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ON PERFECTLY EMBEDDED SUBGROUPS

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Abstract. Let G be a group, and let $H \leq K \leq G$ be subgroups. We say that H is *perfectly embedded* in K if $[H, K] = H$. In the special case $K = G$, we simply say that H is a perfectly embedded subgroup of G.

The aim of this paper is to investigate several aspects of this notion. We first characterize the groups whose only perfectly embedded subgroup is the trivial subgroup, showing that they are exactly the hypocentral groups. At the other extreme, we study groups in which every normal subgroup is perfectly embedded, providing a complete classification of finite $\mathcal{T}$-groups with this feature.

We also examine groups in which every subgroup is perfectly embedded in an appropriate subgroup normalizing it. Finally, attention is restricted to abelian normal subgroups, focusing once more on the two opposite situations. In particular, we show that every finite Frobenius group belongs to the class of groups in which every abelian normal subgroup is perfectly embedded.

1. Introduction

Studying a group through the structure and properties of its subgroups is a classical approach in group theory. In particular, the way in which subgroups interact with the ambient group under conjugation encodes significant global information about the group's structure. This perspective has motivated a substantial body of work on groups in which certain families of subgroups satisfy strong

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self-normalizing or self-centralizing conditions (see for instance [3, 5–9]). Related questions have also been considered in the context of the embeddings of verbal subgroups (cf. [4, 10–12]).

In this paper, we focus on a notion of subgroup embedding purely defined in terms of commutators. Let G be a group, and let $H \leq K \leq G$ be subgroups. We say that H is *perfectly embedded* in K if

$$[H, K] = H;$$

that is, if H is generated by all commutators of elements of H with elements of K. When $K = G$, we simply refer to H as a perfectly embedded subgroup of G. The trivial subgroup is always perfectly embedded, and every perfectly embedded subgroup is clearly normal.

Every non-central minimal normal subgroup H of a group G is perfectly embedded in G. Indeed, $[H, G]$ is a non-trivial normal subgroup of G contained in H, so $[H, G] = H$ by minimality of H.

If G is a perfect group, then $[H, G]$ is perfectly embedded in G for every subgroup H of G. Indeed, since $[H, G]$ is normal in G, we have $[H, G, G] \leq [H, G]$. On the other hand, $[H, G, G] = [G, H, G]$, and by the Three Subgroups Lemma it follows that $[G, G, H] \leq [H, G, G]$. As G is perfect, we get $[G, G, H] = [G, H] = [H, G]$.

This work explores several research directions. In Section 2 we prove that the groups in which the trivial subgroup is the unique perfectly embedded subgroup are precisely the hypocentral groups. At the opposite end, in Section 3 we consider groups in which every normal subgroup is perfectly embedded. We show that this class of perfect groups is closed under taking quotients and direct products of its members. Although it is not closed under taking normal subgroups, for finite groups in this class we are able to highlight restrictions on the lattice of normal subgroups. This leads for instance to a complete classification of finite $\mathcal{T}$-groups in this class. Going further, in Section 4, we consider groups in which every subgroup is perfectly embedded in some suitable subgroup normalizing it. Finally, in Section 5 we focus on abelian normal subgroups, again considering the two extreme cases: the class of groups in which the trivial subgroup is the unique perfectly embedded abelian normal subgroup, and the class of groups in which every such subgroup is perfectly embedded. We prove that every finite Frobenius group belongs to the latter class.

Throughout the paper, we exploit concrete examples constructed using GAP [13], in particular making use of the `PerfectGroup` library. The computational verification of several examples was also carried out in GAP; the corresponding procedure is reported in Appendix A.

For standard concepts in group theory, we refer the reader to [14–16].

2. Groups without non-trivial perfectly embedded normal subgroups

We start by investigating groups in which no non-trivial normal subgroup is perfectly embedded. It comes out that the groups satisfying this property are precisely the *hypocentral* groups, that are those groups whose transfinite lower central series terminates at the trivial subgroup. This class of groups

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also coincides with the class of groups in which no non-trivial subgroup is perfectly embedded in its normalizer.

Theorem 2.1. *Let G be a group. The following statements are equivalent:*

(i) $[H, G] < H$ *for every non-trivial normal subgroup* H *of* G*;*

(ii) G *is hypocentral;*

(iii) $[H, N_G(H)] < H$ *for every non-trivial subgroup* H *of* G.

Proof. (i) $\Rightarrow$ (ii). Let G_ω denote the hypocenter of G, that is the intersection of all members of the transfinite lower central series of G. By definition, $[G_\omega, G] = G_\omega$. So $G_\omega = 1$ by hypothesis. Hence G is hypocentral.

(ii) $\Rightarrow$ (i). Assume by contradiction that there exists a non-trivial normal subgroup H of G such that $[H, G] = H$. Arguing by transfinite induction, we will show that H is contained in every term of the transfinite lower central series of G.

The base step is immediate, since $H = [H, G] \leq [G, G]$. Assume that $H \leq \gamma_\alpha(G)$ for some non-limit ordinal α. Then

$$H = [H, G] \leq [\gamma_\alpha(G), G] = \gamma_{\alpha+1}(G).$$

Now let α be a limit ordinal, and assume that $H \leq \gamma_\beta(G)$ for all $\beta < \alpha$. Then

$$H \leq \bigcap_{\beta<\alpha} \gamma_\beta(G) = \gamma_\alpha(G).$$

Thus H is contained in the hypocenter of G, contradicting the assumption that G is hypocentral.

(iii) $\Rightarrow$ (i). This is obvious.

(i) $\Rightarrow$ (iii). Since (i) is equivalent to (ii), the group G is hypocentral. As hypocentrality is inherited by subgroups, $N_G(H)$ is also hypocentral. Since H is normal in $N_G(H)$, it follows that $[H, N_G(H)] < H$. □

Recall that a group is called *hypoabelian* if its transfinite derived series reaches the trivial subgroup, or equivalently, if its perfect core is trivial. Such groups naturally appear when considering groups in which every non-trivial normal subgroup is not perfectly embedded in itself.

Theorem 2.2. *A group G is hypoabelian if and only if $[H, H] < H$ for every non-trivial normal subgroup H of G.*

Proof. Assume that G is hypoabelian and suppose, by contradiction, that there exists a non-trivial normal subgroup H of G such that $[H, H] = H$. Then H is contained in every term of the derived series of G, and hence lies in the perfect core of G, contradicting hypoabelianity.

Conversely, assume that $[H, H] < H$ for every non-trivial normal subgroup H of G. Let G_α denote the perfect core of G. Then $[G_\alpha, G_\alpha] = G_\alpha$, which forces $G_\alpha = 1$. Hence G is hypoabelian. □

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3. Groups in which every normal subgroup is perfectly embedded

Let $\mathcal{X}$ denote the class of groups in which all normal subgroups are perfectly embedded. In the sequel, sometimes we will say that G is an $\mathcal{X}$-group meaning that the group G belongs to the class $\mathcal{X}$.

Of course, the class $\mathcal{X}$ contains no non-trivial soluble groups, and every $\mathcal{X}$-group G necessarily has trivial center.

The class $\mathcal{X}$ contains all non-abelian simple groups and it is contained in the class of perfect groups. Both inclusions are strict: for instance, the group $GF(2^4) \rtimes H$, where H is a subgroup of $GL(4,2)$ isomorphic to A_5 (namely $\text{PerfectGroup}(960,1)$), is a non-simple $\mathcal{X}$-group, whereas $SL(2,5)$ is perfect but does not belong to $\mathcal{X}$.

We point out that there exist perfect groups with trivial center that do not belong to $\mathcal{X}$, such as $\text{PerfectGroup}(1920,4)$.

Using simple groups, one can construct numerous examples of groups in the class $\mathcal{X}$. For instance, if A and H are non-abelian simple groups then the restricted wreath product $G = A wr H$ is a group belonging to the class $\mathcal{X}$. This is true since the normal subgroups of G are only $\{1\}$, the base group $A^{(H)}$ and G itself. In particular, we point out that if A is infinite, then G is an infinite non-simple group belonging to $\mathcal{X}$.

Obviously every non-trivial $\mathcal{X}$-group has subgroups which does not belong to $\mathcal{X}$. Nevertheless, the class $\mathcal{X}$ has remarkable closure properties.

Proposition 3.1. *Every homomorphic image of an $\mathcal{X}$-group is an $\mathcal{X}$-group.*

Proof. Let G be an $\mathcal{X}$-group, and let H be a normal subgroup of G. If K/H is a normal subgroup of G/H, then

$$[K/H, G/H] = [K,G]/H = K/H,$$

showing that G/H is an $\mathcal{X}$-group. $\square$

From Proposition 3.1 we derive the following criterion.

Proposition 3.2. *A group G is an $\mathcal{X}$-group if and only if $\mathrm{Z}(G/N)$ is trivial for every normal subgroup N of G.*

Proof. If G is an $\mathcal{X}$-group, then G/N is an $\mathcal{X}$-group by Proposition 3.1, and hence $\mathrm{Z}(G/N)$ is trivial. Conversely, assume that G does not belong to the class $\mathcal{X}$, so there exists a normal subgroup H of G such that $[H,G] < H$. Then $[H,G]$ is normal in G, hence $\mathrm{Z}(G/[H,G])$ has to be trivial. This is a contradiction, as $H/[H,G] \leq \mathrm{Z}(G/[H,G])$. $\square$

Moreover, since the Frobenius complement of a finite Frobenius group has non-trivial center, Proposition 3.2 implies that the class $\mathcal{X}$ does not contain any finite Frobenius group.

It is well-known that a normal subgroup of the direct product of two groups need not coincide with the direct product of its canonical projections, as shown by Goursat's Lemma (see, for instance, [14,

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Proposition 5.2.5]). However, Proposition 3.3 will show that this is the case when considering any normal subgroup of a direct product of $\mathcal{X}$-groups.

Proposition 3.3. *The class $\mathcal{X}$ is closed under taking direct products of its members.*

Proof. Let G be a group and let G_1, G_2 be $\mathcal{X}$-subgroups of G such that $G = G_1 \times G_2$. Let H be a normal subgroup of G, and denote by π_1 and π_2 the canonical projections of H onto G_1 and G_2, respectively. Clearly,

$$H \leq \pi_1(H) \times \pi_2(H).$$

Let $h_1 \in \pi_1(H)$ and $g_1 \in G_1$. Then there exists $h_2 \in \pi_2(H)$ such that $h_1h_2 \in H$. Thus $[h_1h_2, g_1] \in H$. Since $[h_1, g_1] = [h_1h_2, g_1]$, we get $[\pi_1(H), G_1] \leq H$. Similarly, $[\pi_2(H), G_2] \leq H$, and therefore

$$[\pi_1(H), G_1] \times [\pi_2(H), G_2] \leq H.$$

Now $[\pi_1(H), G_1] = \pi_1(H)$ and $[\pi_2(H), G_2] = \pi_2(H)$, as G_1 and G_2 are $\mathcal{X}$-groups. Hence

$$\pi_1(H) \times \pi_2(H) \leq H,$$

and equality follows. Finally,

$$H = [\pi_1(H), G_1] \times [\pi_2(H), G_2] = [\pi_1(H) \times \pi_2(H), G_1 \times G_2] = [H, G].$$

This shows that G is an $\mathcal{X}$-group, as required. □

Notice that, by Propositions 3.1 and 3.3, the study of $\mathcal{X}$-groups reduces to the indecomposable case. Even though the class $\mathcal{X}$ is not closed under taking normal subgroups, as shown by PerfectGroup$(960, 1)$, we manage to give further details about the normal subgroup lattice of finite $\mathcal{X}$-groups. As usual, for a finite group G and a prime p, we denote by $O_p(G)$ the largest normal p-subgroup of G.

Proposition 3.4. *Let G be a finite $\mathcal{X}$-group with abelian Sylow 2-subgroups. Then $O_2(G)$ is trivial.*

Proof. Set $C = C_G(O_2(G))$. Then C contains a Sylow 2-subgroup of G. By Proposition 3.1, the quotient group G/C is an $\mathcal{X}$-group. Since G/C has odd order, by Feit-Thompson Theorem, it is soluble. Hence $G = C$, so $O_2(G) \leq Z(G) = 1$. □

Proposition 3.5. *Let G be an $\mathcal{X}$-group, and let N be a cyclic normal subgroup of G. Then $N = 1$.*

Proof. By Proposition 3.1, $G/C_G(N)$ is an $\mathcal{X}$-group which embeds into the abelian group $\mathrm{Aut}(N)$. Hence $G/C_G(N) = 1$, so $G = C_G(N)$, giving $N \leq \mathrm{Z}(G) = 1$. □

Next we describe the structure of $\mathcal{T}$-groups belonging to the class $\mathcal{X}$. Recall that a group G is said to be a $\mathcal{T}$-group if normality is a transitive relation in the subgroup lattice of G.

Lemma 3.6. *Let G be a finite perfect group, N a non-abelian simple normal subgroup of G, and $C = C_G(N)$. Then $G = N \times C$.*

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Proof. By the Schreier Conjecture, now proved, $\mathrm{Out}(N)$ is soluble. Since G/C embeds into $\mathrm{Aut}(N)$, the quotient G/CN is also soluble. As G is perfect, it follows that $CN = G$, and hence $G = N \times C$. □

Theorem 3.7. *A finite $\mathcal{T}$-group belongs to $\mathcal{X}$ if and only if it is a direct product of non-abelian simple groups.*

Proof. Let N be a minimal normal subgroup of a $\mathcal{T}$-group G. Then N is simple, and it is non-abelian by Proposition 3.5. By Lemma 3.6, $G = N \times C$ where $C = C_G(N)$. In particular C is an $\mathcal{X}$-group by Proposition 3.1. Since normal subgroups of $\mathcal{T}$-groups are again $\mathcal{T}$-groups, an inductive argument yields the result.

The converse immediately follows by Proposition 3.3. □

4. Groups in which every subgroup is perfectly embedded in some normalizing subgroup

It seems rather natural to consider groups G in which every subgroup H is perfectly embedded in some suitable subgroup K which normalizes H. In this direction, the most obvious options are $K = H$ and $K = N_G(H)$. The former clearly implies that the group G is trivial. Actually, also the second option turns out to be too restrictive. Indeed, we prove that every non-trivial group contains a subgroup that is not perfectly embedded in its normalizer.

Proposition 4.1. *Let G be a group such that $[H, N_G(H)] = H$ for all $H \leq G$. Then G is trivial.*

Proof. First we will show that G is a periodic group. Indeed, assume for a contradiction that G contains an element x of infinite order, and set $H = \langle x \rangle$. The normalizer $N_G(H)$ acts on H by conjugation, which must be non-trivial by hypothesis. Hence, for all $g \in N_G(H)$, we have $x^g = x^{-1}$. For all $n \in \mathbb{Z}$ and $g \in N_G(H)$ we get

$$[x^n, g] = x^{-n} g^{-1} x^n g = x^{-2n}.$$

Thus $[H, \langle g \rangle] = \langle x^2 \rangle$, a proper subgroup of H, and therefore $[H, N_G(H)] = \langle x^2 \rangle \neq H$, contradicting the hypothesis. Then G is periodic, as claimed.

Now we will prove that G is trivial. Indeed, arguing by contradiction suppose that G is non-trivial. As G is periodic, let p be the smallest prime dividing the order of an element of G, and let $x \in G$ have order p. Then $N_G(\langle x \rangle)/C_G(x)$ embeds into $\mathrm{Aut}(\langle x \rangle)$, whose order is $p - 1$. Hence, for every $g \in N_G(\langle x \rangle)$, we have $g^{p-1} \in C_G(x)$. By minimality of p, this implies $g \in C_G(x)$, so $N_G(\langle x \rangle) = C_G(x)$. Consequently,

$$[\langle x \rangle, N_G(\langle x \rangle)] = 1,$$

contradicting the hypothesis. □

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We point out that weakening our requirement applying the above conditions only to normal subgroups leads to already studied classes. Indeed, groups in which every normal subgroup is perfectly embedded in itself are known as *extra-perfect groups*. These groups have been investigated from a different perspective, namely through their composition factors. More precisely, finite extra-perfect groups are characterized as those groups in which every composition factor is a non-abelian simple group. For a proof of this result, see [2]. On the other hand, groups in which every normal subgroup is perfectly embedded in its normalizer are precisely the $\mathcal{X}$-groups.

5. Perfect Embeddings of Abelian Normal Subgroups

We start this section considering the class $\mathcal{W}$ of groups in which no non-trivial abelian normal subgroup is perfectly embedded. Natural examples of $\mathcal{W}$-groups are hypocentral groups (see Theorem 2.1), and *semisimple* groups, that are groups with no non-trivial abelian normal subgroups. Moreover, $SL(2,5)$ also belongs to $\mathcal{W}$, since the only non-trivial abelian normal subgroup is its center. While this class remains not well understood, we provide some useful structural constraints.

Proposition 5.1. *Every abelian minimal normal subgroup of a $\mathcal{W}$-group is central.*

Proof. Let G be a $\mathcal{W}$-group, and let H be an abelian minimal normal subgroup of G. By hypothesis, $[H,G] < H$. Since $[H,G]$ is a normal subgroup of G, the minimality of H implies that $[H,G] = 1$, so $H \leq \mathrm{Z}(G)$. □

Notice that the previous result implies that the abelian part of the socle of a $\mathcal{W}$-group is central. The class $\mathcal{W}$ is not closed under quotients, as the group $SL(2,3)$ belongs to $\mathcal{W}$ while it has a quotient isomorphic to the alternating group A_4 which is not a $\mathcal{W}$-group. Indeed, if a finite soluble group is not nilpotent, then at least one of its quotients is not a $\mathcal{W}$-group.

Proposition 5.2. *Let G be a finite soluble group such that every factor group of G is a $\mathcal{W}$-group. Then G is nilpotent.*

Proof. Let N be an abelian minimal normal subgroup of G. By Proposition 5.1, $N \leq \mathrm{Z}(G)$, so $\mathrm{Z}(G)$ is non-trivial. Applying the same argument iteratively to $G/\mathrm{Z}(G)$, we obtain that the upper central series reaches G, showing that G is nilpotent. □

Finally, we consider the class $\mathcal{Z}$ of groups in which every abelian normal subgroup is perfectly embedded. All semisimple groups are $\mathcal{Z}$-groups, and actually these are the only groups belonging to both $\mathcal{Z}$ and $\mathcal{W}$. Since the symmetric group S_n of degree $n \geq 3$ belongs to $\mathcal{Z}$, this class strictly contains the above class $\mathcal{X}$, as S_n is not in $\mathcal{X}$ for $n > 4$.

A non-trivial $\mathcal{Z}$-group cannot be nilpotent, as every $\mathcal{Z}$-group has trivial center, since $[\mathrm{Z}(G), G] = 1$. However, there exist soluble $\mathcal{Z}$-groups: the smallest example is S_3.

We conclude this section by showing that every Frobenius group is a $\mathcal{Z}$-group.

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Lemma 5.3. *Let $G = A \rtimes H$ be a finite group with A abelian. If there exists $h \in H$ such that $C_A(h) = 1$, then $[A, H] = A$. In particular, every element of A can be expressed as a commutator $[a, h]$ for some $a \in A$.*

Proof. Set $C = \{[a, h] \mid a \in A\}$ and consider the map $f : A \to C$ defined by $f(a) = [a, h]$. This map is injective: if $[a_1, h] = [a_2, h]$, then $a_1 a_2^{-1} \in C_A(h) = 1$, giving $a_1 = a_2$. Since $C \subseteq A$, we get $A = C$ and the result follows. □

Proposition 5.4. *Let $G = K \rtimes H$ be a finite group such that every abelian normal subgroup of G is contained in K. If there exists an element $h \in H$ with $C_K(h) = 1$, then G is a $\mathcal{Z}$-group.*

Proof. Let A be an abelian normal subgroup of G. By hypothesis $A \leq K$, and thus $C_A(h) = 1$. By Lemma 5.3, $[A, H] = A$. Since $[A, H] \leq [A, G] \leq A$, we obtain $[A, G] = A$. Therefore G is a $\mathcal{Z}$-group. □

Corollary 5.5. *Every finite Frobenius group is a $\mathcal{Z}$-group.*

Proof. Let G be a finite Frobenius group. Since the Frobenius kernel of G is the Fitting subgroup of G, then every abelian normal subgroup of G lies in the Frobenius kernel of G. Now the result follows by Proposition 5.4. □

Appendix

We include below the GAP procedure used to test whether a finite group belongs to the class $\mathcal{X}$ considered in this paper.

```
IsClassX := function(G)
    local N;

    for N in NormalSubgroups(G) do
       if CommutatorSubgroup(N, G) <> N then
            return false;
        fi;
    od;

    return true;
end;
```

Listing 1. Procedure to test membership in $\mathcal{X}$

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