International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. xx No. x (202x), pp. xx–xx.

https://ijgt.ui.ac.ir
https://www.ui.ac.ir

PRODUCTS OF CONJUGACY CLASSES AND IRREDUCIBLE CHARACTERS IN FINITE CAMINA GROUPS

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Abstract. This paper investigates two problems concerning products of conjugacy classes and irreducible characters in finite groups. First, we provide a simplified proof of Dade and Yadav's theorem concerning groups in which the conjugacy class product $x^G y^G = (xy)^G$ holds for all $x \in G, y \in G - (x^{-1})^G$. By using the classification of Camina groups due to Dark and Scoppola, we demonstrate that such groups are either abelian, non-abelian Camina p-groups, or specific Frobenius groups of the form $F^+ \rtimes F^\times$ for some finite field F with $|F| > 2$ or $E_9 \rtimes Q_8$. Second, we explore the dual problem for characters: groups in which the product of any two non-conjugate irreducible characters has exactly one irreducible constituent. We establish that such groups are precisely the Camina p-groups and the two families of Frobenius groups mentioned above.

1. Introduction

Let G be a finite group. A classic problem in the study of finite groups is to investigate the group structure by using the information encoded in the product of conjugacy classes. For example, Arad and Herzog's conjecture asserts that in a non-abelian simple group the product of two conjugacy classes cannot be a conjugacy class. Thompson's conjecture claims every finite non-abelian simple group G contains a conjugacy class $C \subset G$ such that $C^2 = G$. These two conjectures have been verified for many cases but remain open in general. See [2, 9, 14, 15] for some partial results.

MSC(2010): Primary: 20E15; Secondary: 20C15.

Keywords: Camina group, conjugacy class, character product.

Article Type: Research Paper.

Communicated by Alireza Moghaddamfar.

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Received: 07 February 2026, Accepted: 03 July 2026, Published Online: 15 July 2026.

Cite this article: Y. Liu and S. Yang, Products of conjugacy classes and irreducible characters in finite camina groups, *Int. J. Group Theory*, **x** no. x (202x) xx-xx. https://dx.doi.org/10.22108/ijgt.2026.148288.2015 .

In [6], Dade and Yadav studied groups in which the product of two conjugacy classes is itself a conjugacy class, except in some special cases. They proved the following theorem:

Theorem 1.1. *Let G be a finite group. Then $x^G y^G = (xy)^G$ for any $x \in G$ and $y \in G - (x^{-1})^G$ if and only if G is isomorphic to one of the following groups:*

(1) *Any finite abelian groups.*
(2) *A non-abelian Camina p-group, for some prime p.*
(3) *The group $F^+ \rtimes F^\times$ for some finite field F with $|F| > 2$.*
(4) *The group $E_9 \rtimes Q_8$.*

Our motivation for writing this paper arose while reading their proof: when Dade and Yadav analyzed the structure of groups satisfying the condition, they showed that such a group must be a Camina group. This class of groups has been extensively studied, and according to the classification by Dark and Scoppola, it suffices to examine each family separately. The first contribution of the present article is to provide a concise proof of the aforementioned theorem.

In [10], Isaacs conjectured that if the product of two irreducible faithful characters of a solvable group is irreducible, then the group is cyclic. Isaacs's conjecture was further studied in [1, 8, 13]. The investigation of irreducible products of ordinary (and Brauer) characters of quasisimple groups was launched by I. Zisser in [20] and by Bessenrodt and Kleshchev for alternating groups and their covering groups in [3, 4]; it was subsequently extended to groups of Lie type by K. Magaard and P. H. Tiep in [18]. Recently, G. Navarro and P. H. Tiep considered the problem when the product of two non-linear Galois conjugate characters of a finite group is irreducible in [19].

Our second result addresses the analogous problem, in the character-theoretic setting, of the conjugacy-class problem studied by Dade and Yadav. For any two irreducible characters χ, ϕ of G, denote by $n(\chi\phi)$ the number of distinct irreducible constituents in the product $\chi\phi$. We consider the finite group G in which the product of any two non-conjugate irreducible characters has exactly one irreducible constituent.

Theorem 1.2. *Let G be a finite group. Then $n(\chi\phi) = 1$ for any irreducible characters $\chi, \phi \neq \overline{\chi}$ of G if and only if G is isomorphic to one of the following groups:*

(1) *Any finite abelian groups.*
(2) *A non-abelian Camina p-group, for some prime p.*
(3) *The group $F^+ \rtimes F^\times$ for some finite field F with $|F| > 2$.*
(4) *The group $E_9 \rtimes Q_8$.*

To conclude this section, we establish some basic notation.

If N is a normal subgroup of G, for any $\lambda \in \mathrm{Irr}(N)$, $\chi \in \mathrm{Irr}(G)$, we denote by χ_N the restriction of χ to N and by λ^G the induced character of λ from N to G. Let $\overline{\chi}$ be the conjugate of χ, i.e. $\overline{\chi}(g) = \overline{\chi(g)}$ for any $g \in G$. Write $\mathrm{Irr}(\lambda^G) = \mathrm{Irr}(G|\lambda) = \{\chi \in \mathrm{Irr}(G) | [\chi, \lambda^G] \neq 0\}$ and $\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) | N \nsubseteq$

https://dx.doi.org/10.22108/ijgt.2026.148288.2015

$\ker(\chi)\}$. For $n \in N$ and $g \in G$, define $\lambda^g : N \to \mathbb{C}$ by $\lambda^g(n) = \lambda(gng^{-1})$. For two sets A and B, let $A - B = \{x \in A | x \notin B\}$.

2. On product of conjugacy classes

A nonabelian and non-perfect group G is a Camina group if each nontrivial coset $xG' \in (G/G')^{\#}$ is a single G-conjugacy class $x^G = x[x, G]$, i.e., $[x, G] = G'$ for all $x \in G - G'$. The study of Camina group and related Camina pairs originated in [5], where Camina introduced two conditions that generalized properties of Frobenius groups and extra-special groups. First we provide an equivalent condition characterizing Camina groups by characters.

Lemma 2.1. *Let G be a finite nonabelian and non-perfect group. Then G is a Camina group if and only if every nonlinear character $\chi \in \mathrm{Irr}(G)$ vanishes on $G - G'$.*

Proof. By character orthogonal relations,

$$|C_G(g)| = \sum_{\chi \in \mathrm{Irr}(G)} |\chi(g)|^2 = \sum_{\chi \in \mathrm{Irr}(G/G')} |\chi(g)|^2 + \sum_{\chi \in \mathrm{Irr}(G|G')} |\chi(g)|^2 = |G/G'| + \sum_{\chi \in \mathrm{Irr}(G|G')} |\chi(g)|^2.$$

So for any element $g \in G - G'$, $|C_G(g)| = |G/G'|$ if and only if $\chi(g) = 0$ for any $\chi \in \mathrm{Irr}(G|G')$. The Lemma holds. □

The classification of Camina groups was given by Dark and Scoppola in [7]. More comprehensive studies on Camina groups and Camina pairs can be found in [12, 17] or the survey [16].

Theorem 2.2 (Dark and Scoppola)**.** *Let G be a Camina group. Then one of the following occurs:*

(1) *G is a p-group for some prime p.*
(2) *G is a Frobenius group whose Frobenius complement is cyclic.*
(3) *G is a Frobenius group whose Frobenius complement is isomorphic to the quaternion group Q_8.*

The following three results are from [6, section 3]. Lemma 2.3 is [6, Proposition 3.2], Lemma 2.4 is a special case of [6, Proposition 3.3] and Lemma 2.5 is [6, Proposition 3.6].

Lemma 2.3. *Let G be a finite nonabelian group. If $x^G y^G = (xy)^G$ for any $y \in G - (x^{-1})^G$, then $x^G Z(G) = x^G$ for every element $x \in G - Z(G)$ and $Z(G) \leqslant G'$.*

Lemma 2.4. *Let G be a finite group with $Z(G) \leq G'$. If $x^G y^G = (xy)^G$ for any $y \in G - (x^{-1})^G$, then $[x, G] = G'$ for any element $x \in G - G'$.*

Lemma 2.5. *Let G be a finite group such that $x^G y^G = (xy)^G$ for any $y \in G - (x^{-1})^G$. Suppose that $1 < N = [N, G] \trianglelefteq G$ and $Z(G) \cap N = 1$. If N/M is a chief factor of G for some $M \trianglelefteq G$ with $M < N$, then $N - M$ is a single conjugacy class in G.*

At the end of this section, we give a concise proof of Theorem 1.1.

https://dx.doi.org/10.22108/ijgt.2026.148288.2015

Proof. The sufficiency can be checked directly. Next we prove the necessity by induction on the order of G. Now assume $x^G y^G = (xy)^G$ for any $y \in G - (x^{-1})^G$. If G is abelian, then G is the group of type (1). Next we assume G is non-abelian. By Lemmas 2.3 and 2.4 we have that G is a Camina group. So G is a Camina p-group (i.e. type (2)), a Frobenius group whose Frobenius complement is cyclic or a Frobenius group whose Frobenius complement is isomorphic to Q_8 by Theorem 2.2. Next we assume that $G = N \rtimes H$ where N is the Frobenius kernel and H is a Frobenius complement. Because of the properties of Frobenius group, it can be easily checked that $Z(G) = 1$ and $K = [K, G]$ for any normal subgroup $K \leq N$ of G. If N is not a minimal normal subgroup, there exists a normal subgroup M of G such that N/M is a chief factor of G. Next we show M is a minimal normal subgroup of G. Otherwise, there exists a normal subgroup L of G such that $L < M$. Since $\overline{G} = G/L$ satisfies the condition: $\overline{x}^{\overline{G}}\overline{y}^{\overline{G}} = (\overline{xy})^{\overline{G}}$ for any $\overline{y} \in \overline{G} - (\overline{x}^{-1})^{\overline{G}}$, we have that G/L is isomorphic to one of groups in the list of theorem by induction and it implies N/L is a minimal normal subgroup of G/L, a contradiction. So M is a minimal normal subgroup of G. By Lemma 2.5, $N - M$ and $M - \{1\}$ are two conjugacy classes of G and $N/M - \{\overline{1}\}$ is a single conjugacy class of G/M. Denote by $|H| = h$. Then $|N/M| = |M| = h+1$ and $|N - M| = h(h+1)$. Choose $x \in N - M$. Then $M < C_G(x) \leq N$ for N is nilpotent and $M \leq Z(N)$ and $|x^G| < |G/M| = h(h+1)$, a contradiction with $N - M = x^G$. Hence N is a minimal normal subgroup of G. Furthermore $|N| = h+1$ and $|G| = h(h+1)$.

If H is cyclic, then G is the group of type (3). If $H \cong Q_8$, then G is the group of type (4). Hence the theorem holds. $\square$

3. **On product of irreducible characters**

We first give a result on product of characters.

Lemma 3.1. *Let G be a finite group and ϕ be an irreducible character of G. If $\chi\phi$ is a multiple of ϕ for some $\chi \in \mathrm{Irr}(G)$, then χ is not faithful.*

Proof. Since $\chi\phi$ is a multiple of ϕ, we have that $\chi\phi = \chi(1)\phi$. If χ is faithful, then $\chi(g) \neq \chi(1)$ for any nontrivial element $g \in G$. So $\chi(g)\phi(g) = \chi(1)\phi(g)$ implies $\phi(g) = 0$, a contradiction. $\square$

Now we show that the groups we wish to study are precisely the Camina groups.

Lemma 3.2. *Let G be a finite non-abelian group. If $n(\chi\phi) = 1$ for any irreducible characters $\chi, \phi \neq \overline{\chi}$ of G, then G is a Camina group.*

Proof. First we prove $\lambda\chi = \chi$ for any linear character λ and nonlinear character $\chi \in \mathrm{Irr}(G)$. Otherwise, we assume $\lambda\chi \neq \chi$, then $n(\lambda\chi\overline{\chi}) = 1$. Since the trivial character 1_G is a constituent of $\chi\overline{\chi}$, we have that $\lambda\chi\overline{\chi} = \chi(1)^2\lambda$. A contradiction with $[\lambda\chi\overline{\chi}, \lambda] = [\lambda\chi, \lambda\chi] = 1$. Hence $\lambda\chi = \chi$ for any linear character λ and nonlinear character $\chi \in \mathrm{Irr}(G)$. For any element $g \in G - G'$, there exists a linear character θ with $\theta(g) \neq 1$. Since $\theta(g)\chi(g) = \chi(g)$ for any nonlinear character $\chi \in \mathrm{Irr}(G)$, it can be obtained that $\chi(g) = 0$ for any element $g \in G - G'$ and it means G is a Camina group by Lemma 2.1. $\square$

https://dx.doi.org/10.22108/ijgt.2026.148288.2015

According to the classification of Camina groups, we need to investigate each case separately. We first examine the case of p-groups.

Lemma 3.3. *Let G be a p-group. Then G is a Camina p-group if and only if $n(\chi\phi) = 1$ for any irreducible characters $\chi, \phi \neq \overline{\chi}$ of G.*

Proof. The "sufficient" condition is from Lemma 3.2. So we assume G is a Camina p-group. Set $G_1 = G$ and $G_{i+1} = [G_i, G]$. By Dark, MacDonald, Mann and Scoppola's results ([16, Theorems 3.1, 3.2 and 3.3]), G has nilpotence class 2 or 3.

If G has nilpotence class 2, then $G_2 = G'$ is in the center of G. For any nonlinear $\chi \in \mathrm{Irr}(G)$, we have that χ vanishes on $G - G'$. So $|G| = \sum_{g\in G'} |\chi(g)|^2 = |G'| \cdot |\chi(g)|^2$ and $\chi(1) = |G/G'|^{\frac{1}{2}}$. Next we show $n(\chi\phi) = 1$ for any nonlinear irreducible characters $\chi, \phi \neq \overline{\chi}$ of G. Assume $\chi_{G'} = \chi(1)\lambda_1$ and $\phi_{G'} = \phi(1)\lambda_2$. Then $\lambda_1 \neq \overline{\lambda_2}$ and $\chi\phi = \chi(1)\varphi$ where φ is the unique irreducible constituent of $(\lambda_1\lambda_2)^G$.

If G has nilpotence class 3, then every nonlinear irreducible character of G is in $\mathrm{Irr}((G/G_3)|(G'/G_3))$ or $\mathrm{Irr}(G|G_3)$. Furthermore for any $\theta \in \mathrm{Irr}((G/G_3)|(G'/G_3))$ and $\chi \in \mathrm{Irr}(G|G_3)$, θ is fully ramified with respect to G/G' and χ is fully ramified with respect to G/G_3 by [16, Theorem 3.5]. As χ vanishes on $G - G_3$, then $\theta\chi = \theta(1)\chi$. For any $\theta_1, \theta_2 \neq \overline{\theta_1} \in \mathrm{Irr}((G/G_3)|(G'/G_3))$ and $\chi_1, \chi_2 \neq \overline{\chi_1} \in \mathrm{Irr}(G|G_3)$, it can be checked that $n(\theta_1\theta_2) = n(\chi_1\chi_2) = 1$ by a similar discussion to the above paragraph. □

When handling the case of Frobenius groups, we first examine the situation in which the Frobenius kernel is a minimal normal subgroup.

Lemma 3.4. *Let $G = N \rtimes H$ be a Frobenius group with the Frobenius kernel N and a Frobenius complement H. Suppose $n(\chi\phi) = 1$ for any irreducible characters $\chi, \phi \neq \overline{\chi}$ of G. If N is a minimal normal subgroup of G, then $\mathrm{Irr}(N) - \{1_N\} = \{\lambda^h | h \in H\}$ for any nontrivial character $\lambda \in \mathrm{Irr}(N)$.*

Proof. Assume $\mathrm{Irr}(N) - \{1_N\} \neq \{\lambda^h | h \in H\}$. Then there exists two nontrivial characters $\lambda_1, \lambda_2 \in \mathrm{Irr}(N)$ such that $\{\lambda_1^h | h \in H\} \cap \{\lambda_2^h | h \in H\} = \varnothing$. So there are at least two nonlinear irreducible characters in $\mathrm{Irr}(G|N)$. Choose two nonlinear characters $\chi_1, \chi_2 \in \mathrm{Irr}(G|N)$ such that $\chi_1 \neq \overline{\chi_2}$ (here χ_1 may equal to χ_2). Then $\chi_1\chi_2 = k\chi_3$ for some irreducible character $\chi_3 \in \mathrm{Irr}(G)$. By the properties of Frobenius group, $\chi_1(g) = 0$ for any nontrivial element g in H. Then $\chi_3(g) = 0$ which means χ_3 is a non-linear character. Furthermore, $\chi_3 \neq \chi_2$ by Lemma 3.1. By the condition $n(\chi_3\overline{\chi_2}) = 1$, assume $\chi_3\overline{\chi_2} = t\chi_4$ for some irreducible character $\chi_4 \in \mathrm{Irr}(G)$. Hence $\chi_1\chi_2\overline{\chi_2} = k\chi_3\overline{\chi_2} = kt\chi_4$. Since 1_G is a constituent of $\chi_2\overline{\chi_2}$, we have that $\chi_4 = \chi_1$.

If all irreducible constituents of $\chi_2\overline{\chi_2}$ are in $\mathrm{Irr}(G/N)$, then N is in the kernel of $\chi_2\overline{\chi_2}$ and $N \leq Z(\chi_2)$. Assume $\chi_2 = \mu^G$ for some $\mu \in \mathrm{Irr}(N)$. Then $(\chi_2)_N = \sum_{g\in H} \mu^g$. Because $\chi_2(1) = |\chi_2(n)| = |\sum_{g\in H} \mu^g(n)|$ for any $n \in N$, we have that $\mu^g(n) = \mu(n)$ for any element $g \in H$ and $n \in N$. A contradiction. So there exists an irreducible constituent θ of $\chi_2\overline{\chi_2}$ such that $\theta \in \mathrm{Irr}(G|N)$. Furthermore, $\theta\chi_1 = \theta(1)\chi_1$ for $\chi_1\chi_2\overline{\chi_2} = kt\chi_1$. Then θ is not faithful by Lemma 3.1 and $N \leq \ker(\theta)$ for N is the unique minimal normal subgroup. A contradiction. □

https://dx.doi.org/10.22108/ijgt.2026.148288.2015

For the general case, we have the following result.

Lemma 3.5. *Let G be a finite non-nilpotent group. Suppose $n(\chi\phi) = 1$ for any irreducible characters $\chi, \phi \neq \overline{\chi}$ of G. Then G is isomorphic to one of the following groups:*

(1) *The group $F^+ \rtimes F^\times$ for some finite field F with $|F| > 2$.*

(2) *The group $E_9 \rtimes Q_8$.*

Proof. We prove the Lemma by induction on the order of G. By Lemma 3.2, G is a Camina group, thus G is a Frobenius group by Theorem 2.2. Assume that $G = N \rtimes H$ where N is the Frobenius kernel and H is a Frobenius complement. Set $h = |H|$. If N is a minimal normal subgroup of G, then $|N| = h+1$ by Lemma 3.4. By Theorem 2.2, H is cyclic or the quaternion group Q_8. If H is cyclic, then G is isomorphic to $F^+ \rtimes F^\times$ for some finite field F with $|F| > 2$. If $H \cong Q_8$, then $G \cong E_9 \rtimes Q_8$.

Next we assume N is not a minimal normal subgroup of G and there exists a normal subgroup M of G such that $M \subseteq N$ and N/M is a minimal normal subgroup of G/M. If M is not a minimal subgroup of G, choose a nontrivial normal L of G such that $L \subseteq M$ and M/L is a minimal normal subgroup of G/L. Then G/L is isomorphic to the groups in theorem by induction and the Frobenius kernel N/L is a minimal normal subgroup in Frobenius group G/L, a contradiction. So M is a minimal subgroup of G.

Case (1) N is not abelian.

In this case, $M = Z(N) = N'$ is the unique minimal normal subgroup of G.

Subcase (1i) $|\mathrm{Irr}(G|M)| > 1$.

Choose $\chi_1, \chi_2 \in \mathrm{Irr}(G|M)$ such that $\chi_1 \neq \overline{\chi_2}$ (here χ_1 may equal to χ_2). Assume $\chi_1\chi_2 = m\chi_3$. By Lemma 3.1, $\chi_3 \neq \chi_2$. Then $\chi_3\overline{\chi_2} = n\chi_4$ for some $\chi_4 \in \mathrm{Irr}(G)$. So $\chi_1\chi_2\overline{\chi_2} = mn\chi_4$ which implies $\chi_4 = \chi_1$. If every irreducible constituent of $\chi_2\overline{\chi_2}$ is in $\mathrm{Irr}(G/M)$, then $M \subseteq Z(\chi_2)$. Assume $\chi_2 = \theta^G$ and $\theta_M = e\phi$ where $\theta \in \mathrm{Irr}(N)$ and $\phi \in \mathrm{Irr}(M)$. Hence $\chi_2(1) = |\chi_2(m)| = |\sum_{g\in H}\theta^g(m)|$ for any element $m \in M$ which means $\theta^g(m) = \theta(m)$ for any element $m \in M$ and $g \in H$. Furthermore $\phi^g(m) = \phi(m)$ and it implies $M = [M,H] \subseteq \ker(\phi)$, a contradiction with $\chi_2 \in \mathrm{Irr}(G|M)$. So exists an irreducible constituent τ of $\chi_2\overline{\chi_2}$ such that $\tau \in \mathrm{Irr}(G|M)$ and $\chi_1\tau = \tau(1)\chi_1$, a contradiction with Lemma 3.1.

Subcase (1ii) $|\mathrm{Irr}(G|M)| = 1$.

Choose $\theta \in \mathrm{Irr}(N|M)$ and assume $\theta_M = e\phi$. Then $\mathrm{Irr}(G|M) = \{\theta^G\}$ and it means $\mathrm{Irr}(N|M) = \{\theta^h | h \in H\}$ and $\mathrm{Irr}(M) - \{1_M\} = \{\phi^h | h \in H\}$. So $|M| = h+1$ and $|G| = h(h+1)^2$ for $|N/M| = h+1$ by induction on G/M. Since $\theta^G(1)^2 = |G| - |G/M| = h^2(h+1)$ and $\theta^G(1) = h\theta(1)$, we have that $\theta(1) = \sqrt{h+1}$ and θ vanishes on $N - M$ (see [11, problem 6.3]). Choose $x \in N - M$. Then $\theta^h(x) = 0$ for any $\theta^h \in \mathrm{Irr}(N|M)$. Thus $|C_N(x)| = \sum_{\lambda\in\mathrm{Irr}(N)}|\lambda(x)|^2 = \sum_{\lambda\in\mathrm{Irr}(N/M)}|\lambda(x)|^2 = |N/M| = h+1$. A contradiction with $C_N(x) > M$.

Case (2) N is abelian.

Subcase (2i) M is not the unique minimal normal subgroup of G.

Assume K is another minimal normal subgroup of G. Then $N = M \times K$. For any $\chi \in \mathrm{Irr}(G|N)$, $\chi = \lambda^G$ for some $\lambda \in \mathrm{Irr}(N)$ and $\chi(1) = h$. Choose nontrivial characters $\lambda_1 \in \mathrm{Irr}(N/M)$ and $\lambda_2 \in \mathrm{Irr}(N/K)$.

https://dx.doi.org/10.22108/ijgt.2026.148288.2015

Then $\lambda_1^G\lambda_2^G = h(\lambda_1\lambda_2)^G$. Since $(\lambda_1^G)_N = \sum_{g\in H}\lambda_1^g$ and $(\lambda_2^G)_N = \sum_{g\in H}\lambda_2^g$, we have that $\lambda_1^{g_1}\lambda_2^{g_2}$ is an irreducible constituent of $[(\lambda_1\lambda_2)^G]_N$ for any $g_1, g_2 \in H$. If $\lambda_1^{g_1}\lambda_2^{g_2} = \lambda_1^{g_3}\lambda_2^{g_4}$ where $g_1, g_2, g_3, g_4 \in H$, then $\lambda_1^{g_1}\overline{\lambda_1}^{g_3} = \lambda_2^{g_4}\overline{\lambda_2}^{g_2}$. Because M is in the kernel of $\lambda_1^{g_1}\overline{\lambda_1}^{g_3}$ and K is in the kernel of $\lambda_2^{g_4}\overline{\lambda_2}^{g_2}$, we have that $\lambda_1^{g_1}\overline{\lambda_1}^{g_3} = \lambda_2^{g_4}\overline{\lambda_2}^{g_2} = 1_N$, i.e. $\lambda_1^{g_1} = \lambda_1^{g_3}$ and $\lambda_2^{g_2} = \lambda_2^{g_4}$. So $[(\lambda_1\lambda_2)^G]_N$ has h^2 different irreducible constituents, a contradiction.

Subcase (2ii) M is the unique minimal normal subgroup of G.

Choose nontrivial characters $\lambda_1 \in \mathrm{Irr}(N|M)$ and $\lambda_2 \in \mathrm{Irr}(N/M)$. Then $\lambda_1^G\lambda_2^G = h(\lambda_1\lambda_2)^G$. If $(\lambda_1\lambda_2)^G = \lambda_1^G$, we have that $\lambda_1\lambda_2 = \lambda_1^g$ for some $g \in H$. So $\lambda_1(m^{-1}gmg^{-1}) = \lambda_2(m) = 1$ and $m^{-1}gmg^{-1} \in \ker(\lambda_1)$ for any $m \in M$. Because $m_1^{-1}gm_1g^{-1} \neq m_2^{-1}gm_2g^{-1}$ for distinct elements $m_1, m_2 \in M$, we have that $M = \{m^{-1}gmg^{-1} | m \in M\}$ and $M \subseteq \ker(\lambda_1)$, a contradiction with $\lambda_1 \in \mathrm{Irr}(N|M)$. So $(\lambda_1\lambda_2)^G \neq \lambda_1^G$ and $\lambda_1^G\lambda_2^G\overline{\lambda_1^G} = h(\lambda_1\lambda_2)^G\overline{\lambda_1^G} = h^2\lambda_2^G$. If every irreducible constituent of $\lambda_1^G\overline{\lambda_1^G}$ is not faithful, then M is in the kernel of $\lambda_1^G\overline{\lambda_1^G}$ and therefore $M \subseteq Z(\lambda_1^G)$. Similar with the discussion in subcase (1i), we have that $h = |\lambda_1^G(m)| = |\sum_{g\in H}\lambda_1^g(m)|$ and $\lambda_1^g(m) = \lambda_1(m)$ for any $m \in M$ and $g \in H$ which implies $\lambda_1(m^{-1}gmg^{-1}) = 1$. So $M = [M, H]$ is the kernel of λ_1, a contradiction. Hence there exists a faithful irreducible character θ in $\lambda_1^G\overline{\lambda_1^G}$ and $\theta\lambda_2^G = h\lambda_2^G$, a contradiction with Lemma 3.1. □

At last, we give the proof of Theorem 1.2.

Proof. Suppose first that G is isomorphic to one of the four types in the classification. If G is of type (1), it is trivial to check the condition. If G is of type (2), the desired conclusion is provided by Lemma 3.3. The cases where G is of type (3) or (4) are immediate, as such a group has a unique nonlinear irreducible character. Finally, if $n(\chi\phi) = 1$ for any irreducible characters $\chi, \phi \neq \overline{\chi}$ of G, then the Theorem is established by Theorem 2.2, Lemmas 3.2, 3.3 and 3.5. □

Conflict of interest

On behalf of all the authors, the corresponding author confirms that there is no conflict of interest.

Acknowledgments

This work is supported by Tianyuan Fund for Mathematics of NSFC (Grant No.12426633).

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