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ON LARGE PROPORTIONS OF CYCLIC SUBGROUPS IN FINITE GROUPS

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Abstract. Let $c(G)$ be the number of cyclic subgroups of the finite group G and $\alpha(G)$ the function $c(G)/|G|$. In this paper, we classify the non 2-groups with $17/24 \leq \alpha(G) < 3/4$.

1. Introduction

Throughout this paper, all groups are assumed to be finite. Let $c(G)$ denote the number of cyclic subgroups of a group G. The study of cyclic subgroups is a classical topic in finite group theory. Miller [1, 2] was among the first to investigate this invariant, he proved that

$$|G| = \sum_{m||G|} c(m)\,\phi(m),$$

where $c(m)$ is the number of cyclic subgroups of order m in G and ϕ denotes Euler's totient function. Later, Richards [3] established the lower bound $c(G) \geq \tau(|G|)$ (with τ the divisor function), with equality if and only if G is cyclic. Further refinements for nilpotent and abelian groups were obtained by Meng and Lu [4] and by Tóth [5], respectively.

A related line of research concerns the classification of groups with a given number of cyclic subgroups. Tarnăuceanu [6, 7] initiated the study of groups satisfying $c(G) = |G| - r$ for small values of r, and posed the general problem of determining all finite groups with $c(G) = |G| - r$. Subsequently, Belshoff,

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Dillstrom and Reid [8, 9] classified such groups for $r = 3, 4, 5$ and later for all $r \leq 32$. In parallel, several authors classified n-cyclic groups (that is, groups with $c(G) = n$) for small n; see, for example, [10]–[13].

Recently, Garonzi and Lima [14] considered the normalized quantity $\alpha(G) = c(G)/|G|$. Since every element $x \in G$ generates a cyclic subgroup $\langle x \rangle$ which has exactly $\phi(o(x))$ generators, we also have

$$c(G) = \sum_{x \in G} \frac{1}{\phi(o(x))},$$

where $o(x)$ is the order of x. Thus $\alpha(G)$ can be interpreted as the average probability that a randomly chosen element of G generates a cyclic subgroup, and it reflects the abundance of cyclic subgroups in G. Garonzi and Lima [14] classified all finite groups with $\alpha(G) > 3/4$ and proved that if G is not supersolvable then $\alpha(G) \leq \alpha(S_4) = 17/24$. Subsequently, we characterized in [15] the groups attaining $\alpha(G) = 3/4$, proving that this equality holds if and only if G is isomorphic to a direct product of an elementary abelian 2-group and either a dihedral group D_{16} or D_{24}, or a group satisfying $I(G) = |G|/2$ and $\exp(G) = 4$, where $I(G)$ denotes the number of elements g of G with $g^2 = 1$. The notation SmallGroup(n, i) refers to the i-th group of order n in the SmallGroup Library of GAP [16].

In this paper, we extend these results by classifying the groups that are not 2-groups and satisfy $17/24 \leq \alpha(G) < 3/4$. Our main theorem is as follows.

Theorem 1.1. *Let G be a non 2-group with $17/24 \leq \alpha(G) < 3/4$. Then $G \cong Z_2^n \times X$ for some non-negative integer n, where X is one of the following groups:*

(1) $S_3 \times S_3$, *and* $\alpha(X) = 13/18$,
(2) $S_3 \times D_8$, *and* $\alpha(X) = 35/48$,
(3) $\mathrm{SmallGroup}(24, 8) \cong \mathrm{Z}_3 \rtimes \mathrm{D}_8$, *and* $\alpha(X) = 17/24$,
(4) $\mathrm{SmallGroup}(72, 33) \cong \mathrm{Z}_3^2 \rtimes \mathrm{D}_8$, *and* $\alpha(X) = 17/24$,
(5) $\mathrm{SmallGroup}(96, 81) \cong \mathrm{Z}_4 \rtimes \mathrm{D}_{24}$, *and* $\alpha(X) = 17/24$,
(6) S_4, *and* $\alpha(X) = 17/24$.

2. Some Lemmas

In this section, we give some lemmas. The first five lemmas are cited from [14] and Lemmas 2.6, 2.7 and 2.8 are cited from [15]. Let $I(G)$ be the number of elements $g \in G$ with $g^2 = 1$.

Lemma 2.1. *If $N \trianglelefteq G$ then $\alpha(G) \leq \alpha(G/N)$. Moreover $\alpha(G) = \alpha(G/N)$ if and only if $\phi(o(g)) = \phi(o(gN))$ for every $g \in G$, where $o(gN)$ denotes the order of the element gN in the group G/N.*

Lemma 2.2. *If $\alpha(G) = \alpha(G/N)$, then N is an elementary abelian 2-group.*

Lemma 2.3. *If G is any finite group, then $\alpha(G) = \alpha(G \times Z_2^n)$ for all $n \geq 0$.*

Lemma 2.4. *Let G be a group. If G is not supersolvable, then $\alpha(G) \leq \alpha(S_4) = 17/24$ with equality if and only if $G \cong Z_2^n \times S_4$ for some non-negative integer n.*

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Lemma 2.5. *Let G be a group with $\alpha(G) > 3/4$. Then $G \cong Z_2^n \times G_1$, G_1 does not have Z_2 as a direct factor, and either G_1 is trivial or one of the following occurs.*

(1) $A \rtimes \langle \varepsilon \rangle$, where $\langle \varepsilon \rangle = Z_2$ acts on A by inversion and there exists an integer $m \geq 1$ such that one of the following occurs: $A \cong Z_3^m, \alpha(G_1) = \frac{3 \cdot 3^m + 1}{4 \cdot 3^m}$ and $I(G_1) = 3^m$, or $A \cong Z_4^m, \alpha(G_1) = \frac{3 \cdot 2^m + 1}{4 \cdot 2^m}$ and $I(G_1) = 4^m + 2^m$.

(2) $Z_2^{2m} \rtimes \langle c \rangle$, where Z_2^{2m} has a basis $\{x_i, y_i \mid i = 1, 2, \ldots, m\}$, c has order 2, $[c, x_i] = y_i$, $[c, y_i] = 1$ for all i and $\alpha(G_1) = \frac{3 \cdot 2^m + 1}{4 \cdot 2^m}, I(G_1) = 4^m + 2^m$.

(3) D_8^2, $\alpha(G_1) = 25/32$ and $I(G_1) = 36$.

(4) D_8^m/N, where $N = \{(a_1, a_2, \ldots, a_m) \in Z(D_8)^m \mid a_1 a_2 \cdots a_m = 1\}$ and $\alpha(G_1) = \frac{3 \cdot 2^m + 1}{4 \cdot 2^m}, I(G_1) = 4^m + 2^m$.

Lemma 2.6. *Suppose that N is a normal Hall subgroup, H a subgroup of G, and $G = N \rtimes H$. Then $\alpha(G) = \frac{1}{|H|} \sum_{h \in H} \frac{\alpha(C_N(h))}{\phi(o(h))}$.*

Lemma 2.7. *Suppose that N is a normal Hall subgroup, H a subgroup of G, and $G = N \rtimes H$. Let ρ: $H \to \mathrm{Aut}(N)$ be the homomorphism associated to the automorphism action of H on N. Denote by K the kernel of ρ and by X a coset representative of K in H. Then $\alpha(G) = \frac{1}{|H|} \sum_{x \in X} \alpha(C_N(x)) c(Kx)$.*

Lemma 2.8. *Suppose that t is an involution of the general linear group $\mathrm{GL}(n, q)$ over a linear space V with q odd prime, and $t \neq -I_n$. Then $C_V(t)$ is not trivial.*

Let $A.B$ denote the extension of A by B. We now prove some preliminary lemmas.

Lemma 2.9. *Let G be an abelian group of odd order, and let $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^2 = 1$. Define the subgroups $G_1 = \{g \in G \mid g^\alpha = g\}$ and $G_2 = \{g \in G \mid g^\alpha = g^{-1}\}$. Then $G = G_1 \times G_2$.*

Proof. Since G is abelian and α is an automorphism, it is straightforward to verify that G_1 and G_2 are subgroups of G. Suppose $g \in G_1 \cap G_2$. Then $g = g^\alpha = g^{-1}$, so $g^2 = 1$. Since $|G|$ is odd, the only element satisfying $g^2 = 1$ is the identity. Thus, $G_1 \cap G_2 = 1$. Next, we claim that every $g \in G$ can be written as $g = g_1 g_2$, where $g_1 \in G_1$ and $g_2 \in G_2$. Define the elements $x = g g^\alpha$ and $y = g(g^{-1})^\alpha$. Since $x^\alpha = (g g^\alpha)^\alpha = g^\alpha g^{\alpha^2} = g^\alpha g = x$ and $y^\alpha = (g(g^{-1})^\alpha)^\alpha = g^\alpha (g^{-1})^{\alpha^2} = g^\alpha g^{-1} = y^{-1}$. Thus, $x \in G_1$ and $y \in G_2$. Since $|G|$ is odd, the squaring map $h \mapsto h^2$ is an automorphism of G. Hence, there exist unique elements $g_1, g_2 \in G$ such that $g_1^2 = x$ and $g_2^2 = y$. Moreover, since $x \in G_1$ and G_1 is a subgroup, we have $g_1 \in G_1$. Similarly, $g_2 \in G_2$. Observe that $(g_1 g_2)^2 = g_1^2 g_2^2 = xy = g g^\alpha g(g^{-1})^\alpha = g^2 (g g^{-1})^\alpha = g^2$. Since the squaring map is injective, it follows that $g_1 g_2 = g$. Therefore, $G = G_1 G_2$. Since G is abelian, and G_1 and G_2 are subgroups with trivial intersection, thus $G = G_1 \times G_2$. □

Lemma 2.10. *Let G be a supersolvable group, and let p be the maximum prime divisor of $|G|$. Then $\alpha(G) \leq 1/2 + 1/p$.*

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Proof. Since G is supersolvable, it admits a chief series $G = G_0 > G_1 > \cdots > G_{i+1} > \cdots > G_n = 1$. Note that G_i/G_{i+1} is a group of prime order for each i. Let i be the smallest integer such that G_i/G_{i+1} is cyclic of order p, that is, $|G_j/G_{j+1}| < p$ for all $0 \leq j < i$, and hence $p \nmid |G/G_i|$. Since $(G/G_{i+1})/(G_i/G_{i+1}) \cong G/G_i$, the Schur-Zassenhaus Theorem implies that $G/G_{i+1} \cong Z_p \rtimes (G/G_i)$. Let $\overline{G} = G/G_{i+1}$, $H = G/G_i$, and let K be the kernel of the action of H on Z_p. Then $\overline{G} \cong Z_p \rtimes H$ and $K = C_H(Z_p)$. As $C_{\overline{G}}(Z_p)$ is normal in $\overline{G}$, the intersection $C_{\overline{G}}(Z_p) \cap H = K$ is normal in H. Now take any $g = nh \in \overline{G}$ with $n \in Z_p$ and $h \in H$, then $K^g = K^{nh} = K^h = K$, so K is normal in $\overline{G}$. Observe that

$$\overline{G}/C_{\overline{G}}(Z_p) \cong (Z_p \rtimes H)/(Z_p \times K) \cong H/K \lesssim \mathrm{Aut}(Z_p) \cong Z_{p-1},$$

hence H/K is a cyclic subgroup of Z_{p-1}. Write $|H/K| = k$, where $k \mid p-1$. Then $\overline{G} \cong (Z_p \times K).Z_k$ and $\overline{G}/K \cong Z_p \rtimes Z_k$. Therefore, $\alpha(G) \leq \alpha(\overline{G}) \leq \alpha(Z_p \rtimes Z_k) \leq (2 + p(d(k) - 1))/pk \leq \frac{p+2}{2p} = \frac{1}{2} + \frac{1}{p}$, where $d(k)$ denotes the number of positive divisors of k. □

Lemma 2.11. *Suppose that $G = P.(Q \times R)$, where Q is an elementary abelian 2-group and $(|P|, |Q|) = 1$. If $[P, Q] = 1$, then $G \cong Q \times (P.R)$.*

Proof. Note that Q is a subgroup of G. Suppose that $zxy \in G$, where $z \in P$, $x \in Q$, and $y \in P.R$. Let $g \in Q$. Since $[P, Q] = 1$, it follows that $g^{zxy} = g^y$. Consider the quotient group $\overline{G} = G/P$, and let $\overline{g}$ and $\overline{y}$ be the images of g and y in $\overline{G}$, respectively. Then $\overline{g}^{\overline{y}} = \overline{g}$, which implies that $g^y \in gP$. Since $(o(g), |P|) = 1$ and $[Q, P] = 1$, we have $o(gp) = o(g) \cdot o(p) = 2o(p)$ for any element p of P. This means that the coset gP contains a unique element of order 2, namely g. Moreover, $o(g^y) = o(g) = 2$, so $g^y = g$ because $g^y \in gP$ and g is the only element of order 2 in gP. Thus, $g^{zxy} = g^y = g$. Therefore, g is in the center of G, and hence $Q \leq Z(G)$. Thus, $G = Q \times (P.R)$. □

Lemma 2.12. *Suppose that G is a 2-group with $I(G)/|G| > 7/12$. If $\alpha(G) \geq 17/24$, then $G \cong Z_2^l \times X$, where X is trivial or isomorphic to one of the following groups.*

(1) D_{16},
(2) D_8,
(3) $(Z_4^2 \rtimes \langle \varepsilon \rangle)$, *where $\langle \varepsilon \rangle$ acts on Z_4^2 by inversion,*
(4) $(Z_2 \times D_8) \rtimes Z_2 \cong \mathrm{SmallGroup}(32, 49)$,
(5) $Z_2^4 \rtimes Z_2 \cong \mathrm{SmallGroup}(32, 27)$.

Proof. Assume that $G \cong Z_2^l \times X$, where $l \geq 0$, and X does not have Z_2 as a direct factor. Obviously, $I(X)/|X| = I(G)/|G|$ and $\alpha(X) = \alpha(G)$ by Lemma 2.3. It is easy to see from $I(G)/|G| > 7/12 > 1/2$ that X is trivial or one of the groups of Type I, II, III and IV in Wall's classification (see [17, Section 7]).

First, we assume that X is isomorphic to a group of Type I, that is, $X \cong A \rtimes \langle \varepsilon \rangle$, where A is an abelian 2-group and $\langle \varepsilon \rangle \cong Z_2$ acts on A by inversion. Since all elements of $X - A$ have order 2, it follows

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that

$$\alpha(X) = \frac{c(A) + |A|}{2|A|} = \frac{1}{2}\alpha(A) + \frac{1}{2}.$$

By condition $\alpha(X) \geq 17/24$, we deduce that $\alpha(A) \geq 5/12$. Note that we can write $A \cong Z_{2^{d_1}} \times Z_{2^{d_2}} \times \cdots \times Z_{2^{d_k}}$, where $d_1 \leq \cdots \leq d_k$. We have $d_1 \geq 2$ since X does not have Z_2 as a direct factor. If $d_k \geq 4$, then A has a homomorphic image isomorphic to Z_{16}, which implies that $\alpha(A) \leq \alpha(Z_{16}) = 5/16 < 5/12$, a contradiction. Thus, $d_k \leq 3$. Suppose that A admits Z_8 as a direct factor. If $d_k = d_{k-1} = 3$, then A has a quotient that is isomorphic to $Z_8 \times Z_8$. However, $\alpha(A) \leq \alpha(Z_8 \times Z_8) = 11/32 < 5/12$, which is a contradiction. It follows that A has at most one Z_8 as a direct factor. In fact, A does not have a homomorphic image isomorphic to $Z_4^2 \times Z_8$ because $\alpha(A) \leq \alpha(Z_4^2 \times Z_8) = 13/32 < 5/12$, a contradiction. Now, if $A \cong Z_4 \times Z_8$, then $\frac{I(X)}{|X|} = 36/64 < 7/12$, which yields a contradiction. Thus, $A \cong Z_8$, and then $X \cong D_{16}$. Next, we assume that A does not have Z_8 as a direct factor, that is, $X \cong Z_4^k \rtimes \langle \varepsilon \rangle$. Note that $I(X) = 4^k + 2^k$ and $|X| = 2 \cdot 4^k$. From the condition $\frac{I(X)}{|X|} = \frac{4^k+2^k}{2\cdot 4^k} > 7/12$, we deduce that $k \leq 2$. Thus, $X \cong D_8$ or $(Z_4 \times Z_4) \rtimes \langle \varepsilon \rangle \cong \text{SmallGroup}(32, 34)$.

We now suppose that X is of Type II, i.e., $X \cong D_8^2$. Then $I(X) = 36$, and hence $\frac{I(X)}{|X|} = 36/64 = 9/16 < 7/12$, a contradiction.

Next, suppose that X is of Type III, namely, $X \cong D_8^m/N$, where $N = \{(a_1, a_2, \ldots, a_m) \in Z(D_8)^m \mid a_1 a_2 \cdots a_m = 1\}$. By Lemma 2.5, we have $I(X) = 4^m + 2^m$. It is easy to see that $m \leq 2$ because $\frac{I(X)}{|X|} = \frac{4^m+2^m}{2\cdot 4^m} > 7/12$. Thus, $X \cong D_8$ or $X \cong (Z_2 \times D_8) \rtimes Z_2 \cong \text{SmallGroup}(32, 49)$.

Finally, assume that X is of Type IV, namely, $X \cong Z_2^{2m} \rtimes \langle c \rangle$, where Z_2^{2m} has a basis $\{x_i, y_i \mid i = 1, 2, \ldots, m\}$, c has order 2, and the action is given by $[c, x_i] = y_i$, $[c, y_i] = 1$ for all i. Similarly to the case of Type III, we get $X \cong D_8$ or $X \cong Z_2^4 \rtimes Z_2 \cong \text{SmallGroup}(32, 27)$. □

Lemma 2.13. *Let M be a normal subgroup of G and $|M| = 3^m$ such that $G/M \cong Z_2^n \times (Z_3^2 \rtimes \langle z \rangle)$, where $o(z) = 2$ and $\langle z \rangle$ acts on Z_3^2 by inversion. If $\alpha(G) \geq 17/24$, then $G \cong Z_2^n \times (Z_3^{m+2} \rtimes \langle z \rangle)$ and $\alpha(G) = \frac{3\cdot 3^{m+2}+1}{4\cdot 3^{m+2}}$.*

Proof. Clearly, G is supersolvable since $G \ncong Z_2^n \times S_4$ for any n. We prove this result by induction on $|M|$. If $m = 1$, then $G \cong Z_3 \cdot (Z_2^n \times (Z_3^2 \rtimes \langle z \rangle))$. Let $H := Z_2^n \times (Z_3^2 \rtimes \langle z \rangle)$. Obviously, $N_G(Z_3)/C_G(Z_3) = H/C_H(Z_3) \lesssim Z_2$ since $\text{Aut}(Z_3) \cong Z_2$. By Lemma 2.11, we have $G \cong Z_2^{n-1} \times (Z_3.(Z_2 \times (Z_3^2 \rtimes \langle z \rangle)))$.Since $\alpha(G) = \alpha(Z_3.(Z_2 \times (Z_3^2 \rtimes \langle z \rangle))) \geq \frac{17}{24}$, we can use GAP [16] to obtain $G \cong Z_2^n \times (Z_3^3 \rtimes \langle z \rangle)$.

Next, we assume that $|M| = 3^{m-1}$. Then by induction, $G \cong Z_2^n \times (Z_3^{m+1} \rtimes \langle z \rangle)$. Now we consider the case $|M| = 3^m$. Let M_1 be a minimal normal subgroup of M such that $M_1 \trianglelefteq G$. Then $M_1 \cong Z_3$ and $|M/M_1| = 3^{m-1}$. Since $(G/M_1)/(M/M_1) \cong G/M$ and $|M/M_1| < 3^m$, it follows that $G/M_1 \cong Z_2^n \times (Z_3^{m+1} \rtimes \langle z \rangle)$. Thus, $G \cong Z_3.(Z_2^n \times (Z_3^{m+1} \rtimes \langle z \rangle))$. Similarly as above, we have $G \cong Z_2^{n-1} \times (Z_3.(Z_2 \times (Z_3^{m+1} \rtimes \langle z \rangle)))$. For convenience, in the following we write

$$G_1 := Z_3.(Z_2 \times (Z_3^{m+1} \rtimes \langle z \rangle)) = \langle c \rangle.(\langle x \rangle \times (Z_3^{m+1} \rtimes \langle z \rangle)),$$

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where $\langle c\rangle \cong Z_3$ and $\langle x\rangle \cong Z_2$. Then $G \cong Z_2^{n-1} \times G_1$ and $\alpha(G) = \alpha(G_1) \geq 17/24$ by Lemma 2.3. Obviously, G_1 is also supersolvable, so its Sylow 3-subgroup N is normal. Let P be a Sylow 2-subgroup of G_1. Then $P \cong \langle x\rangle \times \langle z\rangle$ and $G_1 \cong N \rtimes P$ by the Schur-Zassenhaus Theorem.

First, we assume that x centralizes c. Then $G_1 \cong \langle x\rangle \times (N \rtimes \langle z\rangle)$. Using Lemma 2.6, we have

$$\alpha(G_1) = \frac{1}{2}(\alpha(C_N(1)) + \alpha(C_N(z))) = \frac{1}{2}(\alpha(N) + \alpha(C_N(z))). \qquad (*1)$$

Suppose that $zc = cz$, that is, z centralizes c. Then $C_N(z) = \langle c\rangle$, since c acts on Z_3^{m+1} by inversion. By the formula (*1), we obtain that $\alpha(G_1) = \frac{1}{2}(\alpha(N) + \alpha(\langle c\rangle)) \leq \frac{1}{2}(2/3 + 2/3) = 2/3 < 17/24$, a contradiction. Now suppose z does not centralize c, then $C_N(z) = 1$. By the well-known result on fixed point free action (see 8.1.10, [18]), it follows that N is abelian. Thus, $N \cong Z_3^{m+2}$ or $Z_3^m \times Z_9$. If $N \cong Z_3^m \times Z_9$, then by (*1) we have

$$\alpha(G_1) = \frac{1}{2}\left(\frac{1 + \frac{3^{m+1}-1}{\phi(3)} + \frac{6\cdot 3^m}{\phi(9)}}{3^{m+2}} + 1\right) = \frac{23}{36} + \frac{1}{4\cdot 3^{m+2}} < \frac{17}{24},$$

a contradiction. If $N \cong Z_3^{m+2}$, then (*1) implies that

$$\alpha(G_1) = \frac{1}{2}\left(\frac{1 + \frac{3^{m+2}-1}{\phi(3)}}{3^{m+2}} + 1\right) = \frac{3\cdot 3^{m+2}+1}{4\cdot 3^{m+2}} > \frac{3}{4}.$$

By Lemma 2.5, we conclude that $G_1 \cong Z_2 \times (Z_3^{m+2} \rtimes \langle z\rangle)$, where $\langle z\rangle$ acts on Z_3^{m+2} by inversion.

Next, we assume that x does not centralize c. Then $G_1/C_{G_1}(\langle c\rangle) \cong Z_2$ since $N_{G_1}(\langle c\rangle)/C_{G_1}(\langle c\rangle) \lesssim Z_2$. Thus, there is one element of order 2 in P that commutes with c. Note that the Sylow 2-subgroup $P \cong \langle x\rangle \times \langle z\rangle$ is neither cyclic nor generalized quaternion, so G_1 is not a Frobenius group. Hence, at most two involutions in P act fixed-point-freely on N. Now suppose there are two elements of P that act fixed-point-freely on N. Without loss of generality, we assume that $C_N(x) = C_N(z) = 1$. Applying the well-known result on fixed point free action again, we conclude that N is abelian, i.e., $N \cong Z_3^m \times Z_9$ or Z_3^{m+2}. Let $H \cong Z_3^{m+1}$ be a maximal normal subgroup of N. Clearly, $C_H(x) = C_H(z) = 1$. We claim that xz centralizes H. Indeed, if xz does not centralize H, then $\langle x\rangle \times \langle z\rangle$ acts faithfully on H, which implies that $\langle x\rangle \times \langle z\rangle$ is isomorphic to a subgroup of $\mathrm{Aut}(Z_3^{m+1}) \cong \mathrm{GL}(m+1, 3)$. Since every involution in $\mathrm{GL}(m+1, 3)$ is similar to a diagonal matrix of the form $\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ (see [19]), and since $C_H(x) = C_H(z) = 1$, it follows from Lemma 2.8 that both x and z correspond to the matrix $\mathrm{diag}(-1, \ldots, -1)$, contradicting the fact that $x \neq z$. Therefore, xz centralizes H, and so $C_N(xz)$ equals either N or H. By Lemma 2.6, we obtain

$$\alpha(G_1) = \frac{1}{4}(\alpha(N) + \alpha(C_N(x)) + \alpha(C_N(z)) + \alpha(C_N(xz))).$$

Assume $N \cong Z_3^{m+2}$. Then $\alpha(N) = \frac{3^{m+2}+1}{2\cdot 3^{m+2}} > \frac{1}{2}$. Since $C_N(xz) = N$ or H, we have $\alpha(C_N(xz)) > \frac{1}{2}$ as well. Hence, $\alpha(G_1) > \frac{1}{4}(1 + 1 + \frac{1}{2} + \frac{1}{2}) = \frac{3}{4}$. By Lemma 2.5, it follows that $G \cong Z_2^n \times (Z_3^{m+2} \rtimes \langle z\rangle)$,

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where $\langle z\rangle$ acts on Z_3^{m+2} by inversion. Now suppose $N \cong Z_9 \times Z_3^m$, where $m \geq 2$. Then $\alpha(N) \leq \alpha(Z_3^2 \times Z_9) = 23/81$, and $\alpha(C_N(xz)) \leq \alpha(Z_3^3) = 14/27$ since $C_N(xz)$ is either N or Z_3^{m+1}. This implies that $\alpha(G_1) \leq \frac{1}{4}(1+1+\frac{23}{81}+\frac{14}{27}) < \frac{17}{24}$, a contradiction. We now consider the case that there is at most one involution in P acting fixed-point-freely on N. Let K denote the kernel of the action of P on N. If $|K| = 2$, then by Lemma 2.7,

$$\begin{aligned}\alpha(G_1) &= \frac{1}{|P|}\sum_{y\in P/K}\alpha(C_N(y))c(Ky) = \frac{1}{4}(2\alpha(N)+2\alpha(C_N(y)))\\ &= \frac{1}{2}(\alpha(N)+\alpha(C_N(y))).\end{aligned}$$

Since P has at most one element that acts fixed-point-freely on N, we have $C_N(y) \neq 1$. Otherwise, if $C_N(y) = 1$, then y acts fixed-point-freely on N, meaning that for any $n \in N$, $n^y \neq n$. Since $K = C_P(N) \cong Z_2$, let $k \in C_P(N)$. Then $n^k = n$, and so $n^{ky} = n^y \neq n$. It follows that ky also acts fixed-point-freely on N, a contradiction. Therefore, $\alpha(G_1) \leq \frac{1}{2}\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{2}{3} < \frac{17}{24}$, a contradiction. If $|K| = 1$, we obtain

$$\alpha(G_1) = \frac{1}{4}(\alpha(N)+\alpha(C_N(x))+\alpha(C_N(z))+\alpha(C_N(xz))).$$

We suppose that none of the centralizers $C_N(x)$, $C_N(z)$, and $C_N(xz)$ is trivial. Then we have $\alpha(G_1) \leq \frac{1}{4}(\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}) = \frac{2}{3} < \frac{17}{24}$, a contradiction. Therefore, at least one of these centralizers must be trivial. By the result on fixed-point-free action (see 8.1.10, [18]), N is an abelian group, which implies that $N \cong Z_3^{m+2}$ or $Z_3^m \times Z_9$. Recall that $xc \neq cx$ and there exists one element of order 2 in P that commutes with c. Without loss of generality, assume that xz centralizes c. Then $c^{xz} = c$, and hence $c \neq c^x = c^z$. We claim that $C_N(z) = 1$. Indeed, for all $n \in N \setminus \langle c\rangle$, by our assumption, $(n\langle c\rangle)^{z\langle c\rangle} = (n\langle c\rangle)^{-1}$, we have $n^z \in n^{-1}\langle c\rangle$, and hence $n^z = n^{-1}$, $n^z = n^{-1}c$, or $n^z = n^{-1}c^{-1}$. Obviously, in the first case $n \in C_N(z)$ is impossible because the order of n is not 1 or 2. For remaining cases, if $n^z = n$, then $n^{-1}c = n$ or $n^{-1}c^{-1} = n$, and so $n^2 = c$ or $n^2 = c^{-1}$. Since N is 3-group, the order of n equals one of c, and then n in $\langle c\rangle$, a contradiction. Hence $C_N(z) = 1$. Moreover, since $o(z) = 2$, by 8.1.10 of [18], we have $n^z = n^{-1}$ for any $n \in N$. Next, we claim that $N = C_N(xz) \times C_N(x)$. Indeed, since $N \trianglelefteq G_1$ and $o(xz) = 2$, we have $N^{xz} = N$, and thus xz induces an automorphism of N by conjugation. Define $T = \{n \in N \mid n^{xz} = n^{-1}\}$. Since N is an abelian group of odd order, we have $N = C_N(xz) \times T$ by Lemma 2.9. Note that $n^{xz} = n^{-1} = n^z$ if and only if $n^x = n$, it follows that $T = C_N(x)$. Therefore, $N = C_N(xz) \times C_N(x)$. In particular, $c^{xz} = c$ and $K = 1$, so $1 < C_N(xz) < N$, this implies that $C_N(xz)$ and $C_N(x)$ are not trivial.

We first assume that $N \cong Z_9 \times Z_3^m$. Since $N = C_N(xz) \times C_N(x)$, one of the groups $C_N(xz)$ or $C_N(x)$ has Z_9 as a direct factor. Thus, $\alpha(G_1) \leq \frac{1}{4}(\frac{8}{27}+\frac{2}{9}+\frac{2}{3}+1) = \frac{59}{108} < \frac{17}{24}$, which is impossible.

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Next, suppose that $N \cong Z_3^{m+2}$ and $|C_N(xz)| = 3^{m_1}$. Then $|C_N(x)| = 3^{m+2-m_1}$. Since $1 \neq C_N(xz) < N$, we have $1 \leq m_1 \leq m+1$. It follows that

$$\begin{aligned}\alpha(G_1) &= \frac{1}{4}\left(\frac{1+\frac{3^{m+2}-1}{\phi(3)}}{3^{m+2}} + \frac{1+\frac{3^{m_1}-1}{\phi(3)}}{3^{m_1}} + \frac{1+\frac{3^{m+2-m_1}-1}{\phi(3)}}{3^{m+2-m_1}} + 1\right) \\ &= \frac{1}{4}\left(\frac{5}{2} + \frac{1}{2\cdot 3^{m+2}} + \frac{1}{2}\cdot\frac{\frac{3^{m+2}}{3^{m_1}} + 3^{m_1}}{3^{m+2}}\right).\end{aligned}$$

It is easy to show that the function $f(t) = \frac{3^{m+2}}{t} + t$ for $t = 3^{m_1}$ with $1 \leq m_1 \leq m+1$ is strictly decreasing for $t < \sqrt{3^{m+2}}$ and strictly increasing for $t > \sqrt{3^{m+2}}$. Thus, its maximum value occurs at the endpoints, i.e., $f(3) = f(3^{m+1}) = 3^{m+1} + 3$. Therefore,

$$\alpha(G_1) \leq \frac{1}{4}\left(\frac{5}{2} + \frac{1}{2\cdot 3^{m+2}} + \frac{3^{m+1}+3}{2\cdot 3^{m+2}}\right) = \frac{1}{4}\left(\frac{8}{3} + \frac{2}{3^{m+2}}\right).$$

Since $m \geq 2$, $\alpha(G_1) \leq \frac{1}{4}(\frac{8}{3} + \frac{2}{81}) = \frac{109}{162} < \frac{17}{24}$, a contradiction. □

3. Proof of Theorem 1.1

Since $\alpha(G) \geq 17/24$, it follows from Lemma 2.4 that either $G \cong Z_2^n \times S_4$ for some n or G is supersolvable. Thus, we need only consider the supersolvable case. Let p be the maximum prime divisor of $|G|$. By Lemma 2.10, we have $\frac{17}{24} \leq \frac{1}{2} + \frac{1}{p}$, which implies that $p \leq 3$. If G is a 3-group, then $\alpha(G) \leq 2/3 < 17/24$, a contradiction. Since G is not a 2-group by hypothesis, the prime divisors of $|G|$ are exactly 2 and 3. Since G is supersolvable, its Sylow 3-subgroup N is normal. Let P be a Sylow 2-subgroup of G, it follows from the Schur-Zassenhaus Theorem that $G = N \rtimes P$. Now, consider a normal series $G \trianglerighteq \cdots \trianglerighteq N = N_0 \trianglerighteq N_1 \trianglerighteq \cdots \trianglerighteq N_s = 1$, where each N_i is a maximal subgroup of N_{i-1} for $i = 1, 2, \ldots, s$. Applying Lemma 2.1, we obtain

$$\frac{17}{24} \leq \alpha(G) = \alpha(G/N_s) \leq \alpha(G/N_{s-1}) \leq \cdots \leq \alpha(G/N_1) \leq \alpha(G/N_0) = \alpha(P).$$

Note that $(G/N_{i+1})/(N_i/N_{i+1}) \cong G/N_i$, so $G/N_{i+1} \cong N_i/N_{i+1}.G/N_i$. Moreover, since N is a 3-group, we have $G/N_1 \cong Z_3 \rtimes P$ and $G/N_{i+1} \cong Z_3.G/N_i$. For convenience, throughout this section we denote G/N_i by G_i, where $i = 1, 2, \ldots, s$. Next, we seek an index $i \in \{0, 1, \ldots, s\}$ such that $\alpha(G_{i+1}) < 17/24$ and $\alpha(G_i) \geq 17/24$. This implies $N_i = 1$, and hence $G = G_i \cong Z_3.G_{i-1}$.

Step 1. Consider the quotient group $G_1 = Z_3 \rtimes P$. Let K be the kernel of the action of P on Z_3, i.e., $K = C_P(Z_3)$. Observe that $N_{G_1}(Z_3)/C_{G_1}(Z_3) \cong P/K \lesssim \mathrm{Aut}(Z_3)$, which implies that K is a normal subgroup of P of index at most 2. If $|P : K| = 1$, then $P = K \trianglelefteq G_1$, and it follows that $G_1 \cong Z_3 \times P$. Therefore, $\alpha(G_1) = \alpha(Z_3) \times \alpha(P) \leq 2/3 < 17/24$, a contradiction. Thus, $|P : K| = 2$, that is, K is a

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maximal subgroup of P, and hence there are three Sylow 2-subgroups in G_1. We therefore have

$$\begin{aligned}\alpha(G_1) &= \frac{c(Z_3\times K)+3c(P-K)}{3|P|} = \frac{1}{2}\alpha(Z_3\times K)+\frac{c(P-K)}{|P|}\\ &= \frac{1}{3}\alpha(K)+\frac{c(P-K)}{|P|}.\end{aligned}$$

This implies that

$$\begin{aligned}\alpha(G_1) &\le \frac{1}{3}\cdot\frac{I(K)+\frac{|K|-I(K)}{\phi(4)}}{|K|}+\frac{I(P)-I(K)+\frac{|K|-(I(P)-I(K))}{\phi(4)}}{|P|}\\ &= \frac{5}{12}+\frac{I(P)}{2|P|}-\frac{I(K)}{12|K|},\end{aligned}$$

with equality if and only if $\exp(P)=4$. We may assume that $\frac{I(P)}{|P|}\le\frac{7}{12}$, then $\alpha(G_1)\le\frac{5}{12}+\frac{1}{2}\cdot\frac{7}{12}-\frac{I(K)}{12|K|}<\frac{5}{12}+\frac{7}{24}=\frac{17}{24}$, a contradiction. Thus, $\frac{I(P)}{|P|}>\frac{7}{12}$. It follows that P is isomorphic to one of the groups in Lemma 2.12. We will discuss one by one.

Case 1. Let $P\cong Z_2^l$. Then $G_1\cong Z_3\rtimes Z_2^l$.

Since $N_{G_1}(Z_3)/C_{G_1}(Z_3)\cong P/C_P(Z_3)\lesssim \mathrm{Aut}(Z_3)\cong Z_2$, we have $G_1\cong Z_2^{l-1}\times(Z_3\rtimes Z_2)$. Note that $\alpha(Z_6)=2/3<17/24$, so $G_1\cong Z_2^{l-1}\times S_3$. In particular, $\alpha(G_1)=5/6$.

Case 2. Let $P\cong Z_2^l\times D_{16}$. Then $G_1\cong Z_3\rtimes(Z_2^l\times D_{16})$.

Since $N_{G_1}(Z_3)/C_{G_1}(Z_3)\cong G_1/C_{G_1}(Z_3)\lesssim Z_2$, we have Z_2^{l-1} commutes with Z_3. Lemma 2.11 implies that $G_1\cong Z_2^{l-1}\times(Z_3\rtimes(Z_2\times D_{16}))\coloneqq Z_2^{l-1}\times H$, where $H=Z_3\rtimes(Z_2\times D_{16})$. Clearly, $\alpha(G_1)=\alpha(H)\ge 17/24$. Using GAP [16], we can get that there are five supersolvable groups of order 96 satisfying $\alpha\ge 17/24$: $Z_2^2\times D_{24}$, $Z_2\times D_8\times S_3$, $Z_2^4\times S_3$, SmallGroup$(96,81)\cong \mathrm{Z}_4\rtimes\mathrm{D}_{24}$, and SmallGroup$(96,219)\cong \mathrm{Z}_2^2\times(\mathrm{Z}_3\rtimes\mathrm{D}_8)$. However, none of these groups admits a homomorphic image isomorphic to $Z_2\times D_{16}$, which leads to a contradiction.

Case 3. Let $P\cong Z_2^l\times D_8$. Then $G_1\cong Z_3\rtimes(Z_2^l\times D_8)$.

Similarly as above, we have $G_1\cong Z_2^{l-1}\times(Z_3\rtimes(Z_2\times D_8))\coloneqq Z_2^{l-1}\times H$, where $H=Z_3\rtimes(Z_2\times D_8)$. Clearly, $\alpha(G_1)=\alpha(H)\ge 17/24$. Using GAP [16], we obtain that there are precisely three supersolvable groups A of order 48 satisfying $\alpha(A)\ge 17/24$ and $A/Z_3\cong Z_2\times D_8$, namely, $Z_2\times D_{24}$, $S_3\times D_8$, and $Z_2\times$ SmallGroup$(24,8)\cong Z_2\times(Z_3\rtimes D_8)$. Thus, $G_1\cong Z_2^{l-1}\times H$, where H is one of the groups: $Z_2\times D_{24}, S_3\times D_8$ or $Z_2\times$ SmallGroup$(24,8)\cong Z_2\times(Z_3\rtimes D_8)$. In particular, the values of $\alpha(G_1)$ are 3/4, 35/48, and 17/24, respectively.

Case 4. Let $P\cong Z_2^l\times(Z_4^2\rtimes\langle\varepsilon\rangle)$, where $\langle\varepsilon\rangle$ acts on Z_4^2 by inversion. Then $G_1\cong Z_3\rtimes(Z_2^l\times(Z_4^2\rtimes\langle\varepsilon\rangle))$.

Similarly as in Case 2, we have $G_1\cong Z_2^{l-1}\times(Z_3\rtimes(Z_2\times(Z_4^2\rtimes\langle\varepsilon\rangle)))\coloneqq Z_2^{l-1}\times H$, where $H=Z_3\rtimes(Z_2\times(Z_4^2\rtimes\langle\varepsilon\rangle))$. Clearly, $\alpha(G_1)=\alpha(H)\ge 17/24$. By GAP [16], there is a unique supersolvable group $A\cong Z_2\times$ SmallGroup$(96,81)$ of order 192 satisfying $\alpha(A)\ge 17/24$ and $A/Z_3\cong Z_2\times(Z_4^2\rtimes\langle\varepsilon\rangle)$. Thus, $H\cong Z_2\times$SmallGroup$(96,81)$ and $G_1\cong Z_2^l\times$SmallGroup$(96,81)\cong \mathrm{Z}_2^{\mathrm{l}}\times(\mathrm{Z}_4\rtimes\mathrm{D}_{24})$. In particular, $\alpha(G_1)=17/24$.

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Case 5. Let $P \cong Z_2^l \times ((Z_2 \times D_8) \rtimes Z_2) \cong Z_2^l \times \text{SmallGroup}(32, 49)$. Then $G_1 \cong Z_3 \rtimes (Z_2^l \times ((Z_2 \times D_8) \rtimes Z_2))$.

Similarly as Case 2, $G_1 \cong Z_2^{l-1} \times (Z_3 \rtimes (Z_2 \times ((Z_2 \times D_8) \rtimes Z_2))) := Z_2^{l-1} \times H$, where $H = Z_3 \rtimes (Z_2 \times ((Z_2 \times D_8) \rtimes Z_2))$. Clearly, $\alpha(G_1) = \alpha(H) \geq 17/24$. A computation using GAP [16] shows that no supersolvable group A of order 192 satisfies both $\alpha(A) \geq 17/24$ and $A/Z_3 \cong Z_2 \times ((Z_2 \times D_8) \rtimes Z_2)$. This leads to a contradiction.

Case 6. Let $P \cong Z_2^l \times (Z_2^4 \rtimes Z_2) \cong Z_2^l \times \text{SmallGroup}(32, 27)$. Then $G_1 \cong Z_3 \rtimes (Z_2^l \times (Z_2^4 \rtimes Z_2))$.

Similarly as Case 2, we have $G_1 \cong Z_2^{l-1} \times (Z_3 \rtimes (Z_2 \times (Z_2^4 \rtimes Z_2))) := Z_2^{l-1} \times H$, where $H = Z_3 \rtimes (Z_2 \times (Z_2^4 \rtimes Z_2))$. It is obvious that $\alpha(G_1) = \alpha(H) \geq 17/24$. With the help of GAP [16], there is no supersolvable group A of order 192 satisfying both $\alpha(A) \geq 17/24$ and $A/Z_3 \cong Z_2 \times (Z_2^4 \rtimes Z_2)$, which is a contradiction. Hence, $G_1 \cong Z_2^{l-1} \times H$, where H is isomorphic to one of S_3, $S_3 \times D_8$, $Z_2 \times D_{24}$, $Z_2 \times \text{SmallGroup}(24, 8)$, or $Z_2 \times \text{SmallGroup}(96, 81)$. The corresponding values of $\alpha(G_1)$ are 5/6, 35/48, 3/4, 17/24, and 17/24, respectively.

Recall that $G_2 = G/N_2 \cong N_1/N_2.G/N_1 \cong Z_3.G_1$ and $17/24 \leq \alpha(G) \leq \alpha(G_2) \leq \alpha(G_1)$. If $\alpha(G_1) = 17/24$, then $\alpha(G) = 17/24$. By Lemma 2.2, this implies that N_1 is an elementary abelian 2-group. But N_1 is a 3-group, so this forces $N_1 = 1$ and thus $G = G_1$. Therefore, it suffices to consider the cases that G_1 is isomorphic to $Z_2^{l-1} \times S_3$, $Z_2^{l-1} \times S_3 \times D_8$, or $Z_2^l \times D_{24}$.

Step 2. Consider the quotient group $G_2 = Z_3.G_1$. First we let $G_1 \cong Z_2^{l-1} \times S_3$. Then $G_2 \cong Z_3.(Z_2^{l-1} \times S_3)$. Since $N_{G_2}(Z_3)/C_{G_2}(Z_3) \cong G_2/C_{G_2}(Z_3) \lesssim Z_2$, it follows that Z_2^{l-2} and Z_3 commute. Hence, by Lemma 2.11, we have $G_2 \cong Z_2^{l-2} \times (Z_3.(Z_2 \times S_3)) := Z_2^{l-2} \times H$, where $H = Z_3.(Z_2 \times S_3)$. Note that $\alpha(G_2) = \alpha(H) \geq 17/24$. In the light of GAP [16], we have $H \cong S_3 \times S_3$ or $Z_2 \times (Z_3^2 \rtimes \langle\varepsilon\rangle)$, where $\langle\varepsilon\rangle \cong Z_2$ acts on Z_3^2 by inversion. Consequently, $G_2 \cong Z_2^{l-2} \times S_3 \times S_3$ or $Z_2^{l-1} \times (Z_3^2 \rtimes \langle\varepsilon\rangle)$, with $\alpha(G_2) = 13/18$ in the first case and 7/9 in the second.

Next, suppose $G_1 \cong Z_2^{l-1} \times S_3 \times D_8$. Then $G_2 \cong Z_3.(Z_2^{l-1} \times S_3 \times D_8)$. Similarly, we have $G_2 \cong Z_2^{l-2} \times (Z_3.(Z_2 \times S_3 \times D_8)) := Z_2^{l-2} \times H$, where $H = Z_3.(Z_2 \times S_3 \times D_8)$. Hence, $\alpha(G_2) = \alpha(H) \geq 17/24$. However, a calculation in GAP [16] shows that no supersolvable group A of order 288 satisfies both $\alpha(A) \geq 17/24$ and $A/Z_3 \cong Z_2 \times S_3 \times D_8$. This is a contradiction.

Finally, we let $G_1 \cong Z_2^l \times D_{24}$. Then $G_2 \cong Z_3.(Z_2^l \times D_{24})$. Similarly as the first case, we have $G_2 \cong Z_2^{l-1} \times (Z_3.(Z_2 \times D_{24})) := Z_2^{l-1} \times H$, where $H \cong Z_3.(Z_2 \times D_{24})$. Since $\alpha(G_2) = \alpha(H) \geq 17/24$, it is not difficult to obtain, with the help of GAP [16], that $H \cong Z_2 \times \text{SmallGroup}(72, 33)$ and $\alpha(H) = 17/24$. Consequently, $G_2 \cong Z_2^l \times \text{SmallGroup}(72, 33) \cong \mathrm{Z}_2^{\mathrm{l}} \times (\mathrm{Z}_{12} \rtimes \mathrm{S}_3)$ and $\alpha(G_2) = 17/24$.

Therefore, $G_2 \cong Z_2^{l-2} \times H$, where H is isomorphic to one of the groups: $S_3 \times S_3$, $Z_2^2 \times \text{SmallGroup}(72, 33)$ and $Z_2 \times (Z_3^2 \rtimes \langle\varepsilon\rangle)$, where $\langle\varepsilon\rangle$ acts on Z_3^2 by inversion. The corresponding values of $\alpha(G_2)$ are 13/18, 17/24, and 7/9, respectively.

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Recall that $G_3 = G/N_3 \cong N_2/N_3.G/N_2 \cong Z_3.G_2$ and $17/24 \leq \alpha(G) \leq \alpha(G_3) \leq \alpha(G_2)$. If $\alpha(G_2) = 17/24$, then, by Lemma 2.2, $N_2 = 1$ since N_2 is a 3-group, and hence $G = G_2$. Assume $G_2 \cong Z_2^{l-1} \times (Z_3^2 \rtimes \langle\varepsilon\rangle)$, where $\langle\varepsilon\rangle$ acts on Z_3^2 by inversion. Since N_2 is a 3-group and $\alpha(G) \geq 17/24$, we have $G \cong Z_2^{l-1} \times (Z_3^m \rtimes \langle\varepsilon\rangle)$ for some m and $\alpha(G) = \frac{3\cdot 3^m+1}{4\cdot 3^m} \geq 3/4$ by Lemma 2.13, a contradiction. Thus, it suffices to consider the case that $G_2 \cong Z_2^{l-2} \times S_3 \times S_3$.

Step 3. Consider the quotient group $G_3 = Z_3.G_2$. Since $G_2 \cong Z_2^{l-2} \times S_3 \times S_3$, it follows that $G_3 \cong Z_3.(Z_2^{l-2} \times S_3 \times S_3)$. Note that $N_{G_3}(Z_3)/C_{G_3}(Z_3) \cong G_3/C_{G_3}(Z_3) \lesssim Z_2$. Therefore, by Lemma 2.11, we have $G_3 \cong Z_2^{l-3} \times (Z_3.(Z_2 \times S_3 \times S_3)) := Z_2^{l-3} \times H$, where $H = Z_3.(Z_2 \times S_3 \times S_3)$. Clearly, $\alpha(G_3) = \alpha(H)$. However, using GAP [16], we can see that there does not exist a group A of order 216 with the properties that $\alpha(A) \geq 17/24$ and $A/Z_3 \cong Z_2 \times S_3 \times S_3$ for any normal subgroup Z_3 of A, which yields a contradiction.

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