



<https://toc.ui.ac.ir>

Transactions on Combinatorics

ISSN (print): 2251-8657, ISSN (on-line): 2251-8665

Vol. xx No. x (202x), pp. xx-xx.

© 202x University of Isfahan



<https://www.ui.ac.ir>

A UNIFIED APPROACH TO TOTAL VERTEX IRREGULARITY STRENGTH

ALEAMS BARRA[✉]

ABSTRACT. We develop a unified framework for computing the total vertex irregularity strength (tvs) of diverse graph classes, significantly extending our prior work in Barra et al. This comprehensive approach generalizes and enhances earlier methods, offering a robust technique applicable to a wide range of graph structures. A notable achievement is resolving an open problem by determining the tvs for simple 2-regular graphs, confirming the conjecture by Ahmad et al. Furthermore, our framework provides unified proofs for the tvs of cycles, paths, prisms, wheels, helm graphs, and friendship graphs. Importantly, the versatility of this method suggests its potential utility in computing tvs for additional graph classes, broadening its impact in graph theory and computational applications.

1. Introduction

A network can be modeled as a graph, where multiple connections between vertices are represented by multiple edges. To represent the number of edges between vertices, Chartrand et al. [6] proposed replacing multiple edges with a single edge labeled with a positive integer.

The *vertex sum* of a vertex in such a graph is defined as the sum of the labels on all edges incident to that vertex. Chartrand et al. further introduced the concept of an *irregular network*, characterized by a graph in which all vertices have distinct vertex sums.

This framework defines the *irregularity strength* of a graph G , denoted $s(G)$, as the smallest positive integer k for which there exists an edge labeling of $E(G)$ with $\{1, 2, \dots, k\}$ ensuring distinct vertex sums.

MSC(2010): Primary: 05C78; Secondary: 05C15, 05C85.

Keywords: total vertex irregularity strength, graph labeling, irregular labeling, edge labeling, $\{1, s\}$ -labeling.

Manuscript Type: Research Paper.

Communicated by Ebrahim Ghorbani.

Received: 6 May 2025, Accepted: 07 September 2026, Published Online: 07 October 2026.

Cite this article: A. Barra, A unified approach to total vertex irregularity strength, *Trans. Comb.*, x no. x (202x) xx-xx.

<https://dx.doi.org/10.22108/toc.2026.145174.2276>.

For d -regular graphs of order n , one trivially has $s(G) \geq n/d$. Faudree and Lehel [7] conjectured the existence of an absolute constant c such that $s(G) \leq n/d + c$. A strong asymptotic form proved by Przybyło and Wei [12] states that for any fixed $\varepsilon \in (0, 1/4)$, there exist constants c_1, c_2 such that for graphs on n vertices with minimum degree $\delta > 1$ and no isolated edge,

$$s(G) \leq \frac{n}{\delta} \left(1 + \frac{c_1}{\delta^\varepsilon}\right) + c_2.$$

This result refines earlier bounds, including the general estimate of Kalkowski et al. [8] and dense-graph bounds of Majerski and Przybyło [10]; a short follow-up [13] further provides explicit additive constants in regular-graph regimes.

Baca et al. [4] extended the concept of irregularity strength $s(G)$ to its *total-labeling version*, where both vertices and edges of G are assigned positive integers. The *vertex sum* of a vertex $v \in V(G)$ is given by

$$\omega_E(v) := \sum_{vw \in E(G)} \lambda(vw),$$

where the sum includes all edges incident to v . The *weight* of v is then defined as

$$\text{wt}(v) = \omega_E(v) + \lambda(v).$$

A total labeling λ is deemed k -irregular if $\text{range}(\lambda) \subseteq \{1, 2, \dots, k\}$ and all vertices in G have distinct weights. The smallest positive integer k for which λ is k -irregular is termed the *total vertex irregularity strength* of G , denoted $\text{tvs}(G)$.

Exact values of $\text{tvs}(G)$ have been established for several graph classes, including cycles, paths, prisms, trees, wheels, helm graphs, friendship graphs, complete graphs, and complete bipartite graphs (see [4, 11, 14, 1, 15]).

For general graphs, a widely used upper bound due to Anholcer et al. [3] states that for every graph G of order n and minimum degree $\delta > 0$,

$$\text{tvs}(G) \leq 3 \left\lceil \frac{n}{\delta} \right\rceil + 1.$$

For dense graphs, Majerski and Przybyło obtained further asymptotic improvements [9].

Despite these developments, the standard approach to determining $\text{tvs}(G)$ still consists of two separate steps: establishing a lower bound and then proving its attainability via an explicit construction. While the lower-bound arguments are relatively consistent, the constructive part is often graph-specific, and no unified construction strategy has proved effective across diverse families. In many cases, the labeling itself is intricate and requires careful, lengthy verification.

Motivated by this gap, we introduce a construction technique for total vertex irregular labelings based on a shared underlying principle that applies to a broad range of graph classes.

2. Preliminaries

The concept of $\{1, s\}$ -edge labeling was introduced by Barra et al. [5] to determine the total vertex irregularity strength of cubic graphs with a perfect matching. Earlier work, including Baca et al. [4], also used two-value edge-label patterns (notably $\{1, 2\}$) in specific constructions; our approach treats this idea systematically through a variable set $\{1, s\}$. A $\{1, s\}$ -edge labeling on a graph G is defined as a function

$$\lambda : E(G) \rightarrow \{1, s\}.$$

Using this labeling, the vertex sums of all vertices in G can be computed. This labeling can be extended to an s -irregular labeling by suitably defining

$$\lambda : V(G) \rightarrow \{1, 2, \dots, s\}$$

such that, for any distinct vertices $v, w \in V(G)$, the weights satisfy $\text{wt}(v) \neq \text{wt}(w)$.

For a given $\{1, s\}$ -edge labeling λ and a subset $U \subseteq V(G)$, we denote by $\mathbf{U}$ the vector of length $|U|$, listing the vertices in U from left to right in non-decreasing order of their vertex sums.

For any positive integer a , the boldface notation $\mathbf{a}$ represents a constant vector with all entries equal to a , where the length of the vector is implied by the context. Likewise, for positive integers a and b with $a \leq b$, the notation $[\mathbf{a}, \mathbf{b}]$ denotes the vector

$$[\mathbf{a}, \mathbf{b}] = \langle a, a + 1, \dots, b \rangle.$$

Given a vector $\mathbf{U} = \langle u_1, u_2, \dots, u_m \rangle$, we define

$$\begin{aligned} \omega_E(\mathbf{U}) &= \langle \omega_E(u_1), \omega_E(u_2), \dots, \omega_E(u_m) \rangle, \\ \lambda(\mathbf{U}) &= \langle \lambda(u_1), \lambda(u_2), \dots, \lambda(u_m) \rangle, \quad \text{and} \\ \text{wt}(\mathbf{U}) &= \langle \text{wt}(u_1), \text{wt}(u_2), \dots, \text{wt}(u_m) \rangle. \end{aligned}$$

We frequently rely on the following lower bound, established in [4].

Theorem 2.1. *Let G be a graph with n vertices, where δ and Δ denote the minimum and maximum degrees of its vertices, respectively. Then*

$$(2.1) \quad \text{tvs}(G) \geq \left\lceil \frac{n + \delta}{\Delta + 1} \right\rceil.$$

3. Main Results

In this section, we utilize the $\{1, s\}$ -edge labeling method to compute several established results regarding the total vertex irregularity strength of various graphs. Our focus is on establishing a unified approach to determine the tvs values of these graphs that simplifies their construction and enhances the transparency of verifying their correctness. While verifying the correctness of constructions in earlier

studies can be intricate, our method ensures that the validation process is both straightforward and transparent.

As a culmination of this approach, we apply it at the end of this section to settle an open problem concerning the total vertex irregularity strength of simple 2-regular graphs.

3.1. Cycles and Paths. The construction for cycles presented here will later be adapted to determine the total vertex irregularity strength of simple 2-regular graphs.

Theorem 3.1. *Let C_n be the cycle graph with n vertices. Then*

$$\text{tvs}(C_n) = \lceil (n+2)/3 \rceil.$$

Proof. In C_n , both the minimum degree δ and maximum degree Δ are 2. Applying the lower bound from (2.1), we obtain

$$\text{tvs}(C_n) \geq \left\lceil \frac{n+\delta}{\Delta+1} \right\rceil = \left\lceil \frac{n+2}{3} \right\rceil.$$

Let $n = a + 2b + c$, where $a, c \geq 1$ and $b \geq 0$. From an arbitrary starting edge, assign 1 to the first a consecutive edges of the cycle, label the next $2b$ edges alternately with s and 1, and mark the remaining c edges with s .

This labeling yields distinct vertex sums: inner vertices of the 1-labeled segment have a vertex sum of 2 (being incident to two edges labeled 1), inner vertices of the alternating $(s, 1)$ -labeled segment have a vertex sum of $s + 1$, and inner vertices of the s -labeled segment have a vertex sum of $2s$. Vertices at boundaries between segments also have a vertex sum of $s + 1$.

Define V_2 , V_{s+1} , and V_{2s} as the sets of vertices with vertex sums 2, $s + 1$, and $2s$, respectively. The distribution of vertex sums under λ is given by

$$D_\lambda(G) = (|V_2|, |V_{s+1}|, |V_{2s}|).$$

From our construction, this distribution is

$$(3.1) \quad D_\lambda = (a - 1, 2b + 2, c - 1).$$

Set $s = \lceil (n+2)/3 \rceil$. We proceed with two cases based on the parity of s :

Case 1: s is even. Choose a and b such that $a - 1 = s - 1$ (i.e., $a = s$) and $2b + 2 = s$ (i.e., $b = (s - 2)/2$). Then $c = n - a - 2b = n + 2 - 2s$, yielding

$$D_\lambda = (s - 1, s, n + 1 - 2s).$$

Extend λ to $V(C_n)$ by defining

$$\lambda(\mathbf{V}_2) = [\mathbf{1}, \mathbf{s} - \mathbf{1}], \quad \lambda(\mathbf{V}_{s+1}) = [\mathbf{1}, \mathbf{s}], \quad \text{and} \quad \lambda(\mathbf{V}_{2s}) = [\mathbf{2}, \mathbf{n} + \mathbf{2} - \mathbf{2s}].$$

The weights are then

$$\begin{aligned}\text{wt}(\mathbf{V}_2) &= \mathbf{2} + [\mathbf{1}, \mathbf{s} - \mathbf{1}] = [\mathbf{3}, \mathbf{s} + \mathbf{1}], \\ \text{wt}(\mathbf{V}_{\mathbf{s}+1}) &= \mathbf{s} + \mathbf{1} + [\mathbf{1}, \mathbf{s}] = [\mathbf{s} + \mathbf{2}, \mathbf{2s} + \mathbf{1}], \\ \text{wt}(\mathbf{V}_{2\mathbf{s}}) &= \mathbf{2s} + [\mathbf{2}, \mathbf{n} + \mathbf{2} - \mathbf{2s}] = [\mathbf{2s} + \mathbf{2}, \mathbf{n} + \mathbf{2}].\end{aligned}$$

These intervals are clearly pairwise disjoint. Moreover, since $s = \lceil \frac{n+2}{3} \rceil$, we have $n + 2 - 2s \leq s$. Hence all vertex labels are at most s , and the resulting weights are distinct.

Case 2: s is odd. Choose a and b such that $a - 1 = s$ (i.e., $a = s + 1$) and $2b + 2 = s - 1$ (i.e., $b = (s - 3)/2$), so $c = n + 2 - 2s$. This gives

$$D_\lambda = (s, s - 1, n + 1 - 2s).$$

Define

$$\lambda(\mathbf{V}_2) = [\mathbf{1}, \mathbf{s}], \quad \lambda(\mathbf{V}_{\mathbf{s}+1}) = [\mathbf{2}, \mathbf{s}], \quad \text{and} \quad \lambda(\mathbf{V}_{2\mathbf{s}}) = [\mathbf{2}, \mathbf{n} + \mathbf{2} - \mathbf{2s}].$$

The weights become

$$\begin{aligned}\text{wt}(\mathbf{V}_2) &= \mathbf{2} + [\mathbf{1}, \mathbf{s}] = [\mathbf{3}, \mathbf{s} + \mathbf{2}], \\ \text{wt}(\mathbf{V}_{\mathbf{s}+1}) &= \mathbf{s} + \mathbf{1} + [\mathbf{2}, \mathbf{s}] = [\mathbf{s} + \mathbf{3}, \mathbf{2s} + \mathbf{1}], \\ \text{wt}(\mathbf{V}_{2\mathbf{s}}) &= \mathbf{2s} + [\mathbf{2}, \mathbf{n} + \mathbf{2} - \mathbf{2s}] = [\mathbf{2s} + \mathbf{2}, \mathbf{n} + \mathbf{2}].\end{aligned}$$

Since $n + 2 - 2s \leq s$ holds as before, all vertex labels are at most s . The weight intervals are disjoint, ensuring all vertices have distinct weights.

In both cases, λ is s -irregular with $s = \lceil (n + 2)/3 \rceil$, matching the lower bound. Thus, $\text{tvs}(C_n) = \lceil (n + 2)/3 \rceil$. $\square$

Having established the result for cycles, we now apply a similar approach to compute the total vertex irregularity strength of paths

Theorem 3.2. *Let P_n be the path with $n \geq 3$ vertices. Then*

$$\text{tvs}(P_n) = \left\lceil \frac{n + 1}{3} \right\rceil.$$

Proof. For P_n , the minimum degree $\delta = 1$ (at one endpoint) and maximum degree $\Delta = 2$ (at inner vertices). Applying the lower bound from (2.1), we get

$$\text{tvs}(P_n) \geq \left\lceil \frac{n + \delta}{\Delta + 1} \right\rceil = \left\lceil \frac{n + 1}{3} \right\rceil.$$

To prove this bound is achievable, we construct a total labeling that ensures all vertex weights are distinct while matching the bound. Since P_n has $n - 1$ edges, let $n - 1 = a + 2b + c$. Starting from the leftmost edge, define a $\{1, s\}$ -edge labeling λ : assign 1 to the first a consecutive edges, label the next $2b$ edges alternately with s and 1, and label the remaining c edges with s . We require $a, c \geq 1$, while $b \geq 0$.

The labeling yields vertex sums similar to the cycle case, with sums 1 at one endpoint, s at the other endpoint, 2 in the 1-labeled segment, $s + 1$ in the alternating and boundary vertices, and $2s$ in the s -labeled segment. Define V_2 as the set of vertices with vertex sum 1 or 2, V_{s+1} as those with sum s or $s + 1$, and V_{2s} as those with sum $2s$. The distribution is

$$D_\lambda = (|V_2|, |V_{s+1}|, |V_{2s}|) = (a, 2b + 2, c - 1).$$

Set $s = \lceil (n + 1)/3 \rceil$. We consider two cases based on the parity of s :

Case 1: s is even. Choose a and b such that $a = s - 1$ and $2b + 2 = s$. Then consequently, $c = n - 1 - a - 2b = n + 2 - 2s$, so

$$D_\lambda = (s - 1, s, n + 1 - 2s).$$

Extend λ to $V(P_n)$ as follows:

$$\lambda(\mathbf{V}_2) = \langle 1, [\mathbf{1}, \mathbf{s} - \mathbf{2}] \rangle, \quad \lambda(\mathbf{V}_{s+1}) = \langle 1, [\mathbf{1}, \mathbf{s} - \mathbf{1}] \rangle, \quad \lambda(\mathbf{V}_{2s}) = [\mathbf{1}, \mathbf{n} + \mathbf{1} - \mathbf{2s}],$$

yielding weights

$$\begin{aligned} \text{wt}(\mathbf{V}_2) &= \langle 1, \mathbf{2} \rangle + \langle 1, [\mathbf{1}, \mathbf{s} - \mathbf{2}] \rangle = [\mathbf{2}, \mathbf{s}], \\ \text{wt}(\mathbf{V}_{s+1}) &= \langle s, \mathbf{s} + \mathbf{1} \rangle + \langle 1, [\mathbf{1}, \mathbf{s} - \mathbf{1}] \rangle = [\mathbf{s} + \mathbf{1}, \mathbf{2s}], \\ \text{wt}(\mathbf{V}_{2s}) &= \mathbf{2s} + [\mathbf{1}, \mathbf{n} + \mathbf{1} - \mathbf{2s}] = [\mathbf{2s} + \mathbf{1}, \mathbf{n} + \mathbf{1}]. \end{aligned}$$

Since $s = \lceil (n + 1)/3 \rceil$, we have $n + 1 \leq 3s$, so $n + 1 - 2s \leq s$, ensuring all labels are at most s . The weight intervals $[\mathbf{2}, \mathbf{s}]$, $[\mathbf{s} + \mathbf{1}, \mathbf{2s}]$, and $[\mathbf{2s} + \mathbf{1}, \mathbf{n} + \mathbf{1}]$ are disjoint and cover all n vertices distinctly.

Case 2: s is odd. Choose a and b such that $a = s$ and $2b + 2 = s - 1$. Then consequently, $c = n + 2 - 2s$, so

$$D_\lambda = (s, s - 1, n + 1 - 2s).$$

Define

$$\lambda(\mathbf{V}_2) = \langle 1, [\mathbf{1}, \mathbf{s} - \mathbf{1}] \rangle, \quad \lambda(\mathbf{V}_{s+1}) = \langle 2, [\mathbf{2}, \mathbf{s} - \mathbf{1}] \rangle, \quad \lambda(\mathbf{V}_{2s}) = [\mathbf{1}, \mathbf{n} + \mathbf{1} - \mathbf{2s}],$$

yielding

$$\begin{aligned} \text{wt}(\mathbf{V}_2) &= \langle 1, \mathbf{2} \rangle + \langle 1, [\mathbf{1}, \mathbf{s} - \mathbf{1}] \rangle = [\mathbf{2}, \mathbf{s} + \mathbf{1}], \\ \text{wt}(\mathbf{V}_{s+1}) &= \langle s, \mathbf{s} + \mathbf{1} \rangle + \langle 2, [\mathbf{2}, \mathbf{s} - \mathbf{1}] \rangle = [\mathbf{s} + \mathbf{2}, \mathbf{2s}], \\ \text{wt}(\mathbf{V}_{2s}) &= \mathbf{2s} + [\mathbf{1}, \mathbf{n} + \mathbf{1} - \mathbf{2s}] = [\mathbf{2s} + \mathbf{1}, \mathbf{n} + \mathbf{1}]. \end{aligned}$$

Again, $n + 1 - 2s \leq s$ holds, and the weight intervals $[\mathbf{2}, \mathbf{s} + \mathbf{1}]$, $[\mathbf{s} + \mathbf{2}, \mathbf{2s}]$, and $[\mathbf{2s} + \mathbf{1}, \mathbf{n} + \mathbf{1}]$ are disjoint and cover all n vertices distinctly.

In both cases, λ is s -irregular with $s = \lceil (n + 1)/3 \rceil$, matching the lower bound. Thus, $\text{tvs}(P_n) = \lceil (n + 1)/3 \rceil$. $\square$

3.2. Prisms and Wheels. The *prism graph* D_n consists of two disjoint n -cycles with corresponding vertices joined by a perfect matching. We denote the cycles by $v_1v_2 \cdots v_nv_1$ and $w_1w_2 \cdots w_nw_1$, and their matching edges by v_iw_i , where $1 \leq i \leq n$. The *wheel graph* W_n is formed from the cycle $v_1v_2 \cdots v_nv_1$ by adding a center vertex x adjacent to every cycle vertex. Each edge xv_i is called a *spoke* of the wheel.

The total vertex irregularity strengths of prisms and wheels were first computed by Baca et al. [4] and Wijaya et al. [14], respectively. For wheel graphs, our construction simplifies the original approach and enhances the transparency of its verification.

Theorem 3.3. *Let D_n be the prism graph with n vertices in each cycle. Then*

$$\text{tvs}(D_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Proof. The prism D_n has $2n$ vertices, all of degree $\delta = \Delta = 3$. Applying the lower bound from (2.1), we obtain

$$\text{tvs}(D_n) \geq \left\lceil \frac{2n + \delta}{\Delta + 1} \right\rceil = \left\lceil \frac{2n + 3}{4} \right\rceil = \left\lceil \frac{n}{2} \right\rceil + 1.$$

To achieve this bound, we employ our $\{1, s\}$ -labeling on D_n . Set $s = \left\lceil \frac{n}{2} \right\rceil + 1$. Label all edges of the cycle $v_1v_2 \cdots v_nv_1$ with 1 and all edges of the cycle $w_1w_2 \cdots w_nw_1$ with s . For the cross edges v_iw_i , label $s - 1$ of them with 1 and the remaining $n + 1 - s$ with s .

This labeling produces vertex sums of 3 or $s + 2$ for vertices v_i and $2s + 1$ or $3s$ for vertices w_i , based on whether the cross edge v_iw_i is labeled 1 or s . Define V_3 , V_{s+2} , V_{2s+1} , and V_{3s} as the sets of vertices with sums 3, $s + 2$, $2s + 1$, and $3s$, respectively. The distribution is

$$D_\lambda = (|V_3|, |V_{s+2}|, |V_{2s+1}|, |V_{3s}|) = (s - 1, n + 1 - s, s - 1, n + 1 - s).$$

Extend λ to $V(D_n)$ by defining

$$\lambda(\mathbf{V}_3) = \lambda(\mathbf{V}_{2s+1}) = [\mathbf{1}, \mathbf{s} - \mathbf{1}], \quad \lambda(\mathbf{V}_{s+2}) = \lambda(\mathbf{V}_{3s}) = [\mathbf{1}, \mathbf{n} + \mathbf{1} - \mathbf{s}],$$

yielding weights

$$\begin{aligned} \text{wt}(\mathbf{V}_3) &= \mathbf{3} + [\mathbf{1}, \mathbf{s} - \mathbf{1}] = [\mathbf{4}, \mathbf{s} + \mathbf{2}], \\ \text{wt}(\mathbf{V}_{s+2}) &= \mathbf{s} + \mathbf{2} + [\mathbf{1}, \mathbf{n} + \mathbf{1} - \mathbf{s}] = [\mathbf{s} + \mathbf{3}, \mathbf{n} + \mathbf{3}], \\ \text{wt}(\mathbf{V}_{2s+1}) &= \mathbf{2s} + \mathbf{1} + [\mathbf{1}, \mathbf{s} - \mathbf{1}] = [\mathbf{2s} + \mathbf{2}, \mathbf{3s}], \\ \text{wt}(\mathbf{V}_{3s}) &= \mathbf{3s} + [\mathbf{1}, \mathbf{n} + \mathbf{1} - \mathbf{s}] = [\mathbf{3s} + \mathbf{1}, \mathbf{n} + \mathbf{2s} + \mathbf{1}]. \end{aligned}$$

Since $s = \left\lceil n/2 \right\rceil + 1 \geq (n/2) + 1$, we have $n + 1 - s \leq s - 1$ and $n + 3 \leq 2s + 1$, ensuring all labels are at most s and the intervals $[\mathbf{4}, \mathbf{s} + \mathbf{2}]$, $[\mathbf{s} + \mathbf{3}, \mathbf{n} + \mathbf{3}]$, $[\mathbf{2s} + \mathbf{2}, \mathbf{3s}]$, and $[\mathbf{3s} + \mathbf{1}, \mathbf{n} + \mathbf{2s} + \mathbf{1}]$ are disjoint, covering $2n$ distinct weights.

Thus, λ is s -irregular with $s = \left\lceil n/2 \right\rceil + 1$, matching the bound, so $\text{tvs}(D_n) = \left\lceil n/2 \right\rceil + 1$. $\square$

Theorem 3.4. *Let W_n be the wheel graph with $n + 1$ vertices, consisting of an n -cycle and a center vertex, for $n \geq 3$. Then*

$$\text{tvs}(W_n) = \lceil (n + 3)/4 \rceil.$$

Proof. First, we establish a lower bound. For a cycle vertex v (degree 3), the smallest possible weight is 4 when all incident edges and v itself are labeled 1, while the largest possible weight is $4k$ if $k = \text{tvs}(W_n)$, giving $4k - 3$ possible values for cycle-vertex weights. Since there are n cycle vertices, we need

$$4k - 3 \geq n,$$

so

$$k \geq \lceil (n + 3)/4 \rceil.$$

Thus, $\text{tvs}(W_n) \geq \lceil (n + 3)/4 \rceil$.

To attain this bound, we perform a $\{1, s\}$ -edge labeling on W_n . We label all edges in W_n (cycle edges and spokes) with 1 or s . Excluding spoke labels, cycle vertices have partial vertex sums 2, $s + 1$, or $2s$; denote the corresponding sets by U_2 , U_{s+1} , and U_{2s} . Applying the cycle construction together with the general distribution formula (3.1), and choosing $a = n + 3 - 3s$, $b = s - 2$, and $c = s + 1$, we obtain

$$|U_2| = n + 2 - 3s, \quad |U_{s+1}| = 2s - 2, \quad |U_{2s}| = s.$$

Extend λ by labeling the spokes as follows: all edges from vertices in U_2 to the center are labeled 1, all edges from vertices in U_{2s} to the center are labeled s , and among the vertices in U_{s+1} , exactly $s - 1$ spokes are labeled 1 while the remaining $s - 1$ spokes are labeled s .

Then each vertex in U_2 has full vertex sum $2 + 1 = 3$. Since $|U_2| = n + 2 - 3s$, this gives $|V_3| = n + 2 - 3s$. Similarly, the $s - 1$ vertices of U_{s+1} whose spokes are labeled 1 have full vertex sum $s + 2$, so $|V_{s+2}| = s - 1$, while the remaining $s - 1$ vertices of U_{s+1} have full vertex sum $2s + 1$, so $|V_{2s+1}| = s - 1$. Finally, each vertex in U_{2s} has full vertex sum $2s + s = 3s$, and therefore $|V_{3s}| = |U_{2s}| = s$. Hence the distribution of full vertex sums is

$$(|V_3|, |V_{s+2}|, |V_{2s+1}|, |V_{3s}|) = (n + 2 - 3s, s - 1, s - 1, s).$$

Label the cycle vertices:

$$\lambda(\mathbf{V}_3) = [\mathbf{1}, \mathbf{n} + \mathbf{2} - \mathbf{3s}], \quad \lambda(\mathbf{V}_{s+2}) = [\mathbf{1}, \mathbf{s} - \mathbf{1}],$$

$$\lambda(\mathbf{V}_{3s}) = [\mathbf{1}, \mathbf{s}], \quad \lambda(\mathbf{V}_{2s+1}) = [\mathbf{1}, \mathbf{s} - \mathbf{1}].$$

The weights are

$$\text{wt}(\mathbf{V}_3) = \mathbf{3} + [\mathbf{1}, \mathbf{n} + \mathbf{2} - \mathbf{3s}] = [\mathbf{4}, \mathbf{n} + \mathbf{5} - \mathbf{3s}],$$

$$\text{wt}(\mathbf{V}_{s+2}) = \mathbf{s} + \mathbf{2} + [\mathbf{1}, \mathbf{s} - \mathbf{1}] = [\mathbf{s} + \mathbf{3}, \mathbf{2s} + \mathbf{1}],$$

$$\text{wt}(\mathbf{V}_{2s+1}) = \mathbf{2s} + \mathbf{1} + [\mathbf{1}, \mathbf{s} - \mathbf{1}] = [\mathbf{2s} + \mathbf{2}, \mathbf{3s}],$$

$$\text{wt}(\mathbf{V}_{3s}) = \mathbf{3s} + [\mathbf{1}, \mathbf{s}] = [\mathbf{3s} + \mathbf{1}, \mathbf{4s}].$$

Set $s = \lceil (n+3)/4 \rceil$. Then

$$4s \geq n+3 > n+2 \Rightarrow n+5-3s < s+3.$$

It follows that the weights of all cycle vertices are distinct.

For the center vertex x , the number of spokes labeled s is $2s-1$, while the remaining $n-2s+1$ spokes are labeled 1. If we set $\lambda(x) = s$, then

$$\text{wt}(x) = s(2s-1) + 1(n-2s+1) + s = 2s^2 - 2s + n + 1.$$

For $n \geq 4$, we have $s \geq 2$, and

$$\text{wt}(x) - 4s = 2s^2 - 6s + n + 1 \geq 2s^2 - 6s + 5 > 2s^2 - 6s + 4 = 2(s-1)(s-2) \geq 0.$$

Hence $\text{wt}(x) > 4s$, so $\text{wt}(x)$ is distinct from all cycle-vertex weights.

The construction above is valid provided that

$$|U_2| = n+2-3s \geq 0.$$

For $s = \lceil (n+3)/4 \rceil$, this condition holds for every $n \geq 4$ except $n = 6$. We handle this exceptional case directly. Let $v_1v_2 \cdots v_6v_1$ be the outer cycle of W_6 , let x be its center, and take $s = 3$. With subscripts on the cycle interpreted modulo 6, define

$$\lambda(v_iv_{i+1}) = \begin{cases} 1, & i = 2, \\ 3, & i \neq 2, \end{cases} \quad \lambda(xv_i) = \begin{cases} 1, & 1 \leq i \leq 4, \\ 3, & i \in \{5, 6\}. \end{cases}$$

Assign the vertex labels by

$$\lambda(v_1) = \lambda(v_2) = \lambda(v_5) = 1,$$

$$\lambda(v_3) = \lambda(v_4) = \lambda(v_6) = \lambda(x) = 2.$$

The resulting weights, in the order $v_1, \dots, v_6, x$, are

$$(8, 6, 7, 9, 10, 11, 12),$$

which are pairwise distinct. Hence $\text{tvs}(W_6) = 3$.

For $n = 3$, $W_3 \cong K_4$, and this case was already settled by Baca et al. [4]. □

3.3. Helm and Friendship Graphs. The *helm graph* H_n is obtained from the *wheel graph* W_n by attaching a pendant edge to each vertex of the n -cycle in W_n , forming n pendant vertices and resulting in $2n+1$ vertices (n pendant vertices plus the $n+1$ vertices of W_n) and $3n$ edges. The *friendship graph* F_n is constructed by taking n copies of the triangle K_3 and identifying one vertex from each triangle into a single common vertex, called the *center*, such that each triangle shares exactly this center vertex, yielding $2n+1$ vertices and $3n$ edges.

The total vertex irregularity strengths of helm graphs and friendship graphs were first computed by Ahmad et al. [1] and Wijaya et al. [14], respectively.

Theorem 3.5. *Let H_n be the helm graph with $2n + 1$ vertices, consisting of a center vertex, an n -cycle, and n pendant vertices attached to the cycle vertices. Then*

$$\text{tvs}(H_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. First, we establish a lower bound. Suppose $\text{tvs}(H_n) = k$. Consider the n pendant vertices of H_n , each of degree 1. The smallest possible weight for a pendant vertex is 2 (when its incident edge and vertex are labeled 1), and the largest is $2k$ (when labeled k). Since all n pendant vertices must have distinct weights, ranging from 2 to $2k$, we need

$$n \leq 2k - 1,$$

so

$$k \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

Thus, $\text{tvs}(H_n) \geq \left\lceil \frac{n+1}{2} \right\rceil$.

To achieve this value, let $s = \left\lceil \frac{n+1}{2} \right\rceil$. Define a $\{1, s\}$ -edge labeling on H_n by labeling every spoke and exactly s pendant edges with 1, and labeling all remaining edges, namely the other $n - s$ pendant edges and all n cycle edges, with s .

Then each pendant vertex has vertex sum 1 or s , so we obtain s vertices in V_1 and $n - s$ vertices in V_s . Every cycle vertex is incident with two cycle edges labeled s and one spoke labeled 1, so its partial sum from non-pendant edges equals $2s + 1$. Consequently, if its pendant edge is labeled 1, then its full vertex sum is $2s + 2$, whereas if its pendant edge is labeled s , then its full vertex sum is $3s + 1$. Hence

$$D_\lambda = (|V_1|, |V_s|, |V_{2s+2}|, |V_{3s+1}|) = (s, n - s, s, n - s),$$

summing to $2n$, the number of non-central vertices in H_n . Extend λ to the non-central vertices by

$$\lambda(\mathbf{V}_1) = \lambda(\mathbf{V}_{2s+2}) = [\mathbf{1}, \mathbf{s}], \quad \lambda(\mathbf{V}_s) = \lambda(\mathbf{V}_{3s+1}) = [\mathbf{2}, \mathbf{n} + \mathbf{1} - \mathbf{s}].$$

By the definition of s , all these labels are at most s .

The weights of non-central vertices are

$$\begin{aligned} \text{wt}(\mathbf{V}_1) &= \mathbf{1} + [\mathbf{1}, \mathbf{s}] = [\mathbf{2}, \mathbf{s} + \mathbf{1}], \\ \text{wt}(\mathbf{V}_s) &= \mathbf{s} + [\mathbf{2}, \mathbf{n} + \mathbf{1} - \mathbf{s}] = [\mathbf{s} + \mathbf{2}, \mathbf{n} + \mathbf{1}], \\ \text{wt}(\mathbf{V}_{2s+2}) &= \mathbf{2s} + \mathbf{2} + [\mathbf{1}, \mathbf{s}] = [\mathbf{2s} + \mathbf{3}, \mathbf{3s} + \mathbf{2}], \\ \text{wt}(\mathbf{V}_{3s+1}) &= \mathbf{3s} + \mathbf{1} + [\mathbf{2}, \mathbf{n} + \mathbf{1} - \mathbf{s}] = [\mathbf{3s} + \mathbf{3}, \mathbf{n} + \mathbf{2s} + \mathbf{2}]. \end{aligned}$$

By the definition of s , we have $n + 1 - s \leq s$, so all assigned vertex labels are at most s . Moreover, these intervals are disjoint since $n + 1 \leq 2s < 2s + 3$. Since every spoke is labeled 1, the center vertex x has $w_E(x) = n$. Thus assigning $\lambda(x) = 2$ yields $\text{wt}(x) = n + 2$, which is distinct from all other weights because $n + 1 < n + 2 < 2s + 3$. $\square$

Theorem 3.6. Let F_n be the friendship graph with $2n + 1$ vertices, consisting of a center vertex and n triangles sharing this center, for $n \geq 2$. Then

$$\text{tvs}(F_n) = \left\lceil \frac{2n+2}{3} \right\rceil.$$

Proof. Since F_1 is isomorphic to C_3 , with $\text{tvs}(C_3) = 2$ from Theorem 3.1, we assume $n \geq 2$.

First, we establish a lower bound. Suppose $\text{tvs}(F_n) = k$. Consider the $2n$ non-central vertices of F_n , each incident to two edges. The smallest possible weight for a non-central vertex is 3 and the largest is $3k$. Since all $2n$ non-central vertices must have distinct weights, ranging from 3 to $3k$, we need

$$2n \leq 3k - 2,$$

so

$$k \geq \left\lceil \frac{2n+2}{3} \right\rceil.$$

Thus, $\text{tvs}(F_n) \geq \left\lceil \frac{2n+2}{3} \right\rceil$.

To realize this value, we implement our $\{1, s\}$ -labeling on F_n , distributing vertex sums evenly across its structure. Let $s = \left\lceil \frac{2n+2}{3} \right\rceil$ and define a $\{1, s\}$ -edge labeling on F_n . We classify the edge labelings of the n triangles as type I, type II, or type III, corresponding to the first, second, and third triangles from left to right in the figure below. Here the red vertices represent the common center (depicted separately for clarity but forming a single vertex).

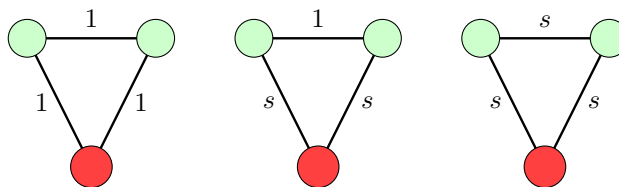


FIGURE 1. Type I, II and III edge labelings

Let V_2 , V_{s+1} , and V_{2s} be the sets of all non-central vertices of F_n with vertex sums 2, $s + 1$, and $2s$, respectively. Notice that labeling a single triangle using type I, type II, or type III contributes exactly 2 vertices to V_2 , V_{s+1} , and V_{2s} , respectively.

If $s = 2a$ is even, label a triangles and $a - 1$ triangles using type I and type II, respectively; then label the rest by using type III. It follows that $|V_2| = s$, $|V_{s+1}| = s - 2$, and $|V_{2s}| = 2n + 2 - 2s$. Then label

$$\lambda(\mathbf{V}_2) = [\mathbf{1}, s], \quad \lambda(\mathbf{V}_{s+1}) = [\mathbf{2}, s - 1], \quad \lambda(\mathbf{V}_{2s}) = [\mathbf{1}, 2n + 2 - 2s].$$

One can easily check that the weights of non-central vertices are all distinct in the range $[3, 2n + 2]$. Lastly, each label is at most s since

$$2n + 2 - 2s \leq s \iff 2n + 2 \leq 3s,$$

which is true by the definition of s .

If s is odd, then $s - 1 = 2a$ for some a . Label a triangles using type I and a triangles using type II; then label the rest using type III. It follows that

$$|V_2| = s - 1, \quad |V_{s+1}| = s - 1, \quad \text{and} \quad |V_{2s}| = 2n + 2 - 2s.$$

Then, similar to the case when s is even, the labeling

$$\lambda(\mathbf{V}_2) = [\mathbf{1}, \mathbf{s} - \mathbf{1}], \quad \lambda(\mathbf{V}_{s+1}) = [\mathbf{1}, \mathbf{s} - \mathbf{1}], \quad \lambda(\mathbf{V}_{2s}) = [\mathbf{1}, \mathbf{2n} + \mathbf{2} - \mathbf{2s}]$$

ensures that the weights of all non-central vertices are distinct, and each label is at most s .

Note that at least one triangle is labeled using either type II or type III. Otherwise, every triangle would be of type I, and hence all non-central vertices would belong to V_2 . However, this is not possible, since for $n \geq 2$ we have $|V_2| \leq s < 2n$.

Consequently, the center vertex is incident to at least two edges labeled s . Since F_n contains at least two triangles when $n \geq 2$, assigning the label s to the center vertex ensures that its weight is at least $3s + 2$. As the weight of any non-central vertex is at most $3s$, the weight of the center vertex is distinct from those of the non-central vertices. $\square$

4. Simple 2-Regular Graphs

The total vertex irregularity strength of simple 2-regular graphs was previously conjectured by Ahmad et al. [2]. They determined the total vertex irregularity strength for the disjoint union of cycles, provided that each cycle has a length of at least 5. However, they were unable to compute it for the general case of simple 2-regular graphs.

Theorem 4.1. *Let G be a simple 2-regular graph of order n . Then*

$$\text{tvs}(G) = \left\lceil \frac{n + 2}{3} \right\rceil.$$

Proof. The graph G is a disjoint union of cycles, $G = \bigcup_{i=1}^m C_i$, where each cycle has length at least 3. Since the case $m = 1$ corresponds to a single cycle (Theorem 3.1), we assume $m \geq 2$, which implies $n \geq 6$. Without loss of generality, assume the cycle lengths are ordered such that $n_1 \leq n_2 \leq \dots \leq n_m$, where $n_i = |V(C_i)|$.

Let $s = \lceil (n + 2)/3 \rceil$. We first consider the case where s is odd. We construct a $\{1, s\}$ -edge labeling such that each vertex belongs to exactly one of the partition sets V_2 , V_{s+1} , or V_{2s} , according to whether its vertex sum is 2, $s + 1$, or $2s$, respectively. We require these sets to have sizes

$$|V_2| = s, \quad |V_{s+1}| = s - 1, \quad |V_{2s}| = n + 1 - 2s.$$

Let $k \geq 0$ be the maximal integer such that $\sum_{i=1}^k n_i \leq s$, where the empty sum is interpreted as 0. We write

$$\sum_{i=1}^k n_i = s - a \quad \text{for some } a \geq 0.$$

We construct the edge labeling in stages to fix $|V_2|$ exactly at s .

Case 1: $n_{k+1} \geq a + 2$.

Label all edges of the first k cycles $C_1, \dots, C_k$ with 1. On C_{k+1} , label a path of $a + 1$ consecutive edges with 1. The remaining edges of C_{k+1} and all edges of subsequent cycles are initially left unlabeled. The cycles $C_1, \dots, C_k$ contribute exactly $\sum_{i=1}^k n_i = s - a$ vertices to V_2 . The path of $a + 1$ edges labeled 1 in C_{k+1} contributes exactly $(a + 1) - 1 = a$ internal vertices to V_2 . Thus, the total count is $|V_2| = (s - a) + a = s$.

Case 2: $n_{k+1} = a + 1$.

If we were to label all $a + 1$ edges of C_{k+1} with 1, the cycle would contribute $a + 1$ vertices to V_2 , yielding a total of $|V_2| = (s - a) + (a + 1) = s + 1$. Since replacing a label 1 with s alters the vertex sum of both endpoints, it reduces $|V_2|$ by exactly 2, making it impossible to achieve $|V_2| = s$ using C_{k+1} alone.

Instead, we label exactly one edge of C_{k+1} with s and the remaining $n_{k+1} - 1$ edges with 1. Then C_{k+1} contributes

$$n_{k+1} - 2 = a - 1$$

vertices of vertex sum 2, so the current total is

$$(s - a) + (a - 1) = s - 1.$$

The two endpoints of the edge labeled s in C_{k+1} now have vertex sum $s + 1$. To recover the missing vertex for V_2 , we use the next cycle, C_{k+2} . Note that

$$\sum_{i=1}^{k+1} n_i = s + 1 = \left\lceil \frac{n + 2}{3} \right\rceil + 1 < \frac{n + 2}{3} + 2 < n \quad \text{for } n \geq 6,$$

ensuring C_{k+2} exists. On C_{k+2} , we label exactly two consecutive edges with 1. This creates exactly one vertex of vertex sum 2, namely the vertex between the two edges labeled 1. Therefore

$$|V_2| = (s - 1) + 1 = s.$$

The remaining edges of C_{k+2} are left unlabeled for now, which completes Case 2.

Thus, in either case, we have fixed $|V_2| = s$. We now determine the labels for the remaining unlabeled edges so as to obtain $|V_{s+1}| = s - 1$.

We now complete the labeling of the current cycle (if it is only partially labeled) and label every subsequent fresh cycle using alternating patterns that produce an even number of vertices with vertex sum $s + 1$, while creating no new pair of consecutive edges labeled 1. We use the following patterns. On a fresh cycle, we use

$$s, 1, s, 1, \dots, s, 1$$

when the cycle length is even and

$$s, 1, s, 1, \dots, s, 1, s$$

when it is odd. On a cycle that already contains a block of edges labeled 1, denoted by $\mathbf{1}$, we use

$$s, \mathbf{1}, s, 1, \dots, s, 1 \quad \text{or} \quad s, \mathbf{1}, s, 1, \dots, s, 1, s,$$

choosing the pattern according to the parity of the remaining unlabeled edges. Among the $n - s$ vertices outside V_2 , at most one vertex in each remaining cycle has vertex sum $2s$. Hence the number of vertices with vertex sum $s + 1$ that is produced is at least

$$n - s - (m - k).$$

In any case, we will show that this quantity is at least $s - 1$. We now distinguish two subcases. If $k = 0$, then necessarily $n_1 \geq s + 1$. It is impossible that $m \geq 3$, for otherwise

$$n = \sum_{i=1}^m n_i \geq n_1 + n_2 + n_3 \geq 3(s + 1) > n,$$

a contradiction. Hence $m \leq 2$. Since we are assuming $m \geq 2$, it follows that $m = 2$. Therefore $n = n_1 + n_2 \geq 2(s + 1)$, and after $|V_2| = s$ has been fixed, the number of vertices with vertex sum $s + 1$ that can be produced is at least

$$n - s - 2 \geq s > s - 1.$$

If $k \geq 1$, then $m - k \leq m - 1$, and since every cycle has length at least 3, we have $m \leq \lfloor n/3 \rfloor$. Hence

$$n - s - (m - k) \geq n - s - \left\lfloor \frac{n}{3} \right\rfloor + 1 \geq s - 1.$$

If the resulting number is larger than $s - 1$, we reduce it by replacing local patterns $s, 1, s$ with s, s, s . This operation decreases $|V_{s+1}|$ by exactly 2 and does not affect $|V_2|$. Since each replacement decreases $|V_{s+1}|$ by 2, the last step before reaching the target $|V_{s+1}| = s - 1$ occurs when $|V_{s+1}| = s + 1$. As $n \geq 6$, we have $s = \lceil \frac{n+2}{3} \rceil \geq 3$, and hence $s + 1 \geq 4$.

In Case 1, where $n_{k+1} \geq a + 2$, at most two vertices of V_{s+1} are boundary vertices adjacent to the initial block of edges labeled 1. Hence, when $|V_{s+1}| = s + 1 \geq 4$, at least two vertices of V_{s+1} arise from a local pattern $s, 1, s$. Replacing this pattern with s, s, s performs the final reduction to $|V_{s+1}| = s - 1$.

In Case 2, where $n_{k+1} = a + 1$, exactly four vertices of V_{s+1} are boundary vertices of the two initial blocks of edges labeled 1, one in C_{k+1} and the other in C_{k+2} . If $s \geq 5$, then $|V_{s+1}| = s + 1 \geq 6$, so at least two vertices of V_{s+1} arise from a local pattern $s, 1, s$, and the final reduction can again be performed. The only remaining possibility is $s = 3$. In this case, $s = \lceil \frac{n+2}{3} \rceil$ and $n \geq 6$ imply $n \in \{6, 7\}$. Since $m \geq 2$, every cycle has length at least 3, and $n_1 \leq n_2$, we must have $n_1 = 3 = s$. Consequently, $k = 1$ and $a = 0$. However, Case 2 would then require $n_2 = a + 1 = 1$, contradicting $n_2 \geq 3$. Thus Case 2 cannot occur when $s = 3$.

Therefore, the final reduction can always be performed. Repeating the operation as needed, we obtain exactly $|V_{s+1}| = s - 1$, while $|V_2|$ remains unchanged.

Consequently, every vertex not in V_2 or V_{s+1} has vertex sum $2s$. Thus we obtain the distribution

$$|V_2| = s, \quad |V_{s+1}| = s - 1, \quad |V_{2s}| = n + 1 - 2s.$$

This is identical to the distribution in Case 2 of Theorem 3.1. Therefore, we assign vertex labels to the ordered sets $\mathbf{V}_2$, $\mathbf{V}_{s+1}$, and $\mathbf{V}_{2s}$ according to the same pattern as in that case. Since this assignment depends only on the distribution of these sets and not on the structure of the underlying graph, it guarantees that all vertex weights are pairwise distinct and all vertex labels are at most s .

It remains to consider the case when s is even. The construction used above to force the size of V_2 does not rely on the parity of s . Hence we may run the same first-stage procedure with target

$$|V_2| = s - 1.$$

Here the parameters k and a are defined relative to $s - 1$: k is maximal such that $\sum_{i=1}^k n_i \leq s - 1$, and $\sum_{i=1}^k n_i = s - 1 - a$ for some $a \geq 0$. After this stage is fixed, there remain $n - (s - 1)$ vertices outside V_2 , distributed among the remaining $(m - k)$ cycle-components under consideration. Using the same alternating completion patterns as in the odd case, at most one vertex per such component fails to contribute to V_{s+1} . Therefore

$$|V_{s+1}| \geq n - (s - 1) - (m - k).$$

Since $k \geq 0$, we have $m - k \leq m \leq \lfloor n/3 \rfloor$. Moreover, the definition $s = \lceil (n + 2)/3 \rceil$ gives

$$n - s - \left\lfloor \frac{n}{3} \right\rfloor \geq s - 2.$$

Therefore,

$$|V_{s+1}| \geq n - (s - 1) - (m - k) \geq n - (s - 1) - \left\lfloor \frac{n}{3} \right\rfloor \geq s - 1.$$

The contribution to V_{s+1} from each cycle is even, so $|V_{s+1}|$ is even. Since s is even, $s - 1$ is odd, and hence the inequality above implies $|V_{s+1}| \geq s$. If this number is larger than s , we repeatedly apply the local replacement

$$(s, 1, s) \longrightarrow (s, s, s),$$

which decreases $|V_{s+1}|$ by exactly 2 and leaves $|V_2|$ unchanged. As in the odd case, at most four vertices of V_{s+1} are boundary vertices that do not arise from a local pattern $s, 1, s$. Hence, whenever $|V_{s+1}| \geq 6$, at least two vertices of V_{s+1} arise from such a pattern, and the local replacement can be performed. Since $|V_{s+1}|$ is even, if $|V_{s+1}| > s$, repeated reductions eventually bring $|V_{s+1}|$ to $s + 2$. Since $s \geq 4$, we then have $|V_{s+1}| = s + 2 \geq 6$, so one final replacement yields $|V_{s+1}| = s$.

Consequently, we obtain an edge labeling such that

$$|V_2| = s - 1, \quad |V_{s+1}| = s, \quad \text{and} \quad |V_{2s}| = n + 1 - 2s.$$

This distribution coincides with that of Case 1 in Theorem 3.1. Although the underlying graphs are different, the final vertex-labeling step depends only on the partition $V(G) = V_2 \sqcup V_{s+1} \sqcup V_{2s}$ and

the cardinalities of these three classes. Since these coincide with those in Case 1 of Theorem 3.1, we assign vertex labels to the ordered sets $\mathbf{V}_2$, $\mathbf{V}_{s+1}$, and $\mathbf{V}_{2s}$ by the same scheme, and the distinct-weight verification applies verbatim. Therefore,

$$\text{tvs}(G) = \left\lceil \frac{n+2}{3} \right\rceil.$$

□

Acknowledgments

The author thanks Institut Teknologi Bandung (ITB) and the Ministry of Higher Education, Science, and Technology (Kementerian Pendidikan Tinggi, Sains dan Teknologi), Republic of Indonesia, for their support in this research. The author is also grateful to the anonymous reviewers for their careful reading and detailed suggestions, which significantly improved the clarity of the paper.

REFERENCES

- [1] A. Ahmad, K. M. Awan, I. Javaid and S. Slamun, Total vertex irregularity strength of wheel related graphs, *Australas. J. Combin.*, **51** (2011) 147–156.
- [2] A. Ahmad, M. Bača and M. K. Siddiqui, Irregular total labeling of disjoint union of prisms and cycles, *Australas. J. Combin.*, **59** (2014) 98–106.
- [3] M. Anholcer, M. Kalkowski and J. Przybylo, A new upper bound for the total vertex irregularity strength of graphs, *Discrete Math.*, **309** no. 21 (2009) 6316–6317.
- [4] M. Bača, S. Jendrol', M. Miller and J. Ryan, On irregular total labellings, *Discrete Math.*, **307** no. 11–12 (2007) 1378–1388.
- [5] A. Barra and M. Afifurrahman, The total vertex irregularity strength for cubic graphs with a perfect matching, *Discrete Math.*, **348** no. 5 (2025), Paper No. 114402, 10 pp.
- [6] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz and F. Saba, *Irregular networks. 250th Anniversary Conference on Graph Theory (Fort Wayne, IN, 1986)*, *Congr. Numer.*, **64** (1988) 197–210.
- [7] R. J. Faudree and J. Lehel, Bound on the irregularity strength of regular graphs, In *Colloq. Math. Soc. János Bolyai*, **52**, North-Holland, Amsterdam, (1988) 247–256.
- [8] M. Kalkowski, M. Karonski and F. Pfender, A new upper bound for the irregularity strength of graphs, *SIAM J. Discrete Math.*, **25** (2011) no. 3 1319–1321.
- [9] P. Majerski and J. Przybylo, Total vertex irregularity strength of dense graphs, *J. Graph Theory*, **76** no. 1 (2013) 34–41.
- [10] P. Majerski and Jakub Przybylo, On the irregularity strength of dense graphs, *SIAM J. Discrete Math.*, **28** no. 1 (2014) 197–205.
- [11] B. Nurdin, E. T. Baskoro, A. N. M. Salman and N. N. Gaos, On the total vertex irregularity strength of trees, *Discrete Math.*, **310** no. 21 (2010) 3043–3048.

- [12] J. Przybylo and F. Wei, On the asymptotic confirmation of the Faudree-Lehel conjecture for general graphs, *Combinatorica*, **43** no. 4 (2023) 791–826.
- [13] J. Przybylo and F. Wei, Short proof of the asymptotic confirmation of the Faudree-Lehel conjecture, *Electron. J. Combin.*, **30** no. 4 Paper No. 4.27 (2023) 13 pp.
- [14] K. Wijaya and Slamin, Total vertex irregular labelings of wheels, fans, suns and friendship graphs, *J. Combin. Math. Combin. Comput.*, **65** (2008) 103–112.
- [15] K. Wijaya, Slamin, Surahmat, and Stanislav Jendrol', Total vertex irregular labeling of complete bipartite graphs, *J. Combin. Math. Combin. Comput.*, **55** (2005) 129–136.

Aleams Barra

Algebra Research Group, Institut Teknologi Bandung, Jl. Ganesa No. 10, Bandung 40132, Indonesia

Email: aleamsbarra@itb.ac.id