



<https://ijgt.ui.ac.ir>

International Journal of Group Theory

ISSN (print): 2251-7650, ISSN (on-line): 2251-7669

Vol. xx No. x (202x), pp. xx-xx.

© 202x University of Isfahan



<https://www.ui.ac.ir>

ON CAMINA PAIRS IN DICYCLIC GROUPS AND THE HEISENBERG GROUP

MARYAM MEHRDAD, MOHAMAD REZA DARAFSHEH[✉] AND HESAM SHARIFI^{✉*}

ABSTRACT. Let G be a finite group and N a proper non-trivial normal subgroup of G . The pair (G, N) is called a Camina pair if every $x \in G \setminus N$ is conjugate to all elements of the coset xN ; when $N = Z(G)$ one speaks of a central Camina pair. In this paper, we classify all Camina pairs arising from dicyclic groups T_{4n} . We prove that $(T_{4n}, Z(T_{4n}))$ is a central Camina pair if and only if $T_{4n} \cong Q_8$, the quaternion group of order 8. More generally, we show that a proper non-trivial normal subgroup $N \triangleleft T_{4n}$ forms a Camina pair (T_{4n}, N) precisely when $T_{4n} \cong Q_8$ and $N = Z(Q_8)$. The proof combines structural lemmas on conjugacy classes of dicyclic groups with the classical Camina dichotomy. We then turn to the Heisenberg group $UT(3, q)$, the Sylow p -subgroup of $GL_3(q)$, and confirm the conjecture $[G : Z(G)] \geq |Z(G)|^2$ for this family, showing that equality holds. As a consequence, we determine that every non-central conjugacy class in $UT(3, q)$ has size q . Finally, we relate central Camina pairs to GVZ-groups via the character-theoretic vanishing condition of Lemma 2.1(d), and discuss how this connection suggests further lines of investigation.

1. Introduction

In [1], Camina introduced a class of groups generalizing Frobenius groups and extra-special p -groups. The theory of Camina pairs and Camina groups was further developed in [4, 5, 6].

Let G be a group and $1 \neq N \triangleleft G$. The pair (G, N) is called a *Camina pair* if every element $x \in G \setminus N$ is conjugate to all elements of the coset xN . The subgroup N is called a *Camina kernel*. If $N = Z(G)$, the pair is called a *central Camina pair*. If (G, G') is a Camina pair, then G is called a *Camina group*.

MSC(2010): Primary: 20D15; Secondary: 20C15.

Keywords: Camina pairs, dicyclic groups, Heisenberg group, GVZ-groups.

Communicated by Alireza Moghaddamfar.

*Corresponding author.

Received: 24 June 2026, Accepted: 18 October 2026, Published Online: xx xxx xxxx.

Cite this article: M. Mehrdad, M. R. Darafsheh and H. Sharifi, On camina pairs in dicyclic groups and the heisenberg group, *Int. J. Group Theory*, x no. x (202x) xx-xx. <https://dx.doi.org/10.22108/ijgt.2026.149447.2030>.

Classical examples include Frobenius groups with abelian kernel, extra-special p -groups, the quaternion group Q_8 , and the dihedral group of order 8.

In this paper, we classify Camina pairs in dicyclic groups T_{4n} . We show that Q_8 is the only dicyclic group admitting a central Camina pair, and we give a complete classification of all Camina pairs (T_{4n}, N) with $N \triangleleft T_{4n}$.

We also study the unitriangular group $UT(3, q)$ and show that all its non-central conjugacy classes have size q , providing evidence for its strong Camina-type structure.

Finally, we give a character-theoretic interpretation of Camina pairs in terms of vanishing properties of irreducible characters lying over the kernel.

2. Preliminary Results

We recall some equivalent characterizations of Camina pairs.

Lemma 2.1. *Let G be a group and $1 \neq N \triangleleft G$. The following are equivalent:*

- (a) *(G, N) is a Camina pair;*
- (b) *for every $x \in G \setminus N$, the element x is conjugate to all elements in the coset xN ;*
- (c) *for every $x \in G \setminus N$, $|x^G| = |xN|$;*
- (d) *for every $\chi \in \text{Irr}(G \mid N)$, $\chi(g) = 0$ for all $g \in G \setminus N$, where $\text{Irr}(G \mid N) = \{\chi \in \text{Irr}(G) \mid N \not\subseteq \ker \chi\}$.*

The following lemma, due to Camina [1], describes the basic structural alternatives for Camina pairs.

Lemma 2.2. *Let (G, N) be a Camina pair. Then either*

- (1) *G is a Frobenius group with kernel N , or*
- (2) *at least one of G/N or N is a p -group for some prime p .*

It is known that in any Camina pair (G, N) one has $Z(G) \leq N \leq G$. The cases $N = Z(G)$ and $N = G'$ are of particular importance in the literature.

Camina groups (G, G') are always solvable, and their structure has been completely determined by Dark and Scoppola [2].

Lemma 2.3. *Let G be a Camina group. Then one of the following holds:*

- (1) *G is a Frobenius group with cyclic complement;*
- (2) *G is a Frobenius group with quaternion complement;*
- (3) *G is a p -group.*

The next two lemmas are due to Macdonald [5] and Verardi [9].

Lemma 2.4. *If G is a Camina 2-group, then G has nilpotency class 2.*

Lemma 2.5. *Let G be a p -group. Then the following are equivalent:*

- (1) G is a Camina group of nilpotency class 2;
- (2) G is a semi-extra-special p -group.

3. Dicyclic groups

The dicyclic group T_{4n} , $n \geq 1$, is defined as follows:

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

This group has order $4n$ and it is easy to see that its center $Z(T_{4n}) = \langle b^2 \rangle$ has order 2 and b^2 is the unique element of order 2 in T_{4n} . As $\langle a \rangle$ has order $2n$, it is normal in T_{4n} with quotient the cyclic group of order 2, hence T_{4n} is a solvable group. For small values of n we have $T_4 \cong \mathbb{Z}_4$ and $T_8 \cong Q_8$ (the quaternion group of order 8).

Lemma 3.1. T_{4n} , $n \geq 1$, is a nilpotent group if and only if n is a power of 2.

Proof. We have $Z_1 = Z(T_{4n}) = \langle b^2 \rangle$. By setting $x = Z_1a$, $y = Z_1b$, we see that $x^n = Z_1$, $y^2 = Z_1$, $y^{-1}xy = x^{-1}$, hence $\frac{T_{4n}}{Z_1} = \langle Z_1a, Z_1b \rangle \cong D_{2n}$, the dihedral group of order $2n$. But if T_{4n} is a nilpotent group, then the center of D_{2n} must be non-trivial, implying that n must be a power of 2. Conversely, if $n = 2^k$ is a power of 2, then $|T_{4n}| = 4n = 2^{k+2}$, and T_{4n} is a 2-group which is nilpotent. $\square$

Remark 3.2. In Lemma 3.1 we proved that $\frac{T_{4n}}{Z(T_{4n})} \cong D_{2n}$. If $n = 2^k$, then $D_{2n} = D_{2^{k+1}}$ is nilpotent of class $k+1$. A standard argument shows that for these values of n the dicyclic group T_{4n} has nilpotency class $k+1$ (and not $k+2$). In particular, T_8 ($k=1$) has class 2, while T_4 ($k=0$) is abelian.

Lemma 3.3. The group T_{4n} , $n \geq 1$, is not a Frobenius group.

Proof. Suppose T_{4n} is a Frobenius group with complement H and kernel K , and reach a contradiction. By [3, pp. 177], since $\langle b^2 \rangle \trianglelefteq T_{4n}$, then either $\langle b^2 \rangle \leq K$ or $K \leq \langle b^2 \rangle$. If $K \leq \langle b^2 \rangle$, then as $\langle b^2 \rangle$ has order 2 we obtain $K = \langle b^2 \rangle$; therefore in any case we may assume K contains the central element b^2 . But again by [3, pp. 173] we must have $C_{T_{4n}}(b^2) = T_{4n} \leq K$, a contradiction. Therefore T_{4n} , $n \geq 1$, is not a Frobenius group. $\square$

The following proposition classifies when the centre of a dicyclic group forms a central Camina pair.

Proposition 3.4. If $(T_{4n}, Z(T_{4n}))$ is a Camina group, then $n = 2$ and $(T_8, Z(T_8)) \cong (Q_8, Z(Q_8))$, where Q_8 is the quaternion group of order 8.

Proof. Suppose $(T_{4n}, Z(T_{4n}))$ is a Camina group. By Lemma 2.2, T_{4n} is a p -group for some prime p , because by Lemma 3.3 it cannot be a Frobenius group. Now since $|T_{4n}| = 4n$, we deduce that $n = 2^k$ is a power of 2 and T_{4n} is a 2-group. Now by Lemma 2.4 the group T_{4n} must have nilpotency class 2. By Remark 3.2, a dicyclic 2-group $T_{2^{k+2}}$ has class $k+1$; hence $k+1 = 2$, so $k = 1$ and $n = 2$. Therefore, $T_8 \cong Q_8$. $\square$

We now extend the previous result to arbitrary proper non-trivial normal subgroups. The following lemmas are needed.

Lemma 3.5. *Let $N \trianglelefteq T_{4n}$ and suppose N contains an element of the form $a^i b$ (with b the generator of order $4n$ outside $\langle a \rangle$). Then $\langle a^2 \rangle \leq N$.*

Proof. Let $a^i b \in N$. Since N is normal, $[a, a^i b] \in N$. A direct computation in T_{4n} gives $[a, a^i b] = a^{-2}$. Hence $a^2 \in N$ and therefore $\langle a^2 \rangle \leq N$. $\square$

Lemma 3.6. *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$ be a dicyclic group, and let $N = \langle a^d \rangle \trianglelefteq T_{4n}$. If (T_{4n}, N) is a Camina pair, then $N = Z(T_{4n})$.*

Proof. Assume that (T_{4n}, N) is a Camina pair and suppose, for a contradiction, that $N \neq Z(T_{4n})$. Since $Z(T_{4n}) = \langle a^n \rangle$ has order 2, the cyclic subgroup N must have order strictly larger than 2.

Since N is a proper subgroup of the cyclic subgroup $\langle a \rangle$ and $Z(T_{4n}) = \langle a^n \rangle \leq N$, there exists a non-central power $a^i \notin N$.

Because (T_{4n}, N) is a Camina pair, Lemma 2.1(b) yields $(a^i)^{T_{4n}} = a^i N$. Taking cardinalities gives $|(a^i)^{T_{4n}}| = |N|$.

On the other hand, for every non-central power of a , the conjugacy class of a^i in T_{4n} is exactly $\{a^i, a^{-i}\}$, which has size 2 (see the proof of Proposition 3.4).

Thus $|N| = 2$. Since N is a cyclic subgroup of $\langle a \rangle$, the unique subgroup of order 2 inside $\langle a \rangle$ is $\langle a^n \rangle = Z(T_{4n})$. Hence $N = Z(T_{4n})$, contradicting the assumption $N \neq Z(T_{4n})$.

Therefore $N = Z(T_{4n})$. $\square$

Lemma 3.7. *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$ be the dicyclic group of order $4n$, and let $N \trianglelefteq T_{4n}$ be a proper normal subgroup with $N \not\leq \langle a \rangle$. Then $\langle a^2 \rangle \leq N$. Moreover, if n is even, then N is one of the two subgroups*

$$\langle a^2, b \rangle \quad \text{or} \quad \langle a^2, ab \rangle.$$

Proof. Since $N \not\leq \langle a \rangle$, there exists an element of the form $a^i b \in N$. By Lemma 3.5, $\langle a^2 \rangle \leq N$.

Now consider the quotient group $\bar{T} = T_{4n}/\langle a^2 \rangle$, which has order 4. If n is even, then $a^n \in \langle a^2 \rangle$, so $\bar{b}^2 = 1$ and $\bar{T} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Lifting the subgroups of $\bar{T}$ gives the two maximal subgroups $\langle a^2, b \rangle$ and $\langle a^2, ab \rangle$.

If n is odd, then $a^n \notin \langle a^2 \rangle$. In the quotient, $\bar{b}^2 = \bar{a}^n \neq \bar{1}$, and since $\bar{a}$ has order 2, the element $\bar{b}$ has order 4. Hence $\bar{T}$ is cyclic of order 4. Consequently, any subgroup strictly containing $\langle a^2 \rangle$ must lie in $\langle a \rangle$, contradicting $N \not\leq \langle a \rangle$. $\square$

We can now prove a complete classification of Camina pairs in dicyclic groups.

Theorem 3.8. *Let T_{4n} be a dicyclic group and let $N \trianglelefteq T_{4n}$ be a proper non-trivial normal subgroup. Then (T_{4n}, N) is a Camina pair if and only if*

$$T_{4n} \cong Q_8 \quad \text{and} \quad N = Z(Q_8).$$

Proof. Assume (T_{4n}, N) is a Camina pair. We distinguish two cases.

Case 1: $N \leq \langle a \rangle$. Then N is cyclic, so by Lemma 3.6, $N = Z(T_{4n})$. Since $(T_{4n}, Z(T_{4n}))$ is a central Camina pair, Proposition 3.4 forces $n = 2$ and $T_{4n} \cong Q_8$.

Case 2: $N \not\leq \langle a \rangle$. By Lemma 3.7, $\langle a^2 \rangle \leq N$. If n is even, then N is one of $\langle a^2, b \rangle$ or $\langle a^2, ab \rangle$, and $|N| = 2n$.

Observe that every element of $\langle a^2, b \rangle$ has the form a^{2k} or $a^{2k}b$, and every element of $\langle a^2, ab \rangle$ has the form a^{2k} or $a^{2k+1}b$. In either case, a itself does not belong to the subgroup. Therefore $a \notin N$.

By the Camina condition and Lemma 2.1(b), $|a^G| = |N|$. But in T_{4n} we have $|a^G| = 2$. Thus $2n = 2$, so $n = 1$, contradicting the assumption that n is even.

If n is odd, then by Lemma 3.7 no proper subgroup $N \not\leq \langle a \rangle$ can strictly contain $\langle a^2 \rangle$ without being contained in $\langle a \rangle$, a contradiction.

Therefore Case 2 cannot occur.

The only remaining possibility is $T_{4n} \cong Q_8$ and $N = Z(Q_8)$. □

4. Heisenberg matrix groups

A well-known conjecture states that for every central Camina pair $(G, Z(G))$ one has $[G : Z(G)] \geq |Z(G)|^2$. Lewis [7] proved that such a group G must be a group of prime power order. We now confirm the conjecture for the Sylow p -subgroup of $GL_3(q)$, which is precisely the unitriangular group $UT(3, q)$. This is a classic example of a semi-extra-special p -group (see Lemma 2.5).

Let $q = p^n$ with p a prime and $n \geq 1$, and let $G = UT(3, q)$ be the Heisenberg group over $\mathbb{F}_q$, i.e., the group of 3×3 upper triangular matrices with 1's on the main diagonal. Then

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \right\}, \quad |G| = q^3,$$

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q \right\}, \quad [G : Z(G)] = q^2,$$

and G is a nilpotent group of class 2.

We first verify that $(G, Z(G))$ is a Camina pair. Take an arbitrary non-central element $x = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in G \setminus Z(G)$. For any $z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z(G)$ we have $xz = \begin{pmatrix} 1 & a & b+d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$. Choose $t = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \in G$; a direct computation gives $t^{-1}xt = \begin{pmatrix} 1 & a & b+a\gamma-\alpha c \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$. Thus $t^{-1}xt = xz$ precisely when $a\gamma - \alpha c = d$. Since

$x \notin Z(G)$, at least one of a or c is non-zero; hence the linear map $(\alpha, \gamma) \mapsto a\gamma - \alpha c$ from $\mathbb{F}_q^2$ to $\mathbb{F}_q$ is surjective. Consequently, α and γ can be chosen to satisfy the equation, making x conjugate to every element of the coset $xZ(G)$. Therefore $(G, Z(G))$ is a Camina pair, and $[G : Z(G)] = q^2 = |Z(G)|^2$.

We record this fact in the following proposition.

Proposition 4.1. *Let G be a Sylow p -subgroup of $GL_3(q)$, where q is a power of the prime p . Then $(G, Z(G))$ is a central Camina pair and $[G : Z(G)] = |Z(G)|^2$.*

A simple but useful general property of central Camina pairs is the following.

Proposition 4.2. *Let $(G, Z(G))$ be a central Camina pair. Then for every $x \in G \setminus Z(G)$,*

$$x^G = xZ(G),$$

and consequently $|x^G| = |Z(G)|$.

Proof. The equality $x^G = xZ(G)$ is exactly condition (b) of Lemma 2.1 for a Camina pair. Taking cardinalities gives $|x^G| = |xZ(G)| = |Z(G)|$. $\square$

Applying this to $UT(3, q)$ immediately yields the size of its non-central conjugacy classes.

Corollary 4.3. *For $G = UT(3, q)$, every non-central conjugacy class has size q .*

Proof. By Proposition 4.1, G is a central Camina pair and $|Z(G)| = q$. The result follows directly from Proposition 4.2. $\square$

5. Character-Theoretic Viewpoint

The character-theoretic condition appearing in Lemma 2.1(d) naturally relates central Camina pairs to vanishing properties of irreducible characters.

Recall that a finite group G is called a *GVZ-group* if every nonlinear irreducible character of G vanishes outside its centre.

If $(G, Z(G))$ is a central Camina pair, then every $\chi \in \text{Irr}(G \mid Z(G))$ vanishes on $G \setminus Z(G)$. This shows that central Camina pairs enforce a strong character vanishing condition on characters lying over the centre, similar in spirit to the defining property of GVZ-groups.

However, these two notions are independent in general, and neither implies the other.

The main examples of this paper illustrate this phenomenon:

- The quaternion group Q_8 is a GVZ-group, and its unique nonlinear irreducible character of degree 2 vanishes on all non-central elements.
- The Heisenberg group $UT(3, q)$ is also a GVZ-group, and its nonlinear characters vanish outside the centre as well.

Thus, Q_8 and $UT(3, q)$ serve as examples where GVZ-type vanishing coincides with the structural condition $x^G = xZ(G)$.

The classification obtained in this paper shows that within the class of dicyclic groups, the Camina property is extremely restrictive, occurring only in the case Q_8 . This reflects the rigidity of Camina-type structures in non-abelian 2-generator 2-groups.

REFERENCES

- [1] A. R. Camina, Some conditions which almost characterize Frobenius groups, *Israel J. Math.*, **31** (1978) 153–160. doi:[10.1007/BF02760546](https://doi.org/10.1007/BF02760546)
- [2] R. Dark and C. M. Scoppola, On Camina groups of prime power order, *J. Algebra*, **181** (1996) 787–802. doi:[10.1006/jabr.1996.0146](https://doi.org/10.1006/jabr.1996.0146)
- [3] L. C. Grove, *Groups and characters*, Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1997.
- [4] I. D. Macdonald, Some p -groups of Frobenius and extra- special type, *Israel J. Math.*, **40** (1982) 350–364. doi:[10.1007/BF02761376](https://doi.org/10.1007/BF02761376)
- [5] I. D. Macdonald, More on p -groups of Frobenius type, *Israel J. Math.*, **56** (1986) 335–344. doi:[10.1007/BF02782940](https://doi.org/10.1007/BF02782940)
- [6] A. Mann and C. M. Scoppola, On p -groups of Frobenius type, *Arch. Math. (Basel)*, **56** (1991) 320–332. doi:[10.1007/BF01198216](https://doi.org/10.1007/BF01198216)
- [7] M. L. Lewis, On p -group Camina pairs, *J. Group Theory*, **15** (2012) 469–483. doi:[10.1515/jgt-2012-0010](https://doi.org/10.1515/jgt-2012-0010)
- [8] D. J. S. Robinson, *A course in the theory of groups*, Second edition. Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1996. doi:[10.1007/978-1-4419-8594-1](https://doi.org/10.1007/978-1-4419-8594-1)
- [9] I. Verardi, Semi-extraspecial groups of exponent p , (Italian), *Ann. Mat. Pura Appl. (4)*, **148** (1987) 131–171. doi:[10.1007/BF01774287](https://doi.org/10.1007/BF01774287)

Maryam Mehrdad

Department of Mathematics and Computer Science, Shahed University, Tehran, Iran

Email: mehrdad@shahed.ac.ir

Mohamad reza Darafsheh

School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

Email: darafsheh@ut.ac.ir

Hesam Sharifi

Department of Mathematics and Computer Science, Shahed University, Tehran, Iran

Email: hsharifi@shahed.ac.ir