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THE RECIPROCAL SERIES OF CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we consider infinite summations derived from the reciprocals of Chebyshev polynomials, and infinite summations derived from the reciprocals of the square of Chebyshev polynomials. Then making use of the floor function to these summations, we obtain several new formulae involving Chebyshev polynomials.

1. Introduction and Motivation

The Chebyshev polynomials of the first kind and the second kind are important classes of orthogonal polynomials that have wide applications in approximation theory, numerical analysis, mathematical physics, and other fields. They are defined by

$$T_n(\cos\theta) = \cos(n\theta) \quad \text{and} \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

and admit several useful properties (see Mason [7] for example):

- Recurrence relations ($n \geq 2$):

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$
$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

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- Initial conditions:

$$T_0(x) = 1, \quad T_1(x) = x,$$
$$U_0(x) = 1, \quad U_1(x) = 2x.$$

- Special value for $x = 0$ and $x = 1$:

$$T_{2n}(0) = (-1)^n, \quad T_{2n+1}(0) = 0, \quad T_n(1) = 1,$$
$$U_{2n}(0) = (-1)^n, \quad U_{2n+1}(0) = 0, \quad U_n(1) = n + 1.$$

- Ordinary generating functions:

$$\frac{1 - \eta x}{1 - 2\eta x + \eta^2} = \sum_{n=0}^{\infty} T_n(x)\eta^n,$$
$$\frac{1}{1 - 2\eta x + \eta^2} = \sum_{n=0}^{\infty} U_n(x)\eta^n.$$

- Binet formulae:

$$\left.\begin{aligned} T_n(x) &= \frac{\alpha^n + \beta^n}{2} \\ U_n(x) &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \end{aligned}\right\}, \quad \alpha, \beta = x \pm \sqrt{x^2 - 1}. \tag{1.1}$$

- Fibonacci and Lucas numbers:

$$U_n(\tfrac{\mathbf{i}}{2}) = F_{n+1}\mathbf{i}^n \quad \text{and} \quad T_n(\tfrac{\mathbf{i}}{2}) = \frac{L_n}{2}\mathbf{i}^n.$$

There exist numerous formulae about Chebyshev polynomials in the literature. For example, the reader can find, in the recent papers [1, 2, 3, 5, 6], more formulae about trigonometric expressions, generating functions, root power sums, integrals and certain combinatorial identities.

For any variable quantity x, the Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are defined by $F_{n+2}(x) = xF_{n+1}(x) + F_n(x), n \geq 0$ with the initial values $F_0(x) = 0$ and $F_1(x) = 1$; $L_{n+2}(x) = xL_{n+1}(x) + L_n(x), n \geq 0$ with the initial values $L_0(x) = 2$ and $L_1(x) = x$. When $x = 1$, the Fibonacci polynomials and Lucas polynomials reduce to Fibonacci numbers and Lucas numbers respectively. Let

$$\phi = \frac{1}{2}(x + \sqrt{x^2 + 4}) \quad \text{and} \quad \psi = \frac{1}{2}(x - \sqrt{x^2 + 4}),$$

which leads to the well-known Binet formulae

$$F_n(x) = \frac{\phi^n - \psi^n}{\phi - \psi} \quad \text{and} \quad L_n(x) = \phi^n + \psi^n. \tag{1.2}$$

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For any positive integer x, Wu and Zhang [8] proved the following four interesting identities:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k(x)} \right)^{-1} \right\rfloor = \begin{cases} F_n(x) - F_{n-1}(x), & n \text{ is even and } n \geq 2; \\ F_n(x) - F_{n-1}(x) - 1, & n \text{ is odd and } n \geq 1; \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2(x)} \right)^{-1} \right\rfloor = \begin{cases} xF_n(x)F_{n-1}(x) - 1, & n \text{ is even and } n \geq 2; \\ xF_n(x)F_{n-1}(x), & n \text{ is odd and } n \geq 1; \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k(x)} \right)^{-1} \right\rfloor = \begin{cases} L_n(x) - L_{n-1}(x) - 1, & n \text{ is even and } n \geq 2; \\ L_n(x) - L_{n-1}(x), & n \text{ is odd and } n \geq 3; \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k^2(x)} \right)^{-1} \right\rfloor = \begin{cases} xL_{2n-1}(x) + 1, & n \text{ is even and } n \geq 2; \\ xL_{2n-1}(x) - 2, & n \text{ is odd and } n \geq 3. \end{cases}$$

In view of Binet formulae (1.1) and (1.2), we can get the following two identities

$$F_n(x) = \frac{U_n(\frac{x\mathbf{i}}{2})}{\mathbf{i}^n} \quad \text{and} \quad L_n(x) = \frac{2T_n(\frac{x\mathbf{i}}{2})}{\mathbf{i}^n}.$$

Furthermore, we can derive reciprocal series involving Chebyshev polynomials from the conclusions of Wu and Zhang. For example ($x = \frac{k\mathbf{i}}{2}, k \in \mathbb{N}$):

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\mathbf{i}^k}{U_k(x)} \right)^{-1} \right\rfloor = \begin{cases} U_n(x)\mathbf{i}^{-n} - U_{n-1}(x)\mathbf{i}^{1-n}, & n \text{ is even and } n \geq 2; \\ U_n(x)\mathbf{i}^{-n} - U_{n-1}(x)\mathbf{i}^{1-n} - 1, & n \text{ is odd and } n \geq 1; \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\mathbf{i}^k}{T_k(x)} \right)^{-1} \right\rfloor = \begin{cases} 4T_n(x)\mathbf{i}^{-n} - 4T_{n-1}(x)\mathbf{i}^{1-n} - 2, & n \text{ is even and } n \geq 2; \\ 4T_n(x)\mathbf{i}^{-n} - 4T_{n-1}(x)\mathbf{i}^{1-n}, & n \text{ is odd and } n \geq 3. \end{cases}$$

The aim of this paper is to examine, for $x, n \in \mathbb{N}$ and $s = 1$ or 2, the following two variants of the Chebyshev polynomials:

$$\mathcal{T}_s(n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_k^s(x)} \right)^{-1} \right\rfloor \quad \text{and} \quad \mathcal{U}_s(n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k^s(x)} \right)^{-1} \right\rfloor.$$

Several closed formulae that express these reciprocal series in terms of the Chebyshev polynomials will be established.

The rest of the paper will be organized as follows. In the next section, the closed forms of Chebyshev polynomials reciprocals series $\mathcal{T}_1(n)$ and $\mathcal{U}_1(n)$ will be examined. Then in Section 3, we shall establish the closed forms of reciprocals series $\mathcal{T}_2(n)$ and $\mathcal{U}_2(n)$, which involves the square of Chebyshev polynomials.

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2. The Case of $s = 1$

For $x, n \in \mathbb{N}$, we shall further examine, in this section, the two reciprocal series involving Chebyshev polynomials:

$$\mathcal{T}_1(n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_k(x)} \right)^{-1} \right\rfloor \quad \text{and} \quad \mathcal{U}_1(n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k(x)} \right)^{-1} \right\rfloor.$$

2.1. Chebyshev polynomials of the first kind.

Theorem 2.1 ($x, n \in \mathbb{N}$)**.**

$$\mathcal{T}_1(n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_k(x)} \right)^{-1} \right\rfloor = T_n(x) - T_{n-1}(x).$$

Proof. When $x = 1$, the conclusion is trivial.

For $x \geq 2$, observing the fact

$$\frac{1}{T_k(x)} = \frac{2}{\alpha^k + \beta^k} = \frac{2}{\alpha^k}\left(1 - \frac{1}{\alpha^{2k}+1}\right),$$

it is easy to obtain that

$$\begin{aligned}\sum_{k=n}^{\infty} \frac{1}{T_k(x)} &= 2\left\{ \frac{1}{\alpha^n - \alpha^{n-1}} - \sum_{k=n}^{\infty} \frac{1}{\alpha^k(\alpha^{2k}+1)} \right\} \\ &= \frac{2}{\alpha^n - \alpha^{n-1}} \left\{ 1 - (\alpha^n - \alpha^{n-1}) \sum_{k=n}^{\infty} \frac{1}{\alpha^k(\alpha^{2k}+1)} \right\}.\end{aligned}$$

Using the above formula, we have

$$\left(\sum_{k=n}^{\infty} \frac{1}{T_k(x)} \right)^{-1} = \frac{\alpha^n - \alpha^{n-1}}{2} \left\{ 1 - (\alpha^n - \alpha^{n-1}) \sum_{k=n}^{\infty} \frac{1}{\alpha^k(\alpha^{2k}+1)} \right\}^{-1}.$$

It is easy to check that

$$\frac{1}{\alpha^k(\alpha^{2k}+1)} < \frac{1}{\alpha^{3k}} \tag{2.1}$$

holds for all $x \geq 2$ and $k \in \mathbb{N}$. Then we can further obtain the following inequality relationship

$$\begin{aligned}\left(\sum_{k=n}^{\infty} \frac{1}{T_k(x)} \right)^{-1} &< \frac{\alpha^n - \alpha^{n-1}}{2} \left\{ 1 - (\alpha^n - \alpha^{n-1}) \sum_{k=n}^{\infty} \frac{1}{\alpha^{3k}} \right\}^{-1} \\ &= \frac{\alpha^n - \alpha^{n-1}}{2} \times \frac{\alpha^{2n-2}(1+\alpha+\alpha^2)}{\alpha^{2n-2}(1+\alpha+\alpha^2) - 1} \\ &= \frac{\alpha^n - \alpha^{n-1}}{2} + \frac{\alpha^n - \alpha^{n-1}}{2(\alpha^{2n} + \alpha^{2n-1} + \alpha^{2n-2} - 1)} \\ &= T_n(x) - T_{n-1}(x) + \varepsilon_3,\end{aligned}$$

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where

$$\varepsilon_3 = -\frac{\beta^n - \beta^{n-1}}{2} + \frac{\alpha^n - \alpha^{n-1}}{2(\alpha^{2n} + \alpha^{2n-1} + \alpha^{2n-2} - 1)}.$$

It is not difficult to verify that

$$\varepsilon_3 = \frac{\alpha^{n+1} + \alpha^n - \alpha^{n-1} - \alpha^{n-2} - \alpha^{1-n} + \alpha^{-n}}{2(\alpha^{2n} + \alpha^{2n-1} + \alpha^{2n-2} - 1)} < 1,$$

which yields

$$\left(\sum_{k=n}^{\infty} \frac{1}{T_k(x)}\right)^{-1} < T_n(x) - T_{n-1}(x) + 1. \tag{2.2}$$

Analogous to (2.1), the following identity

$$\frac{1}{\alpha^k(\alpha^{2k}+1)} > \frac{1}{2\alpha^{3k}}$$

holds for all $x \geq 2$ and $k \in \mathbb{N}$. In further, we can derive the following inequality

$$\begin{aligned}
\left(\sum_{k=n}^{\infty} \frac{1}{T_k(x)}\right)^{-1} &> \frac{\alpha^n - \alpha^{n-1}}{2}\left\{1 - (\alpha^n - \alpha^{n-1})\sum_{k=n}^{\infty} \frac{1}{2\alpha^{3k}}\right\}^{-1} \\
&= \frac{\alpha^n - \alpha^{n-1}}{2} \times \frac{2\alpha^{2n-2}(1+\alpha+\alpha^2)}{2\alpha^{2n-2}(1+\alpha+\alpha^2) - 1} \\
&= \frac{\alpha^n - \alpha^{n-1}}{2} + \frac{\alpha^n - \alpha^{n-1}}{2}\frac{1}{2\alpha^{2n-2}(1+\alpha+\alpha^2) - 1} \\
&= T_n(x) - T_{n-1}(x) + \varepsilon_4,
\end{aligned}$$

where

$$\varepsilon_4 = -\frac{\beta^n - \beta^{n-1}}{2} + \frac{\alpha^n - \alpha^{n-1}}{2}\frac{1}{2\alpha^{2n-2}(1+\alpha+\alpha^2) - 1}.$$

It is not difficult to verify that $\varepsilon_4 > 0$, which yields

$$\left(\sum_{k=n}^{\infty} \frac{1}{T_k(x)}\right)^{-1} > T_n(x) - T_{n-1}(x). \tag{2.3}$$

By combining (2.2) with (2.3), we complete the proof of the Theorem 2.1. □

2.2. Chebyshev polynomials of the second kind.

Theorem 2.2 ($x, n \in \mathbb{N}$)**.**

$$\mathcal{U}_1(n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k(x)}\right)^{-1} \right\rfloor = U_n(x) - U_{n-1}(x) - 1.$$

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Proof. Considering computational convenience, we prove the following equivalent form

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}(x)}\right)^{-1}\right\rfloor = U_{n-1}(x) - U_{n-2}(x) - 1.$$

Similarly to Theorem 2.1, the conclusion is also trivial when $x = 1$.

For $x \geq 2$, observing the fact

$$\frac{1}{U_{k-1}(x)} = \frac{\alpha-\beta}{\alpha^k-\beta^k} = \frac{\alpha-\beta}{\alpha^k}\left(1+\frac{1}{\alpha^{2k}-1}\right),$$

it is easy to obtain that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{U_{k-1}(x)} &= (\alpha-\beta)\left\{\frac{1}{\alpha^n-\alpha^{n-1}}+\sum_{k=n}^{\infty}\frac{1}{\alpha^k(\alpha^{2k}-1)}\right\}\\ &= \frac{\alpha-\beta}{\alpha^n-\alpha^{n-1}}\left\{1+(\alpha^n-\alpha^{n-1})\sum_{k=n}^{\infty}\frac{1}{\alpha^k(\alpha^{2k}-1)}\right\}.\end{aligned}$$

Using the above formula, we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}(x)}\right)^{-1} = \frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\left\{1+(\alpha^n-\alpha^{n-1})\sum_{k=n}^{\infty}\frac{1}{\alpha^k(\alpha^{2k}-1)}\right\}^{-1}.$$

It is easy to check that

$$\frac{1}{\alpha^k(\alpha^{2k}-1)} = \frac{1}{\alpha^{3k}}+\frac{1}{\alpha^{3k}(\alpha^{2k}-1)} > \frac{1}{\alpha^{3k}} \tag{2.4}$$

holds for all $x \geq 2$ and $k \in \mathbb{N}$. Then we can further obtain the following inequality relationship

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}(x)}\right)^{-1} &< \frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\left\{1+(\alpha^n-\alpha^{n-1})\sum_{k=n}^{\infty}\frac{1}{\alpha^{3k}}\right\}^{-1}\\ &= \frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\times\frac{\alpha^{2n-2}(1+\alpha+\alpha^2)}{1+\alpha^{2n-2}(1+\alpha+\alpha^2)}\\ &= \frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}-\frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\frac{1}{1+\alpha^{2n-2}(1+\alpha+\alpha^2)}\\ &= U_{n-1}(x)-U_{n-2}(x)+\varepsilon_1,\end{aligned}$$

where

$$\varepsilon_1 = \frac{\beta^n-\beta^{n-1}}{\alpha-\beta}-\frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\frac{1}{1+\alpha^{2n-2}(1+\alpha+\alpha^2)}.$$

It is not difficult to verify that $\varepsilon_1 < 0$, which yields

$$\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}(x)}\right)^{-1} < U_{n-1}(x)-U_{n-2}(x). \tag{2.5}$$

Analogous to (2.4), the following identity

$$\frac{1}{\alpha^k(\alpha^{2k}-1)} = \frac{1}{\alpha^{3k}}+\frac{1}{\alpha^{3k}(\alpha^{2k}-1)} < \frac{2}{\alpha^{3k}}$$

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holds for all $x \geq 2$ and $k \in \mathbb{N}$. In further, we can derive the following inequality

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}(x)}\right)^{-1} &> \frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\left\{1+(\alpha^n-\alpha^{n-1})\sum_{k=n}^{\infty}\frac{2}{\alpha^{3k}}\right\}^{-1}\\ &= \frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\times\frac{\alpha^{2n-2}(1+\alpha+\alpha^2)}{2+\alpha^{2n-2}(1+\alpha+\alpha^2)}\\ &= \frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}-\frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\frac{2}{2+\alpha^{2n-2}(1+\alpha+\alpha^2)}\\ &= U_{n-1}(x)-U_{n-2}(x)+\varepsilon_2,\end{aligned}$$

where

$$\varepsilon_2=\frac{\beta^n-\beta^{n-1}}{\alpha-\beta}-\frac{\alpha^n-\alpha^{n-1}}{\alpha-\beta}\frac{2}{2+\alpha^{2n-2}(1+\alpha+\alpha^2)}.$$

Keep in mind that $\alpha\beta=1$, the above identity can be computed as

$$\begin{aligned}\varepsilon_2 &= \frac{-\alpha^{n+1}-2\alpha^n+2\alpha^{n-1}+\alpha^{n-2}-2\alpha^{1-n}+2\alpha^{-n}}{(\alpha-\beta)(\alpha^{2n}+\alpha^{2n-1}+\alpha^{2n-2}+2)}\\ &> -\frac{\alpha^{n+1}+2\alpha^n+2\alpha^{1-n}+2\alpha^{-n}}{(\alpha-\beta)(\alpha^{2n}+2)}\\ &> -1,\end{aligned}$$

which indicates that

$$\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}(x)}\right)^{-1} > U_{n-1}(x)-U_{n-2}(x)-1. \tag{2.6}$$

By combining (2.5) with (2.6), we complete the proof of the Theorem 2.2. □

3. The Case of $s=2$

Similarly, we can even consider the reciprocal series involving Chebyshev polynomials with $s=2$:

$$\mathcal{T}_2(n)=\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)}\right)^{-1}\right\rfloor \quad\text{and}\quad \mathcal{U}_2(n)=\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_k^2(x)}\right)^{-1}\right\rfloor,$$

where $x, n \in \mathbb{N}$.

3.1. Chebyshev polynomials of the first kind.

Theorem 3.1 ($x, n \in \mathbb{N}$)**.**

$$\mathcal{T}_2(n)=\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)}\right)^{-1}\right\rfloor = T_n^2(x)-T_{n-1}^2(x).$$

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Proof. When $x=1$, the conclusion is trivial.

For $x \geq 2$, observing the fact

$$\frac{1}{T_k^2(x)} = \frac{4}{\alpha^{2k}+\beta^{2k}+2} = \frac{4}{\alpha^{2k}}\left(1-\frac{2+\alpha^{-2k}}{\alpha^{2k}+2+\alpha^{-2k}}\right),$$

it is easy to obtain that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)} &= 4\left\{\frac{1}{\alpha^{2n}-\alpha^{2n-2}} - \sum_{k=n}^{\infty}\frac{2+\alpha^{-2k}}{\alpha^{2k}(\alpha^{2k}+2+\alpha^{-2k})}\right\}\\ &= \frac{4}{\alpha^{2n}-\alpha^{2n-2}}\left\{1-(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{2+\alpha^{-2k}}{\alpha^{2k}(\alpha^{2k}+2+\alpha^{-2k})}\right\}.\end{aligned}$$

Using the above formula, we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)}\right)^{-1} = \frac{\alpha^{2n}-\alpha^{2n-2}}{4}\left\{1-(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{2+\alpha^{-2k}}{\alpha^{2k}(\alpha^{2k}+2+\alpha^{-2k})}\right\}^{-1}.$$

It is easy to check that

$$\frac{2+\alpha^{-2k}}{\alpha^{2k}+2+\alpha^{-2k}} = \frac{2}{\alpha^{2k}}\times\frac{2+\alpha^{-2k}}{2+4\alpha^{-2k}+2\alpha^{-4k}} < \frac{2}{\alpha^{2k}} \tag{3.1}$$

holds for all $x \geq 2$ and $k \in \mathbb{N}$. Then we can further obtain the following inequality relationship

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)}\right)^{-1} &< \frac{\alpha^{2n}-\alpha^{2n-2}}{4}\left\{1-(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{2}{\alpha^{4k}}\right\}^{-1}\\ &= \frac{\alpha^{2n}-\alpha^{2n-2}}{4}\times\frac{\alpha^{2n}+\alpha^{2n-2}}{\alpha^{2n}+\alpha^{2n-2}-2}\\ &= \frac{\alpha^{2n}-\alpha^{2n-2}}{4}+\frac{\alpha^{2n}-\alpha^{2n-2}}{4}\times\frac{2}{\alpha^{2n}+\alpha^{2n-2}-2}\\ &= T_n^2(x)-T_{n-1}^2(x)+\varepsilon_7,\end{aligned}$$

where

$$\varepsilon_7 = -\frac{\beta^{2n}-\beta^{2n-2}}{4}+\frac{\alpha^{2n}-\alpha^{2n-2}}{4}\times\frac{2}{\alpha^{2n}+\alpha^{2n-2}-2}.$$

It is not difficult to verify that

$$\varepsilon_7 = \frac{2\alpha^{2n}-2\alpha^{2n-2}+\alpha^2-\alpha^{-2}-2\alpha^{2-2n}+2\alpha^{-2n}}{4(\alpha^{2n}+\alpha^{2n-2}-2)} < 1,$$

which yields

$$\left(\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)}\right)^{-1} < T_n^2(x)-T_{n-1}^2(x)+1. \tag{3.2}$$

Analogous to (3.1), the following identity

$$\frac{2+\alpha^{-2k}}{\alpha^{2k}+2+\alpha^{-2k}} = \frac{1}{\alpha^{2k}}+\frac{1-\alpha^{-2k}-\alpha^{-4k}}{\alpha^{2k}+2+\alpha^{-2k}} > \frac{1}{\alpha^{2k}}$$

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holds for all $x \geq 2$ and $k \in \mathbb{N}$. In further, we can derive the following inequality

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)}\right)^{-1} &> \frac{\alpha^{2n}-\alpha^{2n-2}}{4}\left\{1-(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{1}{\alpha^{4k}}\right\}^{-1}\\ &= \frac{\alpha^{2n}-\alpha^{2n-2}}{4}\times\frac{\alpha^{2n}+\alpha^{2n-2}}{\alpha^{2n}+\alpha^{2n-2}-1}\\ &= \frac{\alpha^{2n}-\alpha^{2n-2}}{4}+\frac{\alpha^{2n}-\alpha^{2n-2}}{4}\times\frac{1}{\alpha^{2n}+\alpha^{2n-2}-1}\\ &= T_n^2(x)-T_{n-1}^2(x)+\varepsilon_8,\end{aligned}$$

where

$$\varepsilon_8 = -\frac{\beta^{2n}-\beta^{2n-2}}{4}+\frac{\alpha^{2n}-\alpha^{2n-2}}{4}\times\frac{1}{\alpha^{2n}+\alpha^{2n-2}-1}.$$

It is not difficult to verify that $\varepsilon_8 > 0$, which yields

$$\left(\sum_{k=n}^{\infty}\frac{1}{T_k^2(x)}\right)^{-1} > T_n^2(x)-T_{n-1}^2(x). \tag{3.3}$$

By combining (3.2) with (3.3), we complete the proof of the Theorem 3.1. □

3.2. **Chebyshev polynomials of the second kind.**

Theorem 3.2 ($x, n \in \mathbb{N}$)**.**

$$\mathcal{U}_2(n) = \left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_k^2(x)}\right)^{-1}\right\rfloor = \begin{cases}U_n^2(x)-U_{n-1}^2(x)-n-1, & x=1;\\ U_n^2(x)-U_{n-1}^2(x)-1, & x\geq 2.\end{cases}$$

Proof. According to the following inequality

$$\frac{1}{k+1}-\frac{1}{k+2}=\frac{1}{(k+1)(k+2)}<\frac{1}{(k+1)^2}<\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1},$$

and then summing over k from n to ∞ by telescoping approach (cf. Chu [4]), the above theorem is proved for $x = 1$.

For $x \geq 2$, considering computational convenience, we prove the following equivalent form

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}^2(x)}\right)^{-1}\right\rfloor = U_{n-1}^2(x)-U_{n-2}^2(x)-1.$$

Observing the fact

$$\frac{1}{U_{k-1}^2(x)}=\frac{(\alpha-\beta)^2}{\alpha^{2k}+\beta^{2k}-2}=\frac{(\alpha-\beta)^2}{\alpha^{2k}}\left(1+\frac{2\alpha^{2k}-1}{\alpha^{4k}-2\alpha^{2k}+1}\right),$$

it is easy to obtain that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{U_{k-1}^2(x)}&=(\alpha-\beta)^2\left\{\frac{1}{\alpha^{2n}-\alpha^{2n-2}}+\sum_{k=n}^{\infty}\frac{2\alpha^{2k}-1}{\alpha^{2k}(\alpha^{4k}-2\alpha^{2k}+1)}\right\}\\&=\frac{(\alpha-\beta)^2}{\alpha^{2n}-\alpha^{2n-2}}\left\{1+(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{2\alpha^{2k}-1}{\alpha^{2k}(\alpha^{4k}-2\alpha^{2k}+1)}\right\}.\end{aligned}$$

Using the above formula, we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}^2(x)}\right)^{-1}=\frac{\alpha^{2n}-\alpha^{2n-2}}{(\alpha-\beta)^2}\left\{1+(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{2\alpha^{2k}-1}{\alpha^{2k}(\alpha^{4k}-2\alpha^{2k}+1)}\right\}^{-1}.$$

It is easy to check that

$$\frac{2\alpha^{2k}-1}{\alpha^{4k}-2\alpha^{2k}+1}=\frac{2}{\alpha^{2k}}\times\frac{\alpha^{2k}-\frac{1}{2}}{\alpha^{2k}-2+\alpha^{-2k}}>\frac{2}{\alpha^{2k}} \tag{3.4}$$

holds for all $x\geq 2$ and $k\in\mathbb{N}$. Then we can further obtain the following inequality relationship

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}^2(x)}\right)^{-1}&<\frac{\alpha^{2n}-\alpha^{2n-2}}{(\alpha-\beta)^2}\left\{1+(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{2}{\alpha^{4k}}\right\}^{-1}\\&=\frac{\alpha^{2n}-\alpha^{2n-2}}{(\alpha-\beta)^2}\frac{\alpha^{2n}+\alpha^{2n-2}}{\alpha^{2n}+\alpha^{2n-2}+2}\\&=\frac{\alpha^{2n}-\alpha^{2n-2}}{(\alpha-\beta)^2}-\frac{2(\alpha^{2n}-\alpha^{2n-2})}{(\alpha-\beta)^2(\alpha^{2n}+\alpha^{2n-2}+2)}\\&=U_{n-1}^2(x)-U_{n-2}^2(x)+\varepsilon_5,\end{aligned}$$

where

$$\begin{aligned}\varepsilon_5&=-\frac{\beta^{2n}-\beta^{2n-2}}{(\alpha-\beta)^2}-\frac{2(\alpha^{2n}-\alpha^{2n-2})}{(\alpha-\beta)^2(\alpha^{2n}+\alpha^{2n-2}+2)}\\&=\frac{-2\alpha^{2n}+2\alpha^{2n-2}+\alpha^2-\alpha^{-2}-2\alpha^{-2n}+2\alpha^{2-2n}}{(\alpha-\beta)^2(\alpha^{2n}+\alpha^{2n-2}+2)}.\end{aligned}$$

It is not difficult to verify that $\varepsilon_5<0$, which yields

$$\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}^2(x)}\right)^{-1}<U_{n-1}^2(x)-U_{n-2}^2(x). \tag{3.5}$$

Analogous to (3.4), the following identity

$$\frac{2\alpha^{2k}-1}{\alpha^{4k}-2\alpha^{2k}+1}=\frac{3}{\alpha^{2k}}-\frac{1-5\alpha^{-2k}+3\alpha^{-4k}}{\alpha^{2k}-2+\alpha^{-2k}}<\frac{3}{\alpha^{2k}}$$

holds for all $x \geq 2$ and $k \in \mathbb{N}$. In further, we can derive the following inequality

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}^2(x)}\right)^{-1} &> \frac{\alpha^{2n}-\alpha^{2n-2}}{(\alpha-\beta)^2}\left\{1+(\alpha^{2n}-\alpha^{2n-2})\sum_{k=n}^{\infty}\frac{3}{\alpha^{4k}}\right\}^{-1}\\ &= \frac{\alpha^{2n}-\alpha^{2n-2}}{(\alpha-\beta)^2}\frac{\alpha^{2n}+\alpha^{2n-2}}{\alpha^{2n}+\alpha^{2n-2}+3}\\ &= \frac{\alpha^{2n}-\alpha^{2n-2}}{(\alpha-\beta)^2}-\frac{3(\alpha^{2n}-\alpha^{2n-2})}{(\alpha-\beta)^2(\alpha^{2n}+\alpha^{2n-2}+3)}\\ &= U_{n-1}^2(x)-U_{n-2}^2(x)+\varepsilon_6,\end{aligned}$$

where

$$\varepsilon_6 = -\frac{\beta^{2n}-\beta^{2n-2}}{(\alpha-\beta)^2}-\frac{3(\alpha^{2n}-\alpha^{2n-2})}{(\alpha-\beta)^2(\alpha^{2n}+\alpha^{2n-2}+3)}.$$

Keep in mind that $\alpha\beta = 1$, the above identity can be computed as

$$\begin{aligned}\varepsilon_6 &= \frac{-3\alpha^{2n}+3\alpha^{2n-2}+\alpha^2-\alpha^{-2}-3\alpha^{-2n}+3\alpha^{2-2n}}{(\alpha-\beta)^2(\alpha^{2n}+\alpha^{2n-2}+2)}\\ &> -\frac{3\alpha^{2n}+\alpha^{-2}+3\alpha^{-2n}}{(\alpha-\beta)^2(\alpha^{2n}+\alpha^{2n-2}+2)}\\ &> -1,\end{aligned}$$

which indicates that

$$\left(\sum_{k=n}^{\infty}\frac{1}{U_{k-1}^2(x)}\right)^{-1} > U_{n-1}^2(x)-U_{n-2}^2(x)-1. \tag{3.6}$$

By combining (3.5) with (3.6), we complete the proof of the Theorem 3.2. □

Concluding comments

In contrast to the outcome at $s = 1$ or 2, $\mathcal{U}_s(n)$ and $\mathcal{T}_s(n)$ no longer have clean forms if $s \geq 3$. We also compute the case $s \geq 3$, however, the computations are much more complicated. We hope that these issues can be effectively resolved by us or someone else in the near future.

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