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A NOTE ON THE BINOMIAL FIBONACCI SUMS

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Abstract. In this paper, we extend the recently obtained weighted binomial Fibonacci identities to arbitrary second-order recurrence sequences and to the Fibonacci m-step numbers. This work is inspired by a recently published problem and solution in the journal The Fibonacci Quarterly.

1. Motivation and Preliminaries

The literature on the binomial Fibonacci sums is quite rich. There are many articles on this subject, including the recent ones [3, 11, 12, 16] as well as the classic ones such as [4, 5, 10, 14]. Many problems related to such sums appear in the Elementary Problems and Solutions and Advanced Problems and Solutions of The Fibonacci Quarterly. In [7] the following problem of proving that the following binomial sums hold was suggested:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{k+1}=\frac{F_{2n+1}+L_{2n+1}}{n+1},\qquad \sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{(k+1)(k+2)}=\frac{F_{2n+2}+L_{2n+2}-2}{(n+1)(n+2)}. \tag{1.1}$$

Ventas [18] provided a solution to this problem. In fact, he proved a generalization of these two identities to the Gibonacci sequence. The result is as follows:

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Theorem 1.1. *Let $G_{1,k}$ and $G_{2,k}$ be any two Gibonacci sequences. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{G_{1,k}+G_{2,k}}{k+1}=\frac{G_{1,2n+1}+G_{2,2n+1}+\Delta_1+\Delta_2}{n+1},$$
$$\sum_{k=0}^{n}\binom{n}{k}\frac{G_{1,k}+G_{2,k}}{(k+1)(k+2)}=\frac{G_{1,2n+2}+G_{2,2n+2}-\Delta_{1,n}-\Delta_{2,n}}{(n+1)(n+2)},$$

where

$$\Delta_k=G_{k,1}-G_{k,0},\qquad \Delta_{k,n}=G_{k,1}+(n-2)(G_{k,1}-G_{k,0}).$$

The goal of this note is to extend Theorem 1.1 to Horadam numbers, Fibonacci-like polynomials and to the Fibonacci m-step numbers. These identities are presumably new and they possess the same structure as the original ones.

We note that in parallel to our research, Frontczak [8] also investigated (independently) the extension of (1.1). However, his approach was quite different and has led to different identities, limited to the classic Fibonacci and Lucas numbers, but with the remark that they could be extended to the Gibonacci or the Horadam numbers (we refer to classic books by Koshy [13] and Vajda [17] for more information on these sequences).

To this end, let $H_n=H_n(a,b;p,q)$ for $n\in\mathbb{Z}$ denote the Horadam sequence, which is given by

$$H_n=pH_{n-1}-qH_{n-2}$$

and $H_0=a$, $H_1=b$. We extend Horadam numbers to negative indices via relation

$$H_{-n}=\frac{1}{q}(pH_{-n+1}-H_{-n+2}).$$

If the context is clear, we will shorten the notation to H_n. We also have $F_n=H_n(0,1;1,-1)$, $L_n=H_n(2,1;1,-1)$, and $G_n=H_n(a,b;1,-1)$, the Fibonacci, the Lucas, and the Gibonacci numbers, respectively. We impose the obvious restriction $q\neq 0$ and $(a,b)\neq(0,0)$.

Associated with H_n are the Lucas sequences of the first kind $u_n(p,q)=H_n(0,1;p,q)$ and of the second kind $v_n(p,q)=H_n(2,p;p,q)$.

Horadam numbers in the non-degenerate case $\Delta=p^2-4q>0$ possess a Binet-type formula

$$H_n=\frac{A\sigma^n-B\tau^n}{\sigma-\tau},\qquad n\in\mathbb{Z},$$

where $A=b-a\tau$ and $B=b-a\sigma$, and τ,σ are the roots of the characteristic equation $x^2-px+q=0$. In particular,

$$\sigma=\sigma(p,q)=\frac{p+\sqrt{\Delta}}{2},\quad \tau=\tau(p,q)=\frac{p-\sqrt{\Delta}}{2},$$

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and $\sigma+\tau=p$, $\sigma\tau=q$, $\sigma-\tau=\sqrt{\Delta}$. We also have

$$u_n=\frac{\sigma^n-\tau^n}{\sigma-\tau}, \qquad v_n=\sigma^n+\tau^n.$$

Finally,

$$p\sigma-q=\sigma^2, \qquad p\tau-q=\tau^2.$$

We also have the following simple lemma, which can be easily verified via direct computation.

Lemma 1.2. *For all $r\neq 0$ we have the relations*

$$\sigma^r=\sigma u_r-qu_{r-1}, \qquad \text{and} \qquad \tau^r=\tau u_r-qu_{r-1} \tag{1.2}$$

and

$$\sigma^r\sqrt{\Delta}=\sigma v_r-qv_{r-1}, \qquad \text{and} \qquad \tau^r\sqrt{\Delta}=-\tau v_r+qv_{r-1}. \tag{1.3}$$

The Fibonacci m-step numbers $F_n^{(m)}$ are defined via the recurrence

$$F_n^{(m)}=F_{n-1}^{(m)}+\cdots+F_{n-m}^{(m)}$$

with the initial conditions $F_0^{(m)},\ldots,F_{m-1}^{(m)}$ being arbitrary, but fixed numbers. We extend these sequences to negative indices via the relation

$$F_{-n}^{(m)}=F_{-n+m}^{(m)}-F_{-n+m-1}^{(m)}-\cdots-F_{-n+1}^{(m)}.$$

We use $F_n^{(3)}=T_n$ and $F_n^{(4)}=Q_n$ to denote the Tribonacci and the Tetranacci sequences, respectively.

We let $\alpha_0,\ldots,\alpha_{m-1}$ denote the roots of the characteristic equation of the Fibonacci m-step numbers

$$x^m=x^{m-1}+\cdots+x+1$$

in the ring $\mathbb{C}$. Let $\alpha_{(m)}\in\{\alpha_0,\ldots,\alpha_{m-1}\}$. Then we have the following identity,

$$1+\alpha_{(m)}^{m+1}=2\alpha_{(m)}^m, \tag{1.4}$$

which can be easily verified. Fibonacci m-step numbers satisfy a Binet-type formula: if $\alpha_0,\ldots,\alpha_{m-1}$ are the roots of the characteristic equation, then

$$F_n^{(m)}=\sum_{k=0}^{m-1}A_k\alpha_k^n,$$

where $A_k\in\mathbb{C}$. The numbers A_k are determined by the system of linear equations given by the identity

$$F_j^{(m)}=\sum_{k=0}^{m-1}A_k\alpha_k^j, \qquad j=0,\ldots,m-1.$$

Recall that if $r>0$ is an integer, then the rising factorial of x is defined as

$$x^{(r)}=\prod_{j=0}^{r-1}(x+j)$$

with particular cases $x^{(0)} = 1$ and $x^{(1)} = x$.

2. Extension to Horadam numbers

Before we address the generalization, we give a brief introductory comment. The identities provided in Theorem 1.1 are in fact based on a single sum of weighted Gibonacci numbers of the following form:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{G_k}{k+1}, \qquad \sum_{k=0}^{n} \binom{n}{k} \frac{G_k}{(k+1)(k+2)}.$$

The final result is a combination of these sums for two different Gibonacci sequences. Therefore, to derive any similar identity, it is sufficient to find a closed form for a single Gibonacci sequence.

Let us start with a simple observation related to the binomial expansion.

Lemma 2.1. *Let $0 \leq r < n$ and $s \geq 1$ be integers. Then*

$$\sum_{k=r}^{n} \binom{n}{k} u_s^k (-q)^{n-k} u_{s-1}^{n-k} H_{k-r} = H_{sn-r} - \Delta_{n,s}(r), \tag{2.1}$$

where

$$\Delta_{n,s}(r) = \sum_{k=0}^{r-1} \binom{n}{k} u_s^k (-q)^{n-k} u_{s-1}^{n-k} H_{k-r}. \tag{2.2}$$

Proof. We use (1.2) and calculate

$$\sigma^{sn} = (\sigma u_s - q u_{s-1})^n = \sum_{k=0}^{n} \binom{n}{k} \sigma^k u_s^k (-q)^{n-k} u_{s-1}^{n-k},$$

$$\tau^{sn} = (\tau u_s - q u_{s-1})^n = \sum_{k=0}^{n} \binom{n}{k} \tau^k u_s^k (-q)^{n-k} u_{s-1}^{n-k}.$$

Multiply through by σ^{-r} and τ^{-r}, respectively, and use the Binet formula to get

$$H_{sn-r} = \sum_{k=0}^{n} \binom{n}{k} u_s^k (-q)^{n-k} u_{s-1}^{n-k} H_{k-r}.$$

From that we get (2.1) and (2.2). □

Particular instances of the above lemma were used in [18], where the author considered the Gibonacci sequence.

Example 2.2. For the Gibonacci numbers, we obtain

$$\sum_{k=1}^{n} \binom{n}{k} G_{k-1} = G_{2n-1} - G_{-1} = G_{2n-1} - (G_1 - G_0)$$

and

$$\sum_{k=2}^{n} \binom{n}{k} G_{k-2} = G_{2n-2} - G_{-2} - nG_{-1}.$$

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Using $G_{-1} = G_1 - G_0$ and

$$G_{-2} = G_0 - G_{-1} = 2G_0 - G_1$$

we obtain

$$\sum_{k=2}^{n} \binom{n}{k} G_{k-2} = G_{2n-2} + (n-2)G_0 - (n-1)G_1.$$

This recovers the lemma in [18], where the author provided the identity and suggested induction as a means of proving it. Our consideration shows that this is not necessary.

A similar result to Lemma 2.1 can be stated by using identity (1.3).

Lemma 2.3. *Let $0 \le r < n$, $s \ge 1$ be an integer and let n be even. Then*

$$\sum_{k=r}^{n} \binom{n}{k} v_s^k (-q)^{n-k} v_{s-1}^{n-k} H_{k-r} = \Delta^{\frac{n}{2}} H_{sn-r} - \tilde{\Delta}_{n,s}(r),$$

where

$$\tilde{\Delta}_{n,s}(r) = \sum_{k=0}^{r-1} \binom{n}{k} v_s^k (-q)^{n-k} v_{s-1}^{n-k} H_{k-r}.$$

Proof. Use

$$(\tau^s \sqrt{\Delta})^n = (\tau v_s - q v_{s-1})^n,$$

which follows from (1.3) for even n. □

We now move to the main result of this section, which generalizes the initial identity in two different ways.

Theorem 2.4. *Let $0 \le r < n$ and $s \ge 1$ be integers. Then for any Horadam sequence H_n, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{u_s^k (-q)^{n-k} u_{s-1}^{n-k} H_k}{(k+1)^{(r)}} = \frac{1}{u_s^r (n+1)^{(r)}} \left(H_{s(n+r)-r} - \Delta_{n+r,s}(r) \right).$$

Proof. We have

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \frac{u_s^k (-q)^{n-k} u_{s-1}^{n-k} H_k}{(k+1)^{(r)}} &= \frac{1}{(n+1)^{(r)}} \sum_{k=0}^{n} \binom{n}{k} \frac{(n+1)^{(r)}}{(k+1)^{(r)}} u_s^k (-q)^{n-k} u_{s-1}^{n-k} H_k \\
&= \frac{1}{(n+1)^{(r)}} \sum_{k=0}^{n} \binom{n+r}{k+r} u_s^k (-q)^{n-k} u_{s-1}^{n-k} H_k \\
&= \frac{1}{u_s^r (n+1)^{(r)}} \sum_{k=r}^{n+r} \binom{n+r}{k} u_s^k (-q)^{n+r-k} u_{s-1}^{n+r-k} H_{k-r}.
\end{aligned}$$

The final result follows from Lemma 2.1. □

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Theorem 2.5. *Let $0 \le r < n$ and $s \ge 1$ be integers and let n be even. Then*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{v_s^k(-q)^{n-k} v_{s-1}^{n-k} H_k}{(k+1)^{(r)}} = \frac{1}{v_s^r (n+1)^{(r)}} \left(\Delta^{\frac{n}{2}} H_{s(n+r)-r} - \tilde{\Delta}_{n+r,s}(r) \right).$$

Example 2.6. Consider $J_n = H_n(0,1;1,-2)$ which are the Jacobsthal numbers. We have $p = 1$ and $q = -2$, thus for $r = 1$ and $s = 2$,

$$\Delta_{n+1,2}(1) = \binom{n+1}{0} 2^{n+1} \cdot J_{-1} = 2^n$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{n-k} J_k}{k+1} = \frac{J_{2n+1} - 2^n}{n+1}.$$

Furthermore, setting $r = 2$ we obtain

$$\Delta_{n+2,2}(2) = \binom{n+2}{0} 2^{n+2} J_{-2} + \binom{n+2}{1} 2^{n+1} J_{-1} = (n+1)2^n$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{n-k} J_k}{(k+1)(k+2)} = \frac{J_{2n+2} - (n+1)2^n}{(n+1)(n+2)}.$$

We used $J_{-1} = \frac{1}{2}$ and $J_{-2} = -\frac{1}{4}$. Furthermore, since

$$\tilde{\Delta}_{n+2,2}(2) = \binom{n+2}{0} 2^{n+2} J_{-2} + \binom{n+2}{1} 5 \cdot 2^{n+1} J_{-1} = (5n+9)2^n,$$

(where $v_2 = 5$), this implies

$$\sum_{k=0}^{n} \binom{n}{k} \frac{5^k 2^{n-k} J_k}{(k+1)(k+2)} = \frac{3^{n+2} J_{2n+2} - (5n+9)2^n}{25(n+1)(n+2)},$$

valid for even n.

Example 2.7. Consider the Pell numbers $P_n = H_n(0,1;2,-1)$. Here,

$$\begin{aligned}
\Delta_{n+1,2}(1) &= \binom{n+1}{0} P_{-1} = 1, \\
\Delta_{n+2,2}(2) &= \binom{n+2}{0} P_{-2} + \binom{n+2}{1} 2P_{-1} = -2 + 2(n+2) = 2n+2, \\
\Delta_{n+3,2}(3) &= \binom{n+3}{0} P_{-3} + \binom{n+3}{1} 2P_{-2} + \binom{n+3}{2} 4P_{-1} = 2n^2 + 6n + 5.
\end{aligned}$$

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Thus,

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^kP_k}{k+1}=\frac{P_{2n+1}-1}{2(n+1)},$$
$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^kP_k}{(k+1)(k+2)}=\frac{P_{2n+2}-2n-2}{4(n+1)(n+2)},$$
$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^kP_k}{(k+1)(k+2)(k+3)}=\frac{P_{2n+3}-2n^2-6n-5}{8(n+1)(n+2)(n+3)}.$$

3. Two applications

Since we have evaluated the closed form related to Horadam sequence, we now present two applications.

First, we extend the result to sums along arithmetic progression, that is we consider a subsequence H_{tk+s} of the sequence H_k. Then, it is possible to evaluate

$$\sum_{k=0}^{n}\binom{n}{k}\frac{Q^{n-k}P^kH_{tk+s}}{(k+1)^{(r)}},$$

where $t,s>0$ with $t\neq 0$ and Q,P are such that

$$H_{tn+s}=PH_{t(n-1)+s}+QH_{t(n-2)+s}.$$

Using the matrix form of the Horadam sequence, the numbers P,Q satisfy

$$\begin{bmatrix}H_s & H_{t+s}\\ H_{t+s} & H_{2t+s}\end{bmatrix}\begin{bmatrix}Q\\ P\end{bmatrix}=\begin{bmatrix}H_{2t+s}\\ H_{3t+s}\end{bmatrix},$$

or

$$P=\frac{H_sH_{3t+s}-H_{t+s}H_{2t+s}}{H_sH_{2t+s}-H_{t+s}^2},\qquad Q=\frac{H_{2t+s}^2-H_{t+s}H_{3t+s}}{H_sH_{2t+s}-H_{t+s}^2}.$$

From the above and Theorem 2.4 we can now easily obtain the following.

Corollary 3.1. *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k}3^kF_{2k}}{k+1}=\frac{F_{4n+2}-(-1)^n}{3n+3}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k}3^kF_{2k+1}}{k+1}=\frac{F_{4n+3}-(-1)^{n+1}}{3n+3}.$$

In particular, $3|F_{4n+4}$.

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Proof. Follows from the fact that F_n satisfies the following:

$$F_{2n+\delta} = 3F_{2n-2+\delta} - F_{2n-4+\delta}, \qquad \delta \in \{0,1\},$$
$$F_{-1} = 1, \qquad F_{-2} = -1.$$

To see the divisibility property, add the two identities and multiply them by $n+1$. This makes the left-hand side an integer. □

The second application is to the Fibonacci-like polynomials. These polynomials are given by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 2$$

and $F_0(x) = 0, F_1(x) = 1$. They are the Horadam sequences with $p = x$ and $q = -1$. Similarly, Chebyshev polynomials satisfy the Horadam-like recurrence

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x).$$

If $V_0(x) = 1$ and $V_1(x) = x$, then $V_n = T_n$ are the Chebyshev polynomials of the first kind, while if $V_0(x) = 1$ and $V_1(x) = 2x$, $V_n = U_n$ are the Chebyshev polynomials of the second kind. For example, we can compute the following series of identities.

Corollary 3.2. *We have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x^k F_k(x)}{k+1} = \frac{1}{x(n+1)} \left(F_{2n+1}(x) - 1\right),$$
$$\sum_{k=0}^{n} \binom{n}{k} \frac{x^k F_k(x)}{(k+1)(k+2)} = \frac{1}{x^2(n+1)(n+2)} \left(F_{2n+2}(x) - (n+1)x\right),$$
$$\sum_{k=0}^{n} \binom{n}{k} \frac{(2x)^k(-1)^{n-k} T_k(x)}{k+1} = \frac{1}{2x(n+1)} \left(T_{2n+1}(x) - (-1)^{n+1}x\right),$$
$$\sum_{k=0}^{n} \binom{n}{k} \frac{(2x)^k(-1)^{n-k} T_k(x)}{(k+1)(k+2)} = \frac{1}{4x^2(n+1)(n+2)} \left(T_{2n+2}(x) - (-1)^{n-1}(2(n+1)x^2+1)\right),$$
$$\sum_{k=0}^{n} \binom{n}{k} \frac{(2x)^k(-1)^{n-k} U_k(x)}{k+1} = \frac{1}{2x(n+1)} U_{2n+1}(x),$$
$$\sum_{k=0}^{n} \binom{n}{k} \frac{(2x)^k(-1)^{n-k} U_k(x)}{(k+1)(k+2)} = \frac{1}{4x^2(n+1)(n+2)} \left(U_{2n+2}(x) - (-1)^{n+1}\right).$$

4. Some identities for the Fibonacci m-step numbers

In this section we work with the Fibonacci m-step numbers $F_n^{(m)}$.

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Lemma 4.1. *For any $m > 1$, $n > 0$ and any $s \in \mathbb{Z}$ we have*

$$2^n F^{(m)}_{mn+s} = \sum_{k=0}^{n} \binom{n}{k} F^{(m)}_{k(m+1)+s} \tag{4.1}$$

Proof. We use (1.4) to obtain

$$2^n \alpha^{mn}_{(m)} = (2\alpha^m_{(m)})^n = (1 + \alpha^{m+1}_{(m)})^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^{k(m+1)}_{(m)},$$

valid for any $\alpha_{(m)}$. Combining this with the Binet formula, we obtain (4.1). □

Theorem 4.2. *For any $n > 0$ and $0 < r < n$ integers we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{F^{(m)}_{k(m+1)}}{(k+1)^{(r)}} = \frac{1}{(n+1)^{(r)}} \left(2^{n+r} F^{(m)}_{mn-r} - \Delta^{(m)}_{n+r}(r) \right),$$

where

$$\Delta^{(m)}_{n+r}(r) = \sum_{k=0}^{r-1} \binom{n+r}{k} F^{(m)}_{(k-r)(m+1)}.$$

Proof. Using the previous Lemma with $s = -r(m+1)$ and $n \mapsto n + r$, we get

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)^{(r)}} F^{(m)}_{k(m+1)} &= \frac{1}{(n+1)^{(r)}} \sum_{k=0}^{n} \binom{n+r}{k+r} F^{(m)}_{k(m+1)} \\
&= \frac{1}{(n+1)^{(r)}} \sum_{k=r}^{n+r} \binom{n+r}{k} F^{(m)}_{(k-r)(m+1)} \\
&= \frac{1}{(n+1)^{(r)}} \left(\sum_{k=0}^{n+r} \binom{n+r}{k} F^{(m)}_{(k-r)(m+1)} - \Delta^{(m)}_{n+r}(r) \right) \\
&= \frac{1}{(n+1)^{(r)}} \left(2^{n+r} F^{(m)}_{mn-r} - \Delta^{(m)}_{n+r}(r) \right).
\end{aligned}$$

The proof is completed. □

Corollary 4.3. *The following identities hold:*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{T_{4k}}{k+1} = \frac{1}{n+1} \left(2^{n+1} T_{3n-1} - T_{-4} \right),$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{T_{4k}}{(k+1)(k+2)} = \frac{1}{(n+1)(n+2)} \left(2^{n+2} T_{3n-2} - T_{-8} - (n+2) T_{-4} \right),$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{Q_{5k}}{k+1} = \frac{1}{n+1} \left(2^{n+1} Q_{4n-1} - Q_{-5} \right),$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{Q_{5k}}{(k+1)(k+2)} = \frac{1}{(n+1)(n+2)} \left(2^{n+2} Q_{4n-2} - Q_{-10} - (n+2) Q_{-5} \right).$$

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We recall the following two polynomial identities:

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+2}x^{k+2}(1+x)^{n-k}=\frac{(1+x)^{n+2}-(n+2)x-1}{(n+1)(n+2)},$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+2}x^{k}(1+x)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{(k+1)(k+2)}.$$

These identities come from Dattoli et al. [6] (see also [9, Identities 1.37-1.41]). Combining these we get

$$\sum_{k=0}^{n}\binom{n}{k}\frac{x^{k+2}}{(k+1)(k+2)}=\frac{(1+x)^{n+2}-(n+2)x-1}{(n+1)(n+2)}, \tag{4.2}$$

which is the base for our next result.

Theorem 4.4. *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F^{(m)}_{(k+2)(m+1)+s}}{(k+1)(k+2)}=\frac{2^{n+2}F^{(m)}_{(n+2)(m+1)+s}-(n+2)F^{(m)}_{m+1+s}-F^{(m)}_{s}}{(n+1)(n+2)}.$$

Proof. Substitute $x=\alpha_{(m)}^{m+1}$ to (4.2) and use (1.4). Multiply by $\alpha_{(m)}^{s}$ and combine with the Binet formula. ☐

We note that setting $s=-2(m+1)$ restores Theorem 4.2 in case $r=2$.

5. Further Tribonacci and Tetranacci identities

In this section, we will use the following lemma [15], which is essentially a restatement of the binomial expansion.

Lemma 5.1. *We have*

$$(1+x+xy)^n=\sum_{0\le j\le i\le n}\binom{n}{i}\binom{i}{j}x^iy^j$$

and

$$(1+x+xy+xyz)^n=\sum_{0\le k\le j\le i\le n}\binom{n}{i}\binom{i}{j}\binom{j}{k}x^iy^jz^k.$$

Lemma 5.2. *We have*

$$\sum_{0\le j\le i\le n}\binom{n}{i}\binom{i}{j}T_{i+j+r}=T_{3n+r}$$

and

$$\sum_{0\le k\le j\le i\le n}\binom{n}{i}\binom{i}{j}\binom{j}{k}Q_{i+j+k+r}=Q_{4n+r}$$

for any integers n and r.

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Proof. We only prove the first identity. Set $t = \alpha_{(3)}$. Use $1 + t + t^2 = t^3$ with combination of Lemma 5.1 with $x = y = t$ to get

$$t^{3n} = (1 + t + t^2)^n = \sum_{0\le j\le i\le n} \binom{n}{i}\binom{i}{j} t^{i+j}.$$

Multiply by t^r and apply the Binet formula. □

Using the idea similar to Theorem 2.4 we can obtain the following two results.

Theorem 5.3. *For any $n \ge 1$ we have*

$$\sum_{0\le j\le i\le n} \binom{n}{i}\binom{i}{j} \frac{T_{i+j}}{j+1} = \frac{1}{n+1}\left(T_{3n+1} - \sum_{i=0}^{n+1}\binom{n+1}{i} T_{i-2}\right)$$

and

$$\sum_{0\le k\le j\le i\le n} \binom{n}{i}\binom{i}{j}\binom{j}{k} \frac{Q_{i+j+k}}{j+1} = \frac{1}{n+1}\left(Q_{4n+1} - \sum_{0\le j\le i\le n+1}\binom{n+1}{i}\binom{i}{j} Q_{i+j-3}\right).$$

Proof. We only prove the first identity - the proof of the second one is similar. We have

$$\binom{n}{i}\binom{i}{j}\frac{1}{j+1} = \frac{1}{n+1}\binom{n+1}{i+1}\binom{i+1}{j+1},$$

and thus

$$\begin{aligned}
\sum_{0\le j\le i\le n} \binom{n}{i}\binom{i}{j} \frac{T_{i+j}}{j+1} &= \frac{1}{n+1}\sum_{0\le j\le i\le n}\binom{n+1}{i+1}\binom{i+1}{j+1} T_{i+j} \\
&= \frac{1}{n+1}\sum_{1\le j\le i\le n+1}\binom{n+1}{i}\binom{i}{j} T_{i+j-2} \\
&= \frac{1}{n+1}\left(T_{3n+1} - \sum_{i=0}^{n+1}\binom{n+1}{i} T_{i-2}\right).
\end{aligned}$$

We utilized Lemma 5.2 in the final equality. □

Remark 5.4. *The closed form of the binomial sum*

$$\sum_{k=0}^{n}\binom{n}{k} T_k$$

remains unsolved. Similar formulas are known in various different cases, see for example [1] *and also* [2]*, where the following is evaluated*

$$\sum_{k=0}^{n}\binom{n}{k} T_{4k+r} = 2^n T_{3n+r}.$$

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