic graph G of order n (and hence size $m = n$), we have $EC(G) = m$ if and only if $G \in \Theta$.*

Proof. Let $G \in \Theta$.

For type (a), where $G = C_n$ with $3 \leq n \leq 6$, Observation 4.2 gives the result. So assume G is not a cycle.

For type (b), take the triangle $v_0v_1v_2v_0$ with edges e_1, e_2, e_3. Whenever a star is attached at v_i, each pendant edge at v_i forms an edge coalition with the cycle edge opposite v_i. Every other cycle edge is either full or forms an edge coalition with a pendant edge at its opposite vertex.

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For type (c), let v_0v_1 be the edge attached at v_0, and let P be the pendant edges at v_1. All edges of P meet at v_1, so each $p \in P$ dominates the remaining edges of P and also v_0v_1. Hence p forms an edge coalition with either cycle edge through v_0, while v_0v_1 joins with the cycle edge opposite v_0.

For type (d), let $C_4 = v_0v_1v_2v_3v_0$ with edges $e_1, \ldots, e_4$. When a single star is attached at v_0 with pendant edges $p_1, \ldots, p_r$. The pairs $\{e_1, e_3\}$, $\{e_2, e_4\}$, $\{e_1, e_4\}$, $\{p_j, e_2\}$, and $\{p_j, e_3\}$ are all edge coalitions. When two stars are attached at adjacent vertices, say v_0 with pendant set P and v_1 with Q, the cycle pairs $\{e_1, e_2\}$, $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_2, e_4\}$ are edge coalitions, each $p \in P$ forms one with e_2, and each $q \in Q$ with e_4. When the two stars sit at opposite vertices v_0 and v_2, the pairs $\{e_1, e_3\}$, $\{e_2, e_4\}$, $\{p, e_2\}$, $\{q, e_4\}$ are edge coalitions, where $p \in P$ and $q \in Q$.

For type (e), take $C_5 = v_0v_1v_2v_3v_4v_0$ with edges $e_1, \ldots, e_5$ and pendant edges $p_1, \ldots, p_r$ at v_0. The pairs $\{e_1, e_3\}$, $\{e_2, e_5\}$, $\{e_4, e_1\}$, $\{e_3, e_5\}$, and $\{p_j, e_3\}$ are all edge coalitions. The singleton partition is an ec-partition in every case, so $EC(G) = m$.

Conversely, suppose that $EC(G) = m$. Then the singleton partition is an *ec*-partition, so every non-full edge has an edge coalition partner. Let C be the unique cycle of G, and set $c = |E(C)|$. We proceed in three claims.

Claim 1. If $G = C$, then $c \leq 6$; otherwise, $c \leq 5$.

Proof of Claim 1. If $G = C$, then $c \leq 6$ by Observation 4.2, and hence $G \in \Theta_a$. Suppose that $G \neq C$ and $c \geq 6$. Since G is connected, there is an edge $e \notin E(C)$ with one endpoint v on C and the other outside C. Thus, e is adjacent to exactly two edges of C. For any edge $f \neq e$, if $f \in E(C)$, then f dominates at most three edges of C, namely itself and its two neighbors on C. If $f \notin E(C)$, then f is adjacent to at most two edges of C. Consequently, the set $\{e, f\}$ dominates at most five edges of C. Since $c \geq 6$, at least one edge of C remains undominated by $\{e, f\}$. Hence, e has no edge coalition partner, a contradiction. Therefore, $c \leq 5$.

Claim 2. Let $v_0v_1 \cdots v_t$ be a path in G such that $v_0 \in V(C)$, $v_1, \ldots, v_t \notin V(C)$, v_t is a leaf, and $t \geq 2$. Then $c = 3$ and $t = 2$. Moreover, no pendant edge is attached at v_0.

Proof of Claim 2. Let $v_0v_1 \cdots v_t$ be a path with $v_0 \in V(C)$, $v_1, \ldots, v_t \notin V(C)$, and v_t a leaf, where $t \geq 2$. The edge $v_{t-1}v_t$ is not adjacent to any edge of C, so it is not a full edge. Since $EC(G) = m$, the singleton partition is an *ec*-partition; hence $v_{t-1}v_t$ has an edge coalition partner. Consequently, that partner must dominate every edge of C. As a single edge can dominate at most three edges of a cycle, we obtain $c = 3$. Write $C = v_0abv_0$.

We first show that $t = 2$. Suppose, to the contrary, that $t \geq 3$. The edge v_0v_1 has a coalition partner g that must dominate both ab and $v_{t-1}v_t$. Thus, g joins a vertex of $\{a, b\}$ to a vertex of $\{v_{t-1}, v_t\}$. Such an edge, together with the path from v_0 to the corresponding endpoint of g and an edge of C incident with that endpoint in $\{a, b\}$, creates a cycle distinct from C, a contradiction. Therefore, $t = 2$.

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Finally, suppose that a pendant edge v_0u is attached at v_0. Since v_0u is not adjacent to either ab or v_1v_2, a coalition partner of v_0u must dominate both edges. Hence, it must join a vertex of $\{a, b\}$ to a vertex of $\{v_1, v_2\}$. Such an edge would create a cycle distinct from C, contradicting the assumption that G is unicyclic. Thus, no pendant edge is attached at v_0.

Claim 3. Assume that $G \neq C$. Then either $G \in \Theta_c$, or every edge outside C is a pendant edge, and the attachment vertices satisfy the following: (i) if $c = 5$, at most one vertex of C_5 carries pendant edges; (ii) if $c = 4$, at most two vertices of C_4 carry pendant edges; and (iii) if $c = 3$, pendant edges may be attached to any nonempty subset of $V(C_3)$.

Proof of Claim 3. Suppose first that G has a path of length at least 2 from C to a leaf, with all its internal vertices outside C. Choose such a path $v_0v_1 \cdots v_t$. By Claim 2, $t = 2$, $C = v_0abv_0$, and no pendant edge is attached at v_0. We show that no pendant edge is attached at a or b. Suppose, for example, that au is a pendant edge at a. The cycle edge bv_0 dominates every edge of C and the edge v_0v_1, but it dominates neither au nor v_1v_2. Hence, a coalition partner of bv_0 would have to dominate both au and v_1v_2. Such an edge would join a vertex of $\{a, u\}$ to a vertex of $\{v_1, v_2\}$, creating a cycle distinct from C, a contradiction. The same argument applies to pendant edges at b.

Moreover, no other branch of length 2 can be attached at a vertex of C. Indeed, its terminal edge would be disjoint from v_1v_2. A coalition partner of v_1v_2 must dominate every edge of the triangle C, and hence must be an edge of C; such an edge cannot dominate the terminal edge of the other branch. By Claim 2, no branch of greater length is possible. Consequently, the only edges outside C are v_0v_1 and pendant edges at v_1, so $G \in \Theta_c$.

Now suppose that every edge outside C is a pendant edge. If $c = 5$, we show that at most one vertex of C_5 carries pendant edges. Write $C_5 = v_0v_1v_2v_3v_4v_0$, with $e_i = v_{i-1}v_i$ (indices modulo 5).

Suppose first that two adjacent vertices, say v_0 and v_1, carry pendant edges $p = v_0x$ and $q = v_1y$. Since p does not dominate e_2 or e_4, a coalition partner of p must dominate both e_2 and e_4. The only edge of G that can dominate both is e_3, since any other such edge would be a chord of C_5 and create another cycle. But e_3 does not dominate q, a contradiction.

Suppose next that two nonadjacent vertices, say v_0 and v_2, carry pendant edges $p = v_0x$ and $r = v_2y$. The cycle edge $e_4 = v_3v_4$ dominates neither p nor r. Thus, a coalition partner of e_4 must dominate both pendant edges, and hence must join a vertex of $\{v_0, x\}$ to a vertex of $\{v_2, y\}$. Every such edge would create a cycle distinct from C_5, a contradiction. Therefore, at most one vertex of C_5 carries pendant edges, and $G \in \Theta_e$.

If $c = 4$ and three vertices of C_4 carry pendant edges, there is a vertex v whose two cycle-neighbors u and w also carry pendant edges. Let q be a pendant edge at v , and let p and r be pendant edges at u and w, respectively. Since q dominates neither p nor r, a coalition partner of q must dominate both. Any edge dominating both p and r would have to join an endpoint of p to an endpoint of r. Since the

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two pendant endpoints have degree one, such an edge cannot be incident with either of them. The only remaining possibility is an edge joining u and w, which would be a chord of C_4 and would create a second cycle. This contradicts the assumption that G is unicyclic. Hence at most two vertices of C_4 carry pendant edges, and $G \in \Theta_d$.

Finally, if $c = 3$, pendant edges may be attached at any nonempty subset of $V(C_3)$, giving $G \in \Theta_b$. This completes the proof of Claim 3. By Claims 1–3, every graph G satisfying $EC(G) = m$ belongs to Θ. Together with the first part of the proof, this proves the theorem. □

6.2. **Trees.** Let T be a tree of order n. Then $|E(T)| = m = n - 1$.

Theorem 6.2. *Let T be a tree of order $n \geq 2$ and size m. Then $EC(T) = m = n - 1$ if and only if $T \in \Phi$, where Φ denotes the family consisting of all trees of diameter at most 3, together with all trees of diameter 4 whose center has degree 2, as illustrated in Figure 2.*

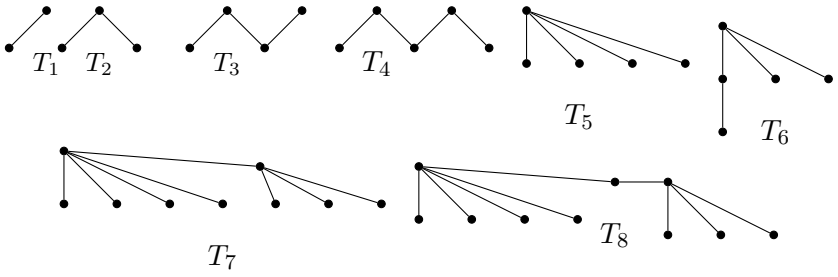


FIGURE 2. Some trees that are in Φ

Proof. Every tree of order n has $m = n - 1$ edges, and no part of an ec-partition is empty, so $EC(T) \leq m$.

First, let $T \in \Phi$. If its diameter is at most 3, then T is P_2, a star, or a double star. In the first two cases every edge is full, so the singleton partition is an ec-partition; for a double star the equality $EC(T) = m$ follows from Observation 4.2. If the diameter equals 4, let v be the center of T with neighbors u_1, u_2, and suppose there are $p \geq 1$ pendant edges at u_1 and $q \geq 1$ at u_2. Then vu_1 forms an edge coalition with any pendant edge at u_2, vu_2 with any pendant edge at u_1, and each pendant edge at u_1 (or u_2) with vu_2 (or vu_1). Hence the singleton partition is an ec-partition, and $EC(T) = m$.

For the reverse, assume $EC(T) = m$. Then the singleton partition is an ec-partition, so every edge has an edge coalition partner. Let $P = v_1v_2 \cdots v_{l+1}$ be a longest path. We show $l \leq 4$.

If $l \geq 5$, then $e_3 = v_3v_4$ is not edge-dominating, because e_1 and e_5 are both nonadjacent to it. So e_3 needs a partner f that dominates both e_1 and e_5. The edge f cannot be e_1 or e_5 itself, since then

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it would fail to dominate the other. Hence f must join a vertex of $\{v_1, v_2\}$ to a vertex of $\{v_5, v_6\}$. But T contains the path $v_2v_3v_4v_5$, so adding f would create a cycle. This is impossible, so $l \leq 4$.

Now take $l = 4$. Assume there is some $w \in N_T(v_3)$ different from v_2, v_4, and set $e = wv_3$. Since e is adjacent to neither e_1 nor e_4, it is not edge-dominating. A partner f for e would have to dominate both e_1 and e_4; as before, f cannot be either of them, so it would have to join $\{v_1, v_2\}$ to $\{v_4, v_5\}$, creating a cycle. Hence no such w exists, and $\deg(v_3) = 2$. The same reasoning applies to every longest path of length 4, so the center of each such path has degree 2.

Finally, if $l = 3$, the tree is a double star; if $l = 2$, it is a star; and if $l = 1$, it is P_2. In every case $T \in \Phi$, and the proof is complete. □

6.3. **Remaining Connected Graphs.** We recall that a graph F satisfies $C(F) = |V(F)|$ if and only if $F \in \mathcal{H}$, where $\mathcal{H}$ is the family of graphs characterized in [12] as follows: either F is a complete graph, or the graph obtained from F by deleting all full vertices can be obtained from $\overline{K_2}$ by a finite sequence of the following three operations:

(i) replacing a vertex by a clique whose vertices have the same neighbors outside the clique as the original vertex;
(ii) adding edges while preserving the absence of full vertices;
(iii) deleting edges while preserving the equality $C(F) = |V(F)|$.

Proposition 6.3 (Reduction)**.** *Let G be a connected graph of order n and size m which is neither a tree nor a unicyclic graph. Then $EC(G) = m$ if and only if $L(G) \in \mathcal{H}$.*

Proof. By Lemma 3.3 and $|V(L(G))| = m$, the equality $EC(G) = m$ is equivalent to $C(L(G)) = |V(L(G))|$. By [12, Main Theorem 2.2], the latter holds if and only if $L(G) \in \mathcal{H}$. Hence $EC(G) = m$ if and only if $L(G) \in \mathcal{H}$. The hypothesis on G excludes the cases already treated in Theorems 6.1 and 6.2. □

Remark 6.4. *Proposition 6.3 is a reduction to the line graph $L(G)$; it does not give a structural characterization of G. The operations defining $\mathcal{H}$ act on $L(G)$, not on G; in particular, for $F = L(G)$, the full vertices removed in the construction of* [12] *correspond exactly to the full edges of G. Operation (iii) in the definition of $\mathcal{H}$ is self-referential as a definitional rule: it deletes edges while preserving the equality $C(F) = |V(F)|$. Nevertheless,* [12] *proves a valid characterization despite this: the "if" direction is direct via* [12, Proposition 2.3]*, and the "only if" direction is proved by induction on $n(G)$. We reproduce the definition for completeness and refer the reader to* [12] *for the proof. A direct characterization of such graphs G remains open (Open Problem 3).*

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7. Edge Coalition Graphs of Some Graph Classes

In this section, we characterize the edge coalition graphs of several graph classes, including complete bipartite graphs $K_{2,s}$, stars, and unicyclic graphs, and we identify all self-edge coalition graphs.

Theorem 7.1. *For every integer $s \geq 2$ and every ec-partition π of $K_{2,s}$, the edge coalition graph $ECG(K_{2,s}, \pi)$ is isomorphic to one of $K_{l,l'}$ for $3 \leq l, l' \leq s$, $K_{2,l'} + e$ for $3 \leq l' \leq s$ (where e joins the two vertices in the part of size 2), or K_4. Moreover, all these graphs occur.*

Proof. Let $X = \{a, b\}$ and $Y = \{c_1, \ldots, c_s\}$ be the two parts of $K_{2,s}$, and split $E(K_{2,s})$ into $Z = \{ac_1, \ldots, ac_s\}$ and $W = \{bc_1, \ldots, bc_s\}$.

A set meeting both Z and W edge-dominates $K_{2,s}$: if it contains ac_i and bc_j, then every ac_k with $k \neq i$ is adjacent to ac_i, and every bc_k with $k \neq j$ is adjacent to bc_j. No proper nonempty subset of Z, and similarly of W, is edge-dominating: if $Z' \subsetneq Z$ misses ac_k, then bc_k has no neighbor in Z'. Since $s \geq 2$, no single edge of $K_{2,s}$ is edge-dominating.

Let π be an ec-partition. A part of π cannot meet both Z and W, since any such part contains at least two edges and is edge-dominating, contrary to Definition 3.2. Since no single edge of $K_{2,s}$ is edge-dominating, a singleton part, if present, is not edge-dominating and must form an edge coalition with another part. Thus every part of π lies entirely inside Z or entirely inside W. Moreover, neither Z nor W can itself be a part of π, since both are non-singleton edge-dominating sets. Therefore, each of Z and W is divided into at least two parts, and we may write $\pi = \{Z_1, \ldots, Z_p, W_1, \ldots, W_q\}, \quad 2 \leq p, q \leq s$.

Any Z_i together with any W_j forms an edge coalition, since their union meets both Z and W while neither Z_i nor W_j alone edge-dominates. Two parts from the same side form an edge coalition only when their union is the whole side, that is, only when $p = 2$ for the Z_i's or $q = 2$ for the W_j's. Consequently:

- if $p \geq 3$ and $q \geq 3$, then $ECG(K_{2,s}, \pi) \cong K_{p,q}$;
- if $p = 2$ and $q \geq 3$, then $ECG(K_{2,s}, \pi) \cong K_{2,q} + e$, where e joins the two vertices in the part of size 2;
- if $p \geq 3$ and $q = 2$, then symmetrically $ECG(K_{2,s}, \pi) \cong K_{p,2} + e$, and since $K_{p,2} + e \cong K_{2,p} + e$, this is the same form as the previous case;
- if $p = q = 2$, then $ECG(K_{2,s}, \pi) \cong K_4$.

The case $s = 4$, $p = q = 2$ is illustrated in Figure 3. Since p, q range over $[2, s]$, and each of the three graph types above is realized by some choice of p, q, the result follows. □

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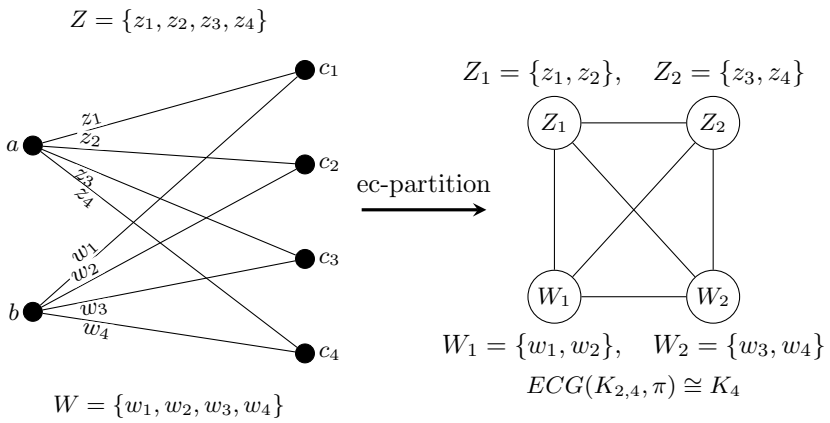


FIGURE 3. The graph $K_{2,4}$ with the ec-partition $\pi = \{Z_1, Z_2, W_1, W_2\}$, yielding $ECG(K_{2,4}, \pi) \cong K_4$.

Theorem 7.2. *For any star $K_{1,n-1}$ of order $n \geq 2$, with its unique ec-partition π_0,*

$$ECG(K_{1,n-1}, \pi_0) \cong (n-1)K_1.$$

Proof. Every edge of a star is full, so every nonempty set of edges is edge-dominating. The singleton partition π_0 is therefore the only ec-partition of $K_{1,n-1}$, and no two of its parts form an edge coalition. It follows that $ECG(K_{1,n-1}, \pi_0)$ has $(n-1)$ isolated vertices, so it is isomorphic to $(n-1)K_1$. □

Let $\mathcal{B}_{r,s,t}$ denote the graph obtained from C_3 by attaching $K_{1,r}$, $K_{1,s}$, and $K_{1,t}$ at its three vertices, where $r, s, t \geq 1$. Also, let $\mathcal{T}_r$ denote the graph obtained from $K_{1,r+2}$ by subdividing two of its pendant edges once.

Theorem 7.3. *Let G be a unicyclic graph of order n and size $m = n$ with $EC(G) = m$, and let π_0 be the singleton edge partition of G. Then $ECG(G, \pi_0)$ is isomorphic to one of the following graphs: $3K_1$, K_4, C_5, $3K_2$, $2K_1 \cup K_{1,r}$, $K_1 \cup K_{r+1,s+1}$, $\mathcal{B}_{r,s,t}$, $K_{3,p+1}$, $K_{r+2,2} + e$, $K_{r+2,s+2}$, $\mathcal{T}_r$, where $r, s, t, p \geq 1$. In $K_{r+2,2} + e$, the edge e joins two vertices in the part of size $r + 2$.*

Proof. Since $EC(G) = m$, the singleton partition π_0 is an ec-partition with m parts. By Theorem 6.1, $G \in \Theta$.

For type (a), a direct inspection of edge adjacency gives

$$ECG(C_3, \pi_0) \cong 3K_1, \quad ECG(C_4, \pi_0) \cong K_4,$$

$$ECG(C_5, \pi_0) \cong C_5, \quad ECG(C_6, \pi_0) \cong 3K_2.$$

For type (b), if one star is attached at a vertex v_0 of C_3, the two cycle edges incident with v_0 are full edges and form isolated vertices in ECG. The third cycle edge forms an edge coalition with every pendant edge, so the ECG is $2K_1 \cup K_{1,r}$. If two stars are attached at two vertices v_0 and v_1 of C_3, the cycle edge v_0v_1 is the unique full edge and is isolated in ECG, while the remaining edges

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form $K_{r+1,s+1}$. Thus the ECG is $K_1 \cup K_{r+1,s+1}$. If three stars are attached, no edge is full, and each pendant edge forms an edge coalition with the cycle edge opposite its attachment vertex. This gives $\mathcal{B}_{r,s,t}$.

For type (c), the three cycle edges form one part of the bipartition and the edge v_0v_1 together with the p pendant edges at v_1 form the other. The result is $K_{3,p+1}$.

For type (d), using the edge coalitions identified in the proof of Theorem 6.1, we obtain the following. If one star is attached at v_0 of $C_4 = v_0v_1v_2v_3v_0$, then $ECG(G, \pi_0) \cong K_{r+2,2} + e$, where the part of size $r+2$ is $\{e_1, e_4, p_1, \ldots, p_r\}$, the part of size 2 is $\{e_2, e_3\}$, with $e_i = v_{i-1}v_i$ (indices modulo 4), and the edge e joins e_1 and e_4. If two stars are attached at adjacent vertices of C_4, then $ECG(G, \pi_0) \cong \mathcal{B}_{1,r,s}$; if the two attachment vertices are opposite, then $ECG(G, \pi_0) \cong K_{r+2,s+2}$.

For type (e), let the cycle be $v_0v_1v_2v_3v_4v_0$ with a star attached at v_0, and let $p_1, \ldots, p_r$ be the pendant edges at v_0. Set $e_1 = v_0v_1$, $e_2 = v_1v_2$, $e_3 = v_2v_3$, $e_4 = v_3v_4$, and $e_5 = v_4v_0$. The edge coalitions are: e_1 with e_3 and e_4; e_2 with e_5; e_3 with e_1, e_5, and every p_i; e_4 with e_1; e_5 with e_2 and e_3; and each p_i only with e_3. Hence $ECG \cong \mathcal{T}_r$.

Thus, all possible edge coalition graphs are among those listed above. Conversely, each graph listed above is realized by the corresponding type in Θ: type (a) realizes $3K_1$, K_4, C_5, $3K_2$; type (b) realizes $2K_1 \cup K_{1,r}$, $K_1 \cup K_{r+1,s+1}$, $\mathcal{B}_{r,s,t}$; type (c) realizes $K_{3,p+1}$; type (d) realizes $K_{r+2,2} + e$, $\mathcal{B}_{1,r,s}$, $K_{r+2,s+2}$; type (e) realizes $\mathcal{T}_r$. □

A graph G is called a self-edge coalition graph if G is isomorphic to $ECG(G, \pi_0)$ for the singleton *ec*-partition π_0 of G. The preceding results yield the following characterization of self-edge coalition graphs.

Theorem 7.4. *Let G be a connected graph. Then G is a self-edge coalition graph if and only if $G \cong C_5$ or $G \cong \mathcal{B}_{r,s,t}$ for some $r, s, t \geq 1$.*

Proof. First consider the case $G \cong C_5$. By Theorem 7.3, $ECG(C_5, \pi_0) \cong C_5$, and hence G is a self-edge coalition graph.

Now let $G \cong \mathcal{B}_{r,s,t}$, where $r, s, t \geq 1$. By definition, G is obtained from C_3 by attaching the stars $K_{1,r}$, $K_{1,s}$, and $K_{1,t}$ at the three vertices of C_3, respectively. By Theorem 7.3, $ECG(G, \pi_0) \cong \mathcal{B}_{r,s,t} \cong G$. Thus G is a self-edge coalition graph.

For the converse, let G be a self-edge coalition graph. Then $G \cong ECG(G, \pi_0)$, so $|V(G)| = |E(G)|$. Since G is connected, it is unicyclic. Moreover, π_0 is a singleton *ec*-partition, so $EC(G) = |E(G)|$. By Theorem 6.1, $G \in \Theta$.

If G is of type (a), then Theorem 7.3 gives $3K_1$, K_4, C_5, or $3K_2$ according as $G \cong C_3$, C_4, C_5, or C_6. Hence only C_5 is possible.

For type (b), if one or two stars are attached, the edge coalition graph is disconnected, whereas G is connected; if three stars are attached, the edge coalition graph is $\mathcal{B}_{r,s,t} \cong G$.

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For type (c), Theorem 7.3 gives $K_{3,p+1}$. This graph has minimum degree at least 2, whereas G has pendant vertices. Hence, these graphs are not isomorphic.

For type (d), if one star is attached, the edge coalition graph is $K_{r+2,2} + e$, which has minimum degree at least 2. If two stars are attached at opposite vertices, the edge coalition graph is $K_{r+2,s+2}$, which also has minimum degree at least 2. If the two stars are attached at adjacent vertices, the edge coalition graph is $\mathcal{B}_{1,r,s}$, which contains a triangle, and therefore cannot be isomorphic to a unicyclic graph whose unique cycle is C_4. Thus, none of these cases is possible.

Finally, for type (e), Theorem 7.3 gives the tree $\mathcal{T}_r$, whereas G is unicyclic. Hence this case is impossible. Therefore, $G \cong C_5$ or $G \cong \mathcal{B}_{r,s,t}$. □

8. **Open Problems**

We conclude with the following open problems.

(1) Characterize the edge coalition graphs arising from edge coalition partitions of paths, cycles, and trees.

(2) Determine the computational complexity of deciding, for a given graph G and integer k, whether G admits an ec-partition with exactly k parts.

(3) Provide a direct structural characterization, in terms of the graph G itself, of the connected graphs that are neither trees nor unicyclic and satisfy $EC(G) = m$. Proposition 6.3 reduces this question to the line graph $L(G)$, but the operations defining the family $\mathcal{H}$ act on $L(G)$ and do not translate directly to operations on G.

(4) Determine whether every connected graph G of order n with $G \neq K_n$, and with $k \geq 1$ vertices of degree $n-1$, satisfies $EC(G) \geq kn - \frac{k(k+1)}{2}$.

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References

[1] S. Alikhani, D. Bakhshesh, H. Golmohammadi and E. V. Konstantinova, Connected coalitions in graphs, *Discuss. Math. Graph Theory*, **44** no. 4 (2024) 1551–1566. https://doi.org/10.7151/dmgt.2509

[2] S. Arumugam and S. Velammal, Edge domination in graphs, *Taiwanese J. Math.*, **2** no. 2 (1998) 173–179.

[3] J. Baste, M. Fürst, M. A. Henning, E. Mohr and D. Rautenbach, Domination versus edge domination, *Discrete Appl. Math.*, **285** (2020) 343–349. https://doi.org/10.1016/j.dam.2020.05.030

https://dx.doi.org/10.22108/toc.2026.150035.2385

[4] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, **244**, Springer, New York, 2008.

[5] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Introduction to coalitions in graphs, *AKCE Int. J. Graphs Comb.*, **17** no. 2 (2020) 653–659. https://doi.org/10.1080/09728600.2020.1832874

[6] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Upper bounds on the coalition number, *Australas. J. Combin.*, **80** no. 3 (2021) 442–453.

[7] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Coalition graphs, *Commun. Comb. Optim.*, **8** no. 2 (2023) 423–430. https://doi.org/10.22049/CCO.2022.27916.1394

[8] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Coalition graphs of paths, cycles and trees, *Discuss. Math. Graph Theory*, **43** no. 4 (2023) 931–946. https://doi.org/10.7151/dmgt.2416

[9] M. A. Henning and D. A. Mojdeh, Double coalitions in graphs, *Bull. Malays. Math. Sci. Soc.*, **48** no. 2 (2025) Article 51 pp. 19. https://doi.org/10.1007/s40840-025-01831-7

[10] M. A. Henning and D. A. Mojdeh, Double coalitions in regular graphs, *Graphs Combin.*, **41** no. 4 (2025), Article 74 pp. 19. https://doi.org/10.1007/s00373-025-02937-2

[11] D. B. West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

[12] F. Yang, Q. Sun and C. Zhang, Characterizations of graphs with equal vertex and coalition number, *Discrete Appl. Math.*, **381** (2026) 112–119. https://doi.org/10.1016/j.dam.2025.11.002

[13] M. Yannakakis and F. Gavril, Edge dominating sets in graphs, *SIAM J. Appl. Math.*, **38** no. 3 (1980) 364–372.

[14] B. Zelinka, Edge-domatic number of a graph, *Czechoslovak Math. J.*, **33(108)** no. 1 (1983) 107–110.

Nazli Besharati

Department of Mathematics, Faculty of Sciences, Payame Noor University, Tehran, Iran

Email: nbesharati@pnu.ac.ir

Azam Sadat Emadi

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

Email: mathemadi2025@gmail.com

Iman Masoumi

Department of Mathematics, Faculty of Mathematical Sciences, University of Tafresh, Tafresh, Iran

Email: i.masoumi1359@gmail.com

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