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International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. xx No. x (202x), pp. xx–xx.

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ON THE NUMBER OF DISCONNECTED CHARACTER DEGREE GRAPHS SATISFYING PÁLFYS INEQUALITY

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Abstract. Let G be a finite solvable group with disconnected character degree graph $\Delta(G)$. Under these conditions, it follows from a result of P. P. Pálfy's that this graph consists of two connected components. Another result of P. P. Pálfy gives an inequality which relates the sizes of these two connected components (in terms of the number of vertices in each component). In this paper, we define the function $c(n)$ which calculates the number of possible component size pairs that satisfy Pálfy's inequality in terms of the order of $\Delta(G)$. Additionally, for a fixed positive integer n, the number of distinct graph orders for which exactly n component size pairs satisfy Pálfy's inequality is shown. Finally, a table of examples is given to further illustrate the number of component size pairs which satisfy the inequality versus the total possible number of component size pairs when the order of the graph gets large.

1. Introduction

Throughout this paper, G is a finite solvable group. Let $\mathrm{Irr}(G)$ denote the set of irreducible ordinary characters of G and let $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ be the set of irreducible character degrees of G. We write $\rho(G)$ for the set of primes dividing the elements of $\mathrm{cd}(G)$. The character degree graph of G, denoted $\Delta(G)$, is defined as the graph having $\rho(G)$ as its vertex set; there is an edge between

MSC(2010): Primary: 20C15; Secondary: 05C25.

Keywords: character degree graphs, Pálfy's inequality, finite solvable groups.

Article Type: Research Paper.

Communicated by Behrooz Khosravi.

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Received: 27 May 2025, Accepted: 13 September 2026, Published Online: 29 September 2026.

Cite this article: M. L. Lewis and A. Summers, On the number of disconnected character degree graphs satisfying Pálfy's inequality, *Int. J. Group Theory*, **x** no. x (202x) xx-xx. https://dx.doi.org/10.22108/ijgt.2026.145369.1964 .

primes $p, q \in \rho(G)$ if pq divides $\chi(1) \in \mathrm{cd}(G)$ for some character $\chi \in \mathrm{Irr}(G)$. Note that each $p \in \rho(G)$ corresponds to a vertex v of $\Delta(G)$, and it is commonplace to refer to them interchangeably. Recall also that the order of a graph is the cardinality of its vertex set. Character degree graphs have been studied in many places (for a summary see [3] and for background see [2] and [6, Chapter 5, Section 18]).

In the [7, main theorem], Pálfy shows that if G is a solvable group, then for any three vertices of $\rho(G)$, at least two must be adjacent in $\Delta(G)$. Equivalently, the complement graph $\overline{\Delta}(G)$ contains no three-cycles. Graphs which satisfy this result are commonly said to satisfy Pálfy's condition, and doing so imposes a great deal of structure on the character degree graphs of solvable groups. For example, when G is solvable and $\Delta(G)$ is disconnected, then by Pálfy's condition, $\Delta(G)$ must have exactly two connected components, each of which is a complete graph. Because of this, we see that each disconnected $\Delta(G)$ is completely determined by the sizes of its two connected components. Throughout this paper, we will often refer to pairs of connected component sizes as a stand-in for the associated disconnected character degree graph.

In [8, Theorem 3], Pálfy proves another result regarding the sizes of these two complete connected components. Suppose that G is solvable and that $\Delta(G)$ has two complete connected components of sizes a and b (without loss of generality, suppose $a \leq b$) then it was shown that $b \geq 2^a - 1$. We will refer to this result as Pálfy's inequality, and it provides a useful tool when investigating disconnected character degree graphs of solvable groups, as well as motivating our main result. First, we establish a bit more background and some notation.

We say that a graph Γ "occurs" as a character degree graph if $\Gamma = \Delta(G)$ for some group G. When studying character degree graphs, it is a common question to ask which graphs of a fixed order can occur as $\Delta(G)$, and which do not. Such classifications have taken place for graphs of several small orders. For example, in [4], Lewis classifies all character degree graphs with five vertices that occur for some solvable group, with one exception. In [1], Bissler et. al. perform a similar classification for graphs of order six; this time with nine graphs escaping classification. In these investigations, it is common to treat connected and disconnected graphs separately. As such, it makes sense to ask the question: how many possible disconnected graphs of a given order n could occur as $\Delta(G)$? To make this question more manageable, particularly as n gets large, we introduce a function.

For a given order $n \geq 2$, we define the function $c(n)$ (c for "connected components") to denote the number of distinct ways in which a graph of order n can be partitioned into connected components whose sizes satisfy Pálfy's inequality. For example, since the only disconnected graph of order two consists of two isolated vertices, and choosing $(a, b) = (1, 1)$ satisfies Pálfy's inequality, we have that $c(2) = 1$. With this notation in mind, we state our main result.

Let α be any positive integer. There are exactly $2^\alpha + 1$ positive integers $n \geq 2$ which satisfy $c(n) = \alpha$.

https://dx.doi.org/10.22108/ijgt.2026.145369.1964

We begin Section 2 with a rough count of the number of possible pairs of connected component sizes for a graph of fixed order n. Afterwards, we count the number of disconnected graphs of order n which may occur as $\Delta(G)$ by calculating the value of $c(n)$ for any $n \geq 2$, and prove that this count is correct. Finally, we prove the main result. In Section 3, calculations are given to put our results in context as well as to illustrate the surprisingly small number of possible disconnected graphs which can occur as the order of the character degree graph gets large. We note that while the results of this paper are motivated by the study of irreducible character degrees of finite solvable groups, they are actually more combinatorial and number-theoretic in nature. This will be discussed further, along with some future directions, in Section 4.

We note that the work in this paper was completed by the second author as a Ph. D. candidate under the supervision of the first author at Kent State University. The contents of this paper may appear as part of the second author's Ph. D. dissertation.

2. **Results**

We begin this section with a simple result that holds for more general graphs.

Lemma 2.1. *Let Γ be a graph of order $n \geq 2$. Suppose Γ is disconnected with two connected components. Then the number of distinct possible pairs of component sizes is $\lfloor n/2 \rfloor$.*

Proof. If n is odd, then $n = 2m + 1$ for some positive integer m. Let (a, b) represent a possible partition of n into two connected components of sizes a and b. We then have that the total number of such partitions is given by the set $\mathcal{C} = \{(a, 2m + 1 - a) : a \in \{1, \ldots, m\}\}$. Thus we have $|\mathcal{C}| = m = \lfloor m + \frac{1}{2} \rfloor = \lfloor (2m + 1)/2 \rfloor = \lfloor n/2 \rfloor$.

If n is even, then $n = 2m$ for some positive integer m, and a similar counting argument yields $\mathcal{C} = \{(a, 2m - a) : a \in \{1, \ldots, m\}\}$. It follows that $|\mathcal{C}| = m = \lfloor 2m/2 \rfloor = \lfloor n/2 \rfloor$, as desired. $\square$

We now look to refine the previous result in the context of character degree graphs of finite solvable groups. We do this by calculating the value of $c(n)$ for any positive integer $n \geq 2$. Namely, we have the following result:

Lemma 2.2. *Let $n \geq 2$ be a positive integer. Then*

$$c(n) = \max\{\alpha \in \mathbb{Z}^+ : n \geq 2^\alpha + \alpha - 1\}$$

yields the number of ways a graph of order n can be partitioned into two connected components whose sizes satisfy Pálfy's inequality.

Proof. Let Γ be a graph of order n and let α and β denote the sizes of the two connected components of Γ. Without loss of generality, suppose $\alpha \leq \beta$. To satisfy Pálfy's inequality, we must have $\beta \geq 2^\alpha - 1$. Noting that $n = \alpha + \beta$, it follows that we must have $n \geq 2^\alpha + \alpha - 1$. It is not hard to see that the right

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side of this inequality is strictly increasing in α, and so if $\alpha = a$ satisfies the inequality, we have that $a-1, a-2, \dots, 1$ are also solutions. Thus, for a given order n, the total number of pairs of component sizes (α, β) which satisfy Pálfy's inequality is given by $\max\{\alpha \in \mathbb{Z}^+ : n \geq 2^\alpha + \alpha - 1\}$. Since this is exactly what $c(n)$ was defined to count, the proof is complete. □

With the notation established by Lemma 2.2 in mind, it follows that for a disconnected character degree graph of order n, once $c(n)$ is known, the possible component size pairs are $(n-1, 1), (n-2, 2), \dots, (n - c(n), c(n))$. We now prove our main result.

Theorem 2.3. *Let α be any positive integer. There are exactly $2^\alpha + 1$ positive integers $n \geq 2$ which satisfy $c(n) = \alpha$.*

Proof. By definition, $c(n)$ is a non-decreasing function of n. It follows that the minimal n such that $c(n) = \alpha$ occurs when equality is achieved in the definition; that is, when $n = 2^\alpha + \alpha - 1$.

Let $n_1, n_2 \in \mathbb{Z}^+$ be minimal such that $c(n_1) = \alpha$ and $c(n_2) = \alpha + 1$. It follows that:

$$\begin{aligned} n_2 - n_1 &= (2^{\alpha+1} + (\alpha + 1) - 1) - (2^\alpha + \alpha - 1) \\ &= 2^{\alpha+1} - 2^\alpha + 1 \\ &= 2^\alpha(2-1) + 1 \\ &= 2^\alpha + 1. \end{aligned}$$

Since n_1, n_2 were chosen minimally, it follows that there are exactly $2^\alpha + 1$ positive integers satisfying $c(n) = \alpha$. □

3. Calculations

In this section, we provide calculations to better illustrate the consequences of our results. Recalling the previous discussion on occurring graphs, we again ask: how many possible disconnected graphs of order n could occur as $\Delta(G)$ when n gets large? While at first glance, the rough count of Lemma **??** seems daunting; calculating $c(n)$ shows that the number of possibilities is surprisingly few, as illustrated in Table 1:

If one wanted to attempt a classification of graphs with one hundred vertices, rather than investigating fifty disconnected graphs with complete components, the calculation $c(100) = 6$ yields that only six graphs actually require attention. For a more extreme example, note that when considering graphs with 10^{30} vertices, Lemma 2.1 provides 5×10^{29} possible pairs of component sizes, yet, computing $c(n)$ reveals that the number of possible component size pairs that can occur as $\Delta(G)$ is still fewer than one hundred.

Table 1. Values of $c(n)$ for increasingly large graph orders

n	$c(n)$
10	3
100	6
$1,000$	9
$10,000$	13
$100,000$	16
$1,000,000$	19

n	$c(n)$
10^9	29
10^{10}	33
10^{15}	49
10^{20}	66
10^{25}	83
10^{30}	99

Additionally, for a fixed number of possible disconnected character degree graphs, our main result yields the possible orders of graph that must be considered, as shown in Table 2.

Table 2. Number of possibilities for $\Delta(G)$ and corresponding graph orders

Number of possibilities for disconnected $\Delta(G)$	Possible graph orders
1	$2-4$
2	$5-9$
3	$10-18$
4	$19-35$
5	$36-68$
6	$69-133$
7	$134-262$
8	$263-519$
9	$520-1,032$
10	$1,033-2,057$

For example, when considering which graphs of order between 36 and 68 can occur as $\Delta(G)$, there will be exactly 5 disconnected possibilities to check. Moreover, one would not have to check more

https://dx.doi.org/10.22108/ijgt.2026.145369.1964

than 10 possible disconnected graphs unless the character degree graphs being considered had at least 2,058 vertices.

4. **Future Directions**

As of the time of writing, it is still unknown whether all pairs of component sizes that satisfy Pálfy's inequality actually occur as $\Delta(G)$ for some finite solvable group G. That is, Pálfy's inequality is a necessary condition for a disconnected graph to occur, but it is unknown whether or not it is also sufficient. As mentioned previously, investigating this question deviates into much more number-theoretic territory, which we discuss in more detail now.

Currently, the most common method for constructing examples of solvable groups whose character degree graphs are disconnected was introduced by the first author in [4]. One begins with the finite field $\mathbb{F}$ of order p^e, where p is a prime and e is a positive integer. Let $\mathbb{F}_p$ denote the base field with p elements. Next, consider the full multiplicative group $\mathbb{F}^\times$ acting on $\mathbb{F}$ in the natural way, followed by the full Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$. Provided a few divisibility conditions are met (outlined in [5, example 2.4]), the resulting affine semi-linear group G has a character degree set consisting of all positive divisors of e, as well as the value $|\mathbb{F}^\times| = p^e - 1$. That is, $\mathrm{cd}(G) = \{p^e - 1\} \cup \{d : d \mid e\}$. Thus, by choosing a value of e with α distinct prime divisors such that $p^e - 1$ has β distinct prime divisors (different from those in α), one can construct a disconnected character degree graph with prescribed component sizes α and β.

At present, we are not aware of any component size pairs which satisfy Pálfy's inequality but do not occur. In fact, all disconnected graphs of order less than nine whose component size pairs satisfy Pálfy's inequality have been shown to occur via the above construction (see [4] and [1] for the examples of orders five and six respectively). While we expect that all such pairs do occur, in general, it has not been proven that an appropriate prime p and exponent e always exist for any desired combination of sizes α and β, with the exception of $\alpha = 1$ and $\beta = N$ with $N \in \mathbb{Z}^+$ (see the extra-special p group construction on [3, page 182]). If this could be shown, then a disconnected graph with component sizes α and β would occur if and only if α and β satisfy Pálfy's inequality. It follows that the $c(n)$ function would yield the exact number of occurring disconnected character degree graphs of order n, rendering the disconnected case trivial in the classification of character degree graphs of any order. Additionally, the structure of solvable groups whose character degree graphs have two connected components was completely classified by the first author in [5]. One wonders if these other possible group structures lend themselves to showing the sufficiency of Pálfy's condition.

Acknowledgments

The authors wish to thank the referees for their time, careful reading, and helpful suggestions which resulted in an improved paper.

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