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ON F-ZARISKI TOPOLOGY IN A POSET

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Abstract. Let Q be a partially ordered set, and let F be an ℓ-filter in Q. An ideal P in a poset Q with $P \cap F = \emptyset$ is called F-prime, if there exists a fixed element $f \in F$ such that whenever $(a,b)^{\ell} \subseteq P$, for some $a, b \in Q$ then $(f,a)^{\ell} \subseteq P$ or $(f,b)^{\ell} \subseteq P$. In this paper, we study a topology on the set $Spec_F(Q)$ of all F-prime ideals in Q, which is a generalization of the prime spectrum $Spec(Q)$ of a poset Q. We also investigate the relationship between order theoretic properties of Q and topological properties of $Spec_F(Q)$.

1. Introduction

The notion of the prime ideal has a significant role in the theory of modules, rings, posets, lattices, etc., and it is used to study certain classes of algebraic structures. For many years, various authors have constructed topologies over algebraic structures and investigated the relations between the algebraic properties of these structures and the topological properties of these topologies(see [1, 3, 6, 12, 14, 15, 17]). The concepts of an S-prime ideal and an S-prime submodule for a non-empty multiplicatively closed subset S of a commutative ring with unity can be found in Sevim et al. [14], Yildiz et al. [17]. Let S be a non-empty multiplicatively closed subset of a ring R with unity. An ideal P in R with $P \cap S = \emptyset$ is called an **S-prime ideal**, if there exists a fixed $s \in S$ and whenever $ab \in P$ for some $a, b \in R$, then either

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$sa \in P$ or $sb \in P$. In this paper, we generalize this concept to a partially ordered set Q. Throughout this article, we focus on the partially ordered set Q with the smallest element 0 and the greatest element 1.

We begin with the necessary preliminaries and terminology in posets.

Let Q be a partially ordered set with least element 0 and the greatest element 1. Let $A \subseteq Q$. The set $A^u = \{x \in Q \mid x \geq a$ for every $a \in A\}$ is called the **upper cone** of A. Dually, we have the concept of the **lower cone**, A^ℓ of A. $A^{u\ell}$ shall mean $\{A^u\}^\ell$ and $A^{\ell u}$ shall mean $\{A^\ell\}^u$. The upper cone $\{a\}^u$ is simply denoted by a^u and $\{a, b\}^u$ is denoted by $(a, b)^u$. Similar notations are used for lower cones. We note that $A \subseteq A^{u\ell}$ and $A \subseteq A^{\ell u}$. If $A \subseteq B$ then $A^\ell \supseteq B^\ell$ and $A^u \supseteq B^u$. Moreover, $A^{\ell u \ell} = A^\ell$, $A^{u\ell u} = A^u$, $\{a^u\}^\ell = a^\ell$ and $\{a^\ell\}^u = a^u$.

Let Q be a poset. A non-empty subset I of Q is a **semi-ideal** if $x \leq y$ and $y \in I$ imply that $x \in I$. A non-empty subset I of Q is said to be an **ideal** if for all $a, b \in I$ implies that $(a, b)^{u\ell} \subseteq I$. An ideal I in Q is called a **u-ideal** if for $x, y \in I$, $(x, y)^u \cap I \neq \emptyset$. Dually, we have concepts of a **filter** and an **ℓ-filter.**

A proper ideal(semi-ideal) I in a poset Q is called **prime** if for all $x, y \in Q$, $(x, y)^\ell \subseteq I$ implies $x \in I$ or $y \in I$. Dually, we have concept of a **prime filter.** The collection of all prime semi-ideals in Q is denoted by $P_s(Q)$ and the collection of all prime ideals in Q is denoted by $Spec(Q)$. If I is a semi-ideal in a poset Q then $V(I) = \{P \in P_s(Q) \mid I \subseteq P\}$. The complement of $V(I)$ is denoted by $D(I)$, that is $D(I) = \{P \in P_s(Q) \mid I \nsubseteq P\}$. The collection $\{D(I_\alpha) \mid I_\alpha$ is a semi-ideal of $Q, \alpha \in \Lambda\}$ forms a topology on $P_s(Q)$ called as the **Zariski** or **hull kernel topology**. For more details, one can see Venkatanarasimhan [16].

Let F be an ℓ-filter in a poset Q, then an ideal P in Q with $P \cap F = \emptyset$ is said to be an **F-prime** ideal, if there exists a fixed element $f \in F$ such that whenever $(a, b)^\ell \subseteq P$, for $a, b \in Q$, then $(f, a)^\ell \subseteq P$ or $(f, b)^\ell \subseteq P$. The set of all F-prime ideals in a poset Q is denoted by $Spec_F(Q)$.

2. F-Zariski topology

In this section, we study F-prime ideals and the Zariski topology on the space $Spec_F(Q)$ of F-prime ideals in a poset Q, where F is an ℓ-filter in Q.

Let a be an element in a poset Q. The **principal ideal** generated by a, denoted by $(a]$ and is defined as: $(a] = \{x \in Q \mid x \leq a\}$. Dually, we have the concept of a **principal filter.**

Example 2.1. *Consider the poset Q depicted in Figure 1. Observe that $I = (b]$ is a prime ideal in Q. If we consider an ℓ-filter $F = [b)$ then $I \cap F = \{b\}$. Therefore, $I = (b]$ is not an F-prime ideal for an ℓ-filter $F = [b)$. Thus, $(b] \in Spec(Q)$ but $(b] \notin Spec_F(Q)$ for an ℓ-filter $F = [b)$.*

Example 2.2. *Consider the poset Q depicted in Figure 2. Observe that, $(a, b)^\ell \subseteq (0]$ but $a \notin (0]$ and $b \notin (0]$. Therefore, $(0]$ is not a prime ideal in Q. If we consider an ℓ-filter $F = [b)$, then observe that $(0]$ is an F-prime ideal. Thus, $(0]$ is an F-prime ideal, but not a prime ideal in Q. Therefore, $(0] \in Spec_F(Q)$ but $(0] \notin Spec(Q)$.*

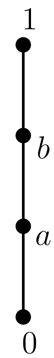

FIGURE 1.

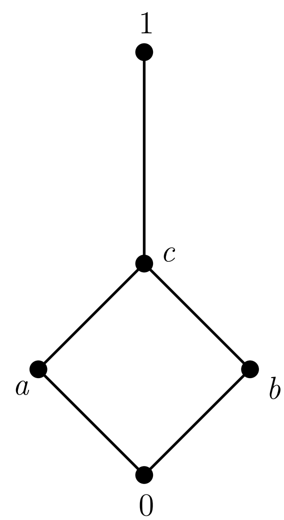

FIGURE 2.

Lemma 2.3. *Let Q be a poset, and let F be an ℓ-filter in Q. If P is a prime ideal in Q with $P \cap F = \emptyset$, then P is an F-prime ideal in Q.*

Proof. Suppose that, P is a prime ideal and F is an ℓ-filter in a poset Q with $P \cap F = \emptyset$. Let $(x, y)^\ell \subseteq P$ for some $x, y \in Q$. This implies that, $x \in P$ or $y \in P$. Therefore, $(x, 1)^\ell = (x] \subseteq P$ or $(y, 1)^\ell = (y] \subseteq P$ and hence P is an F-prime ideal. □

Lemma 2.4. *Let Q be a poset, and let F be an ℓ-filter in Q. If $F = [1)$ then P is a prime ideal if and only if P is an F-prime ideal in Q.*

Proof. Let Q be a poset, and let $F = [1)$. Suppose that, P is a prime ideal in Q. As $F = [1)$, we have $P \cap F = \emptyset$. Then by Lemma 2.3, P is an F-prime ideal.

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Conversely, suppose that P is an F-prime ideal in Q. Let $(x,y)^\ell \subseteq P$ for some $x, y \in Q$. Therefore, $(x,1)^\ell \subseteq P$ or $(y,1)^\ell \subseteq P$. But $x \in (x,1)^\ell$ and $y \in (y,1)^\ell$. This implies that, $x \in P$ or $y \in P$. Therefore, P is a prime ideal in Q. □

Remark 2.5. *Let Q be a poset, and let F be an ℓ-filter in Q. Then, from Example 2.1 and Example 2.2, one can observe that, in general the spaces $Spec(Q)$ and $Spec_F(Q)$ are distinct for an ℓ-filter $F \neq [1)$. If we consider an ℓ-filter $F = [1)$ then P is a prime ideal in Q if and only if P is an F-prime ideal. That is for $F = [1)$, $Spec(Q) = Spec_F(Q)$.*

Let I be an ideal in a poset Q, and let $A \subseteq Q$. Then the set $(I : A)$ is defined as $(I : A) = \{x \in Q \mid (x,a)^\ell \subseteq I$ for all $a \in A\}$. Note that, $I \subseteq (I\ :\ A)$. Further, if $I = (0]$ then $((0]\ :\ A)$ is denoted by $A^\perp$ and is called an **annihilator** of A.

Theorem 2.6. *Let Q be a poset. Then the following statements hold.*

(1) *If I is an ideal in Q, and $x \in Q$ then $(I\ :\ x)$ is a semi-ideal in Q.*
(2) *If P is a prime ideal in Q, and $x \in Q \backslash P$ then $(P\ :\ x)$ is a prime ideal in Q.*

Proof. (1) Let I be an ideal in a poset Q, and let $x \in Q$. Consider $a, b \in (I : x)$ with $a \leq b$ and $b \in (I : x)$. Therefore, $(x,b)^\ell \subseteq I$. As $a \leq b$ implies that $(a,x)^\ell \subseteq (b,x)^\ell$. Therefore, $(x,b)^\ell \subseteq I$ implies that, $(a,x)^\ell \subseteq I$. Hence, we have $a \in (I : x)$.
(2) Let P be a prime ideal in a poset Q, and let $x \in Q \setminus P$. First, we prove that $(P : x)$ is an ideal in Q. Let $a, b \in (P : x)$ and $t \in (a,b)^{u\ell}$. To show $(P : x)$ is an ideal, it is sufficient to show that $t \in (P : x)$, that is, $(t,x)^\ell \subseteq P$. As $a, b \in (P : x)$, we have $(a,x)^\ell \subseteq P$ and $(b,x)^\ell \subseteq P$, but P is a prime ideal with $x \notin P$, implies that $a \in P$ and $b \in P$. Therefore, $(a,b)^{u\ell} \subseteq P$ and so, $t \in P$. This implies that $(t,x)^\ell \subseteq P$, that is $t \in (P : x)$. Therefore $(P : x)$ is an ideal in Q. Now, we prove that $(P : x)$ is a prime ideal. Let $a, b \in Q$ such that $(a,b)^\ell \subseteq (P : x)$. Now, if $t \in (a,b)^\ell$ then $t \in (P : x)$. That is, $(t,x)^\ell \subseteq P$, but $x \notin P$ and P is a prime ideal implies that, $t \in P$. Therefore, if $(a,b)^\ell \subseteq (P : x)$ then $(a,b)^\ell \in P$ implies that $a \in P$ or $b \in P$. As $P \subseteq (P : x)$, we have $a \in (P : x)$ or $b \in (P : x)$. Therefore, $(P : x)$ is a prime ideal. □

Theorem 2.7. *Let I, J be ideals in a poset Q, and let F be an ℓ-filter in Q. If P is an ideal in Q disjoint from F then the following statements are equivalent:*

(1) *P is an F-prime ideal.*
(2) *There exists a fixed element $f \in F$, such that whenever $I \bigcap J \subseteq P$ implies $I \bigcap (f] \subseteq P$ or $J \bigcap (f] \subseteq P$.*

Proof. (1) ⇒ (2) Let F be an ℓ-filter in a poset Q, and let P be an F-prime ideal in Q. Therefore by the definition of an F-prime ideal, there exists a fixed element $f' \in F$ such that whenever $(a,b)^\ell \subseteq P$ for $a, b \in Q$ then $(a,f')^\ell \subseteq P$ or $(b,f')^\ell \subseteq P$. Suppose that I and J are ideals in Q with $I \bigcap J \subseteq P$. On the contrary, suppose that $I \bigcap (f] \nsubseteq P$ and $J \bigcap (f] \nsubseteq P$ for all $f \in F$. Now, choose a fixed $f' \in F$.

Then there exist $x_{f'} \in (I \bigcap (f']) \setminus P$ and $y_{f'} \in (J \bigcap (f']) \setminus P$. Therefore, $x_{f'} \in I \bigcap (f']$ implies that $(x_{f'}, f')^{\ell} \subseteq I \bigcap (f']$ and $y_{f'} \in J \bigcap (f']$ implies that $(y_{f'}, f')^{\ell} \subseteq J \bigcap (f']$. Note that, since $x_{f'} \in I \bigcap (f']$ and $y_{f'} \in J \bigcap (f']$, we have $x_{f'} \leq f'$, $y_{f'} \leq f'$ and so, $x_{f'} \in (x_{f'}, f')^{\ell}$, $y_{f'} \in (y_{f'}, f')^{\ell}$. Now, $x_{f'} \in I \bigcap (f'] \subseteq I$ and $y_{f'} \in J \bigcap (f'] \subseteq J$ implies that $(x_{f'}, y_{f'})^{\ell} \subseteq I \bigcap J$ and $I \bigcap J \subseteq P$. Thus, $(x_{f'}, y_{f'})^{\ell} \subseteq P$. But P is an F-prime ideal and so, $(x_{f'}, f')^{\ell} \subseteq P$ or $(y_{f'}, f')^{\ell} \subseteq P$, a contradiction to the fact that $x_{f'} \notin P$ and $y_{f'} \notin P$. Therefore, $I \bigcap (f'] \subseteq P$ or $J \bigcap (f'] \subseteq P$ for a fixed $f' \in F$.

(2) $\Rightarrow$ (1) Suppose that, there exists a fixed element $f \in F$, such that whenever $I \bigcap J \subseteq P$ then $I \bigcap (f] \subseteq P$ or $J \bigcap (f] \subseteq P$. Let $(x, y)^{\ell} \subseteq P$ for some $x, y \in Q$. Consider $I = (x]$ and $J = (y]$. Then, $(x, y)^{\ell} = I \bigcap J \subseteq P$. Therefore, we have a fixed $f \in F$, such that $I \bigcap (f] = (x] \bigcap (f] = (x, f)^{\ell} \subseteq P$ or $J \bigcap (f] = (y] \bigcap (f] = (y, f)^{\ell} \subseteq P$. Therefore, P is an F-prime ideal. □

Lemma 2.8. *Let I, J be ideals in a poset Q, and let F be an ℓ-filter in Q. Then for $f \in F$, $I \cap (f] \subseteq J$ if and only if $I \subseteq (J : f)$.*

Proof. Let I, J be ideals in a poset Q, and let F be an ℓ-filter in Q. Suppose that, $I \bigcap (f] \subseteq J$ for some $f \in F$. Let $x \in I$. We claim that, $(x, f)^{\ell} \subseteq J$. Choose $t \in (x, f)^{\ell}$. Then, $t \in (x] \subseteq I$ and $t \in (f]$. Therefore, $t \in I \bigcap (f]$ and so $t \in J$. Thus, $(x, f)^{\ell} \subseteq J$, implies that $x \in (J : f)$. Therefore, $I \subseteq (P : f)$.

Conversely, suppose that $I \subseteq (J : f)$ for some $f \in F$. Let $x \in I \bigcap (f]$. This implies that, $x \in I$ and $x \leq f$. As $x \in I$ and $I \subseteq (J : f)$, we have $(x, f)^{\ell} \subseteq J$. Again $x \leq f$, implies that $x \in (x, f)^{\ell}$ and hence $x \in J$. Therefore, $I \bigcap (f] \subseteq J$. □

Theorem 2.9. *Let F be an ℓ-filter in a poset Q, and let P be an F-prime ideal in Q. Then there exists $f' \in F$ such that $(P : f) \subseteq (P : f')$ for all $f \in F$. That is $(P : f')$ is a maximal semi-ideal of the collection $\{(P : f)\}_{f \in F}$.*

Proof. Let F be an ℓ-filter in a poset Q, and let P be an F-prime ideal in Q. By Theorem 2.6, $(P : f)$, where $f \in F$ is a semi-ideal in Q. Now, consider $x \in (P : f)$ an arbitrary element. As $x \in (P : f)$, we have $(x, f)^{\ell} \subseteq P$. But, since P is an F-prime ideal, there exists a fixed $f' \in F$ such that either $(x, f')^{\ell} \subseteq P$ or $(f, f')^{\ell} \subseteq P$. If $(f, f')^{\ell} \subseteq P$ then as F is an ℓ-filter, $(f, f')^{\ell} \cap F \neq \emptyset$, a contradiction to the fact that $F \cap P = \emptyset$. This implies that, $(x, f')^{\ell} \subseteq P$ and so $x \in (P : f')$ for a fixed element $f' \in F$. Therefore $(P : f) \subseteq (P : f')$ for every $f \in F$, consequently $(P : f')$ is a maximal semi-ideal of the collection $\{(P : f)\}_{f \in F}$. □

The following result about $P_s(Q)$ can be found in Venkatanarasimhan [16].

Lemma 2.10 (Venkatanarasimhan [16])**.** *Let Q be a poset, and let I, J, I_{α} be semi-ideals in Q, where α belongs to an indexing set Λ. Then the following statements hold.*

(1) $V(Q) = \emptyset$ *and* $V((0]) = P_s(Q)$;
(2) $V(I) \bigcup V(J) = V(I \cap J)$;

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(3) $\bigcap_{\alpha\in\Lambda} V(I_\alpha) = V\left(\bigvee_{\alpha\in\Lambda} I_\alpha\right)$.

We define the F-variety $V_F(I)$, of a semi-ideal I in a poset Q, with respect to an ℓ-filter F as follows:

$$V_F(I) = \{P \in Spec_F(Q) \mid I \subseteq (P : f) \text{ for some } f \in F\}.$$

The following result will be frequently used in this article.

Lemma 2.11. *Let Q be a poset, F an ℓ-filter and let I, J, I_α be semi-ideals in Q, where α belongs to an indexing set Λ. Then the following statements hold.*

(1) $V_F(Q) = \emptyset$ *and* $V_F((0]) = Spec_F(Q)$;
(2) $V_F(I) \bigcup V_F(J) = V_F(I \bigcap J)$;
(3) $\bigcap_{\alpha\in\Lambda} V_F(I_\alpha) = V_F\left(\bigvee_{\alpha\in\Lambda} I_\alpha\right)$.

Proof. (1) Let F be an ℓ-filter, and let P be an F-prime ideal in a poset Q. If $P \in V_F(Q)$ then there exists $f \in F$ such that $Q \subseteq (P : f)$. This implies that, $1 \in (P : f)$, that is, $(1, f)^\ell \subseteq P$. But as $f \in (1, f)^\ell$, we have $f \in P$, a contradiction to the fact that $P \cap F = \emptyset$. Therefore, $V_F(Q) = \emptyset$. Also, if P is any F-prime ideal in Q, then observe that, $(0] \subseteq (P : f)$ for every $f \in F$. This implies that, $P \in V_F((0])$ and hence $V_F((0]) = Spec_F(Q)$.
(2) Suppose that, $P \in V_F(I) \cup V_F(J)$. Therefore, $P \in V_F(I)$ or $P \in V_F(J)$. Assume that, $P \in V_F(I)$. Then $I \cap J \subseteq I \subseteq (P : f)$ for some $f \in F$. This implies that, $P \in V_F(I \cap J)$.

Conversely, suppose that $P \in V_F(I \cap J)$. That is, $I \cap J \subseteq (P : f)$ for a fixed $f \in F$. Therefore, by Lemma 2.8, $(I \cap J) \cap (f] \subseteq P$ and so $I \cap (J \cap (f]) \subseteq P$. By Theorem 2.7, $I \cap (f_1] \subseteq P$ or $J \cap (f] \cap (f_1] \subseteq P$ for a fixed $f_1 \in F$. As F is an ℓ-filter and $f, f_1 \in F$ we have $((f] \cap (f_1]) \cap F \neq \emptyset$. Choose $f' \in ((f] \cap (f_1]) \cap F$. Therefore, $J \cap (f'] \subseteq J \cap ((f] \cap (f_1]) \subseteq P$. This implies that $I \cap (f_1] \subseteq P$ or $J \cap (f'] \subseteq P$ for $f_1, f' \in F$. Therefore by Lemma 2.8, $I \subseteq (P : f)$ or $J \subseteq (P : f')$, that is $P \in V_F(I)$ or $P \in V_F(J)$.
(3) Let $P \in \bigcap_{\alpha\in\Lambda} V_F(I_\alpha)$. Then $P \in V_F(I_\alpha)$ for every $\alpha \in \Lambda$. This implies that, $I_\alpha \subseteq (P : f_\alpha)$ for every $\alpha \in \Lambda$ and $f_\alpha \in F$. By Theorem 2.9, there exists $f' \in F$ such that $(P : f_\alpha) \subseteq (P : f')$ for every $f_\alpha \in F$. This implies that, $I_\alpha \subseteq (P : f')$ for every $\alpha \in \Lambda$. Therefore, $\bigvee_{\alpha\in\Lambda} I_\alpha \subseteq (P : f')$. Thus, $P \in V_F\left(\bigvee_{\alpha\in\Lambda} I_\alpha\right)$.

Conversely, let $P \in V_F\left(\bigvee_{\alpha\in\Lambda} I_\alpha\right)$. Then $\bigvee_{\alpha\in\Lambda} I_\alpha \subseteq (P : f)$ for some $f \in F$. As $I_\alpha \subseteq \bigvee_{\alpha\in\Lambda} I_\alpha$, we have $I_\alpha \subseteq (P : f)$ for every $\alpha \in \Lambda$. Therefore, $P \in \bigcap_{\alpha\in\Lambda} V_F(I_\alpha)$. $\square$

From Lemma 2.11, it is clear that if F is an ℓ-filter in a poset Q, then the set $\{V_F(I) \mid I \text{ a semi-ideal in } Q\}$ satisfies all axioms of closed sets for a topology τ on $Spec_F(Q)$. This topology is called the **F-Zariski topology** on Q and denoted by $Spec_F(Q)$.

The following theorem gives a basis for the F-Zariski topology on the set of all F-prime ideals in a poset Q.

Theorem 2.12. *Let Q be a poset, and let F be an ℓ-filter in Q. Then collection $D_F((a]) = \{P \in Spec_F(Q) \mid (a] \nsubseteq (P : f)$ for all $f \in F\}$, where $a \in Q$ forms a basis for the F-Zariski topology.*

Proof. Let Q be a poset, and let F be an ℓ-filter in Q. First, we prove that $D_F((a])$ is an open set for every $a \in Q$. Let $P \in Spec_F(Q) \setminus D_F((a])$. Then $P \notin D_F((a])$, that is $(a] \subseteq (P : f)$ for some $f \in F$. This implies that, $P \in V_F((a])$ and so, $Spec_F(Q) \setminus D_F((a]) \subseteq V_F((a])$. Now, suppose that $P \in V_F((a])$. Therefore, $(a] \subseteq (P : f)$ for some $f \in F$. This implies that, $P \notin D_F((a])$. This means that $P \in Spec_F(Q) \setminus D_F((a])$ and so, $V_F((a]) \subseteq Spec_F(Q) \setminus D_F((a])$. Therefore, we have $Spec_F(Q) \setminus D_F((a]) = V_F((a])$. Thus, $D_F((a])$ is an open set for all $a \in Q$.

Now, we prove that for a semi-ideal I in Q, $Spec_F(Q) \setminus V_F(I)$ is union of $D_F((a_i])$ for some $a_i \in Q$. Let $P \in Spec_F(Q) \setminus V_F(I)$. This implies that, $I \nsubseteq (P : f)$ for all $f \in F$. By Theorem 2.9, there exists $f^* \in F$ such that $(P : f) \subseteq (P : f^*)$ for all $f \in F$. Therefore, $I \nsubseteq (P : f^*)$ for $f^* \in F$. Now, suppose that $I = \{a_i\}_{i\in\Lambda}$ and let $\Lambda' = \{i \in \Lambda \mid (a_i] \nsubseteq (P : f^*)\}$. If $\Lambda' = \emptyset$ then $(a_i] \subseteq (P : f^*)$ for all $a_i \in I$ implies that, $\bigvee_{i\in\Lambda}(a_i] \subseteq (P : f^*)$. As $I \subseteq \bigvee_{i\in\Lambda}(a_i]$, we have $I \subseteq (P : f^*)$. This implies that, $P \in V_F(I)$, a contradiction and so $\Lambda' \neq \emptyset$. Now, by Theorem 2.9, $(a_i] \nsubseteq (P : f)$ for all $f \in F$ and $i \in \Lambda'$. This implies that, $P \in D_F((a_i]), i \in \Lambda'$ and so, $P \in \bigcup_{i\in\Lambda'} D_F((a_i])$. Conversely, suppose $P \in \bigcup_{i\in\Lambda'} D_F((a_i])$. Then $P \in D_F((a_i])$ for some $i \in \Lambda'$ and $a_i \in I$. This implies that, $(a_i] \nsubseteq (P : f)$ for all $f \in F$. Therefore, $I \nsubseteq (P : f)$ for all $f \in F$, that is $P \in Spec_F(Q) \setminus V_F(I)$. Hence, we have $Spec_F(Q) \setminus V_F(I) = \bigcup_{i\in\Lambda'} D_F((a_i])$. $\square$

Remark 2.13. *Let F be an ℓ-filter, and let P be a F-prime ideal in a poset Q. Then by Theorem 2.9, there exists $f' \in F$ such that $(P : f) \subseteq (P : f')$ for all $f \in F$. From now on, we denote this $f' \in F$ by f_P. That is, for an F-prime ideal P there exists $f_P \in F$ such that $(P : f) \subseteq (P : f_P)$ for all $f \in F$.*

Theorem 2.14. *Let F be an ℓ-filter in a poset Q. If $Y \subseteq Spec_F(Q)$ then*

$$\overline{Y} = V_F\left(\bigcap_{P\in Y}(P : f_P)\right).$$

Proof. Let $Y \subseteq Spec_F(Q)$. Suppose that, $P_1 \in Y$. Note that, $\bigcap_{P\in Y}(P : f_P) \subseteq (P_1 : f_{P_1})$, for some $f_{P_1} \in F$. This implies that, $P_1 \in V_F\left(\bigcap_{P\in Y}(P : f_P)\right)$. Therefore,

$Y \subseteq V_F\left(\bigcap_{P\in Y}(P : f_P)\right)$ and so, $\overline{Y} \subseteq V_F\left(\bigcap_{P\in Y}(P : f_P)\right)$.

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Now, suppose that $Y \subseteq V_F(I)$ for some semi-ideal I in a poset Q. If $P \in Y$ then $P \in V_F(I)$. This implies that, $I \subseteq (P : f)$ for some $f \in F$. By Theorem 2.9, $(P : f) \subseteq (P : f_P)$ for some $f_P \in F$. Therefore, $I \subseteq (P : f_P)$, for $f_P \in F$. This implies that, $I \subseteq \bigcap\limits_{P \in Y} (P : f_P)$. Therefore, $V_F\left(\bigcap\limits_{P\in Y}(P : f_P)\right) \subseteq V_F(I)$. Since $\overline{Y}$ is the smallest closed set containing Y, we have $V_F\left(\bigcap\limits_{P\in Y}(P : f_P)\right) \subseteq \overline{Y}$. Consequently,

$\overline{Y} = V_F\left(\bigcap\limits_{P\in Y}(P : f_P)\right)$. □

Corollary 2.15. *Let F be an ℓ-filter in a poset Q. If Y is a subset of prime ideals in Q disjoint from F then*

$$\overline{Y} = V_F\left(\bigcap_{P\in Y}(P : f_P)\right).$$

Proof. Let F be an ℓ-filter in a poset Q. By Lemma 2.3, if P is a prime ideal disjoint from an ℓ-filter F, then P is an F-prime ideal. This implies that, $Y \subseteq \mathit{Spec}_F(Q)$. Therefore, by Lemma 2.14, $\overline{Y} = V_F\left(\bigcap\limits_{P\in Y}(P : f_P)\right)$. □

Lemma 2.16. *Let F be an ℓ-filter in a poset Q. If P is a prime ideal in Q disjoint from F then $P = (P : f)$ for all $f \in F$.*

Proof. Suppose that, P is a prime ideal in a poset Q disjoint from an ℓ-filter F. Note that, $P \subseteq (P : f)$ for all $f \in F$. Let $x \in (P : f)$. Then $(x, f)^\ell \subseteq P$. This implies that, $x \in P$ or $f \in P$. As $P \cap F = \emptyset$, we have $x \in P$. Thus, $P = (P : f)$ for all $f \in F$. □

Theorem 2.17. *Let F be an ℓ-filter in a poset Q. If P is a prime ideal in Q disjoint from F, then the following statements hold.*

(1) $\overline{\{P\}} = V_F(P) = V_F(P : f_P)$.
(2) *If $P^\star \in \mathit{Spec}(Q)$ with $P^\star \cap F = \emptyset$. Then $\overline{\{P\}} = \overline{\{P^\star\}}$ if and only if $(P : f) = (P^\star : f)$ for some $f \in F$.*

Proof. (1) Let P be a prime ideal in a poset Q disjoint from an ℓ-filter F. By Lemma 2.16, $P = (P : f_P)$. Furthermore, by Corollary 2.15, we have $\overline{\{P\}} = V_F(P : f_P) = V_F(P)$. Therefore, $\overline{\{P\}} = V_F(P) = V_F(P : f_P)$.

(2) Let $P^\star \in \mathit{Spce}(Q)$ with $P^\star \cap F = \emptyset$. Suppose that, $\overline{\{P\}} = \overline{\{P^\star\}}$. Then by (1), $V_F(P) = V_F(P^\star)$. Since $P \in V_F(P)$, we have $P \in V_F(P^\star)$. This implies that, $P^\star \subseteq (P : f)$ and hence by Lemma 2.16, $P^\star \subseteq P$. Similarly, we can conclude that $P \subseteq P^\star$. Consequently, $(P : f) = (P^\star : f)$ for $f \in F$.

Conversely, suppose that $(P^\star : f) = (P : f)$ for some $f \in F$. By Lemma 2.16, $P^\star = P$. Therefore, $(P : f_P) = (P^\star : f_{P^\star})$. This implies that, $V_F((P : f_P)) = V_F((P^\star : f_{P^\star}))$ and hence by (1), we have $\overline{\{P\}} = \overline{\{P^\star\}}$. □

Definition 2.18 (Chajda [4]). *Let Q be a poset with the smallest element 0 and the greatest element 1. An element $y \in Q$ is said to be a **complement** of $x \in Q$, if $(x, y)^{u\ell} = (x, y)^{\ell u} = Q$. A poset Q is said to be **complemented** if each element of Q has a complement in Q.*

Definition 2.19 (Larmerová and Rachunek [11]). *A poset Q is called **distributive** if for $a, b, c \in Q, \{(a, b)^u, c\}^\ell = \{(a, c)^\ell, (b, c)^\ell\}^{u\ell}$.*

Lemma 2.20. *Let Q be a distributive poset, and let F be an ℓ-filter in Q. If $x \in Q$ is a complement of $y \in Q$ then $V_F((x]) = Spec_F(Q) \setminus V_F((y])$.*

Proof. Let Q be a distributive poset, and let F be an ℓ-filter in Q. Suppose that $x \in Q$ is a complement of $y \in Q$. Let $P \in V_F((x])$. Therefore, $(x] \subseteq (P : f)$ for some $f \in F$. By Theorem 2.9, $(P : f) \subseteq (P : f_P)$ for some $f_P \in F$. This implies that, $(x] \subseteq (P : f_P)$. If $P \in V_F((y])$ then $(y] \in (P : f_1)$, for some $f_1 \in F$. Again, by Theorem 2.9, $(y] \subseteq (P : f_P)$. Thus, $(x] \subseteq (P : f_P)$ and $(y] \subseteq (P : f_P)$. Then $(x, f_P)^\ell \subseteq P$ and $(y, f_P)^\ell \subseteq P$. This implies that, $\{(x, f_P)^\ell, (y, f_P)^\ell\}^{u\ell} \subseteq P$. As Q is a distributive poset, we have $\{(x, f_P)^\ell, (y, f_P)^\ell\}^{u\ell} = \{(x, y)^u, f_p\}^\ell$ and so, $\{(x, y)^u, f_p\}^\ell \subseteq P$. But an element x is a complement of y, that is $(x, y)^{u\ell} = Q$. Therefore, we have $\{(x, y)^u, f_P\}^\ell = (f_P]$. This implies that, $f_P \in P$, a contradiction to the fact that $F \cap P = \emptyset$. Hence, $P \notin V_F((y])$ and so, $P \in Spec_F(Q) \setminus V_F((y])$.

Conversely, suppose that $P \in Spec_F(Q) \setminus V_F((y])$. That is, $(y] \nsubseteq (P : f)$ for all $f \in F$. Note that, $(x, y)^{\ell u} = Q$ implies that $(x, y)^\ell = \{0\} \subseteq P$. As P is an F-prime ideal, we have $(x, f)^\ell \subseteq P$ or $(y, f)^\ell \subseteq P$ for some $f \in F$. If $(y, f)^\ell \subseteq P$ then $(y] \subseteq (P : f)$, a contradiction. Therefore $(x, f)^\ell \subseteq P$ for some $f \in F$ and so, $P \in V_F((x])$. Consequently, $V_F((x]) = Spec_F(Q) \setminus V_F((y])$. □

Theorem 2.21. *Let Q be a distributive complemented poset, and let F be an ℓ-filter in Q. Then $D_F((a])$ is a clopen set in $Spec_F(Q)$.*

Proof. Let F be an ℓ-filter in a distributive complemented poset Q, and let $a \in Q$. Then $(a, b)^{\ell u} = (a, b)^{u\ell} = Q$ for some $b \in Q$. Therefore, by Lemma 2.20, $V_F((b]) = Spec_F(Q) \setminus V_F((a])$. This implies that, $D_F((a]) = V_F((b])$ is a closed set in $Spec_F(Q)$. Also, by Theorem 2.12, $D_F((a])$ is a open set in $Spec_F(Q)$. Consequently, $D_F((a])$ is a clopen set in $Spec_F(Q)$. □

3. Irreducibility and compactness in $Spec_F(Q)$

In this section, we prove that if F is an ℓ-filter and P is an F-prime ideal in a poset Q, then $V_F(P)$ is an irreducible subset of $Spec_F(Q)$. Moreover, we prove that $Spec_F(Q)$ is a compact space.

Lemma 3.1 (Fulton [5]). *Let X be a topological space. A closed subset V of X is said to be **reducible** if $V = V_1 \cup V_2$, where V_1 and V_2 are closed sets in X, and V_1, V_2 are proper subsets of V . Otherwise V is called **irreducible**.*

Theorem 3.2. *Let F be an ℓ-filter in a poset Q. If P is a prime ideal in Q with $P \cap F = \emptyset$, then $V_F(P)$ is an irreducible set in $Spec_F(Q)$.*

Proof. Let F be an ℓ-filter, and let P be a prime ideal in a poset Q with $P \cap F = \emptyset$. Suppose that, $V_F(P) = V_F(I) \cup V_F(J)$ for semi-ideals I and J in Q. Then $V_F(I) \subseteq V_F(P)$ and $V_F(J) \subseteq V_F(P)$. Note that, $P \in V_F(P)$ implies that $P \in V_F(I)$ or $P \in V_F(J)$. Assume that $P \in V_F(I)$, that is, $I \subseteq (P : f)$ for some $f \in F$. If $P_1 \in V_F(P)$ then $P \subseteq (P_1 : f_1)$ for some $f_1 \in F$. Now, we prove that $I \subseteq (P_1 : f_1)$. Let $x \in I$. Since, $I \subseteq (P : f)$, we have $(x, f)^\ell \subseteq P$. As P is a prime ideal with $P \cap F = \emptyset$, we have $x \in P$. Therefore, $x \in (P_1 : f_1)$. This implies that, $I \subseteq (P_1 : f_1)$. That is, $P_1 \in V_F(I)$. Consequently, $V_F(P) \subseteq V_F(I)$ concluding that $V_F(I) = V_F(P)$. Thus, $V_F(P)$ can not be written as a union of two proper closed sets. Therefore, $V_F(P)$ is an irreducible set in $Spec_F(Q)$. □

Now, we prove that for an ℓ-filter F in a poset Q, $Spec_F(Q)$ is a compact space. For this purpose, we need the following few results.

Theorem 3.3 (Joshi and Mundlik [8])**.** *Let Q be a poset. If F is a prime ℓ-filter in Q then $Q \setminus F$ is a prime ideal in Q.*

Lemma 3.4. *Let Q be a poset, and let F be a prime ℓ-filter in Q. Then for a semi-ideal I in Q, $V_F(I) = \emptyset$ if and only if $I \cap F \neq \emptyset$.*

Proof. Let I be a semi-ideal, and let F be a prime ℓ-filter in a poset Q. Suppose that, $V_F(I) = \emptyset$. On the contrary, suppose that $I \cap F = \emptyset$. By Theorem 3.3, $P = Q \setminus F$ is a prime ideal in Q containing I. We know that, $P \subseteq (P : f)$ for $f \in F$. This implies $I \subseteq (P : f)$. Therefore, $P \in V_F(I)$, a contradiction to $V_F(I) = \emptyset$. Hence, $I \cap F \neq \emptyset$.

Conversely, suppose that $I \cap F \neq \emptyset$. Then there exists $x \in I \cap F$. If $V_F(I) \neq \emptyset$ then there exists an F-prime ideal $P \in V_F(I)$. This implies that, $I \subseteq (P : f)$ for some $f \in F$. In particular $(x, f)^\ell \subseteq P$. As F is an ℓ-filter and $x, f \in F$, we have $(x, f)^\ell \cap F \neq \emptyset$. Choose $f_1 \in (x, f)^\ell \cap F$. This implies that, $f_1 \in P \cap F$, a contradiction to the fact that $P \cap F = \emptyset$. Therefore, $V_F(I) = \emptyset$. □

Lemma 3.5 (Grätzer [7])**.** *The set $Id(Q)$ of all ideals in a poset Q forms an algebraic lattice concerning the set inclusion, and every principal ideal is a compact element of $Id(Q)$.*

Theorem 3.6. *Let Q be a poset, and let F be an ℓ-filter in Q. Then $Spec_F(Q)$ is a compact topological space.*

Proof. Let Q be a poset, and let F be an ℓ-filter in Q. Let $Spec_F(Q) = \bigcup_{i\in\Lambda} D_F((a_i])$, which is an open cover of $Spec_F(Q)$. Therefore, we have,

$$\begin{aligned} Spec_F(Q) &= \bigcup_{i\in\Lambda} D_F((a_i]) = \bigcup_{i\in\Lambda} (Spec_F(Q) \setminus V_F((a_i])) \\ &= Spec_F(Q) \setminus \left(\bigcap_{i\in\Lambda} V_F((a_i]) \right) \end{aligned}$$

By Lemma 2.11, $\bigcap_{i\in\Lambda} V_F((a_i]) = V_F\left(\bigvee_{i\in\Lambda}(a_i]\right)$. Therefore,

$$Spec_F(Q) = Spec_F(Q) \setminus V_F\left(\bigvee_{i\in\Lambda}(a_i]\right).$$

Let $I = \bigvee_{i\in\Lambda}(a_i]$. Then, $Spec_F(Q) = Spec_F(Q) \setminus V_F(I)$. This implies that, $V_F(I) = \emptyset$. Therefore, by Lemma 3.4, we have $I \cap F \neq \emptyset$. Choose $t_1, t_2, \ldots, t_n \in I \cap F$. Then $\left(\bigvee_{i=1}^{n}(t_i]\right) \cap F \neq \emptyset$. Again, by Lemma 3.4 , $V_F\left(\bigvee_{i=1}^{n}(t_i]\right) = \emptyset$. Therefore, $V_F((t_1] \vee (t_2] \vee \cdots \vee (t_n]) = \bigcap_{i=1}^{n} V_F((t_i]) = \emptyset$. By taking complements, implies that $Spec_F(Q) = \bigcup_{i=1}^{n} D_F((t_i])$. Therefore, $Spec_F(Q)$ is a compact topological space.

□

4. Separation axioms in F-prime spectrum

A topological space X is called a T_0**-space** if for any two distinct points $x, y \in X$, there exists an open set O in X such that $x \in O, y \notin O$ or $x \notin O, y \in O$. It is well known that the prime spectrum, $Spec(R)$ of a commutative ring R is always a T_0-space. In contrast, an F-prime spectrum $Spec_F(Q)$ may not be a T_0-space. The following Example 4.2, shows that for a poset Q and an ℓ-filter F in Q, $Spec_F(Q)$ is not a T_0-space in general. However, we prove that if every F-prime ideal is a prime ideal in Q, then $Spec_F(Q)$ is a T_0-space.

Theorem 4.1. [13] *Let X be a topological space. Then X is a T_0-space if and only if for each $x, y \in X, \overline{\{x\}} = \overline{\{y\}}$ implies $x = y$.*

Example 4.2. *Consider the poset Q depicted in Figure 3 and an ℓ-filter $F = [c)$. Note that,*

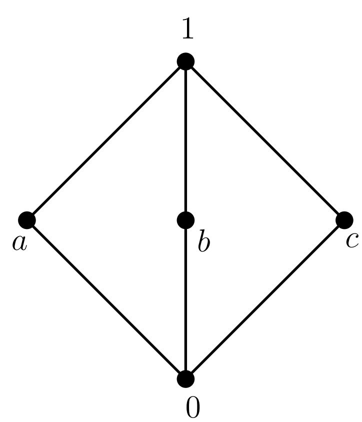

FIGURE 3.

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$\overline{\{(a]\}} = \bigcap V_F(I)$, *where* $\{(a]\} \subseteq V_F(I)$. *Observe that,* $Spec_F Q = \{(0], (a], (b]\}$. *Now,* $V_F((0]) = \{P \in Spec_F(Q) : (0] \subseteq (P : f)$ *for some* $f \in [c)\}$. *Therefore,* $V_F((0]) = \{(0], (a], (b]\}$. *Similarly,* $V_F((a]) = \{(0], (a], (b]\}$ *and* $V_F((b]) = \{(0], (a], (b]\}$. *This implies that,* $\overline{\{(a]\}} = \{(0], (a], (b]\}$. *Also, observe that* $\overline{\{(b]\}} = \{(0], (a], (b]\}$. *Thus,* $\overline{\{(a]\}} = \overline{\{(b]\}}$ *but* $a \neq b$. *Therefore, by Theorem 4.1,* $Spec_F(Q)$ *is not a* T_0*-space.*

Example 4.3. *Consider the poset* Q *depicted in Figure 4. Observe that* $(0]$ *is the only prime ideal in* Q. *Therefore,* $Spec(Q) = \{(0]\}$. *If we consider an* ℓ*-filter* $F = [a)$ *then* $(0]$ *is the only ideal disjoint from* F. *Also, observe that* $(0]$ *is an* F*-prime ideal in* Q. *Therefore,* $Spec_F(Q) = \{(0]\}$.

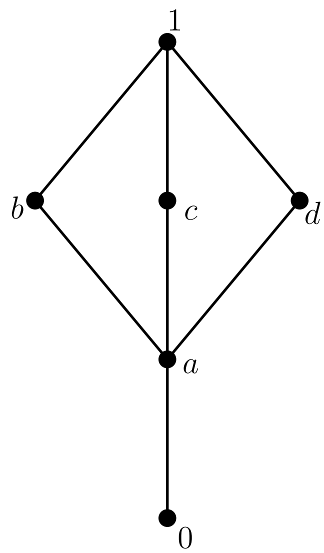

FIGURE 4.

Theorem 4.4. *Let* F *be an* ℓ*-filter in a poset* Q. *If every* F*-prime ideal in* Q *is a prime ideal then* $Spec_F(Q)$ *is a* T_0*-space.*

Proof. Let F be an ℓ-filter in a poset Q. Suppose that, every F-prime ideal in Q is a prime ideal. Consider $P, P^\star \in Spec_F(Q)$ with $\overline{\{P\}} = \overline{\{P^\star\}}$. Then by Theorem 2.17, $(P : f_P) = (P^\star : f_P)$ for $f_P \in F$. As $P, P^\star \in Spec_F(Q) \cap Spec(Q)$, by Lemma 2.16, we have $P = (P : f_P)$ and $P^\star = (P^\star : f_P)$. Thus $P = (P : f_P) = (P^\star : f_P) = P^\star$. Consequently, $\overline{\{P\}} = \overline{\{P^\star\}}$ implies that $P = P^\star$. Therefore, by Theorem, 4.1, $Spec_F(Q)$ is a T_0-space. □

Note that, for a poset Q, if an ℓ-filter $F = [1)$, then every F-prime ideal is a prime ideal in Q. Therefore, the following theorem of Mundlik, Joshi, and Halaš [12] becomes a corollary of Theorem 4.4.

Corollary 4.5 (Joshi, Mundlik and Halaš [12])**.** *Let* Q *be a poset. Then, the topological space* $Spec(Q)$ *of prime ideals is a* T_0*-space.*

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Definition 4.6. *Let Q be a poset, and let F be an ℓ-filter in Q. Then the F-krull dimension of Q is the supremum of the length of all chains of F-prime ideals and it is denoted by $Dim_F(Q)$.*

Example 4.7. *Let Q be a poset depicted in Figure 5. Consider an ℓ-filter $F = [a)$. Note that, $P = (x_i]$ is not a prime ideal in Q, but it is an F-prime ideal in Q. It is easy to observe that,*

$$(0] \subset (x_1] \subset \cdots (x_n] \subset \cdots$$

Therefore, $Dim_F(Q) = \infty$.

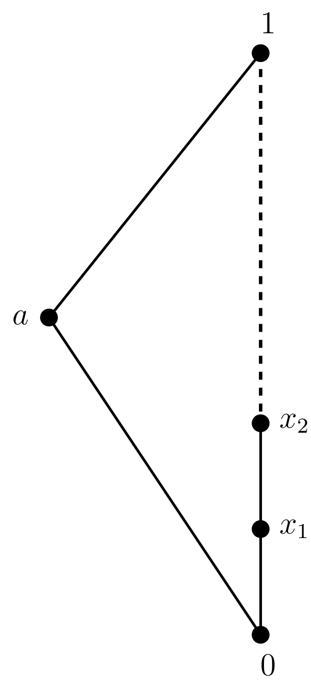

FIGURE 5. Example showing F-krull dimension, $Dim_F(Q) = \infty$ in a poset Q.

A topological space X satisfies T_1**-separation axiom** or is called T_1**-space** if for each pair of distinct elements $x, y \in X$, there exists open sets U_1, U_2 such that $x \in U_1, y \notin U_1$ and $y \in U_2, x \notin U_2$.

Theorem 4.8. [13] *Let X be a topological space. Then X is T_1-space if and only if $\overline{\{x\}} = \{x\}$ for each $x \in X$.*

Theorem 4.9. *Let Q be a poset, and let F be an ℓ-filter in Q. If every F-prime ideal in Q is a prime ideal, then the following statements are equivalent.*

(1) *$Spec_F(Q)$ is a T_1-space.*
(2) *Every F-prime ideal P in Q is a maximal element of the set $\{P' \in Spec(Q) \mid P' \cap F = \emptyset\}$.*
(3) *$Dim_F(Q) = 0$.*

Proof. Let Q be a poset, and let F be an ℓ-filter in Q. Suppose that every F-prime ideal in Q is a prime ideal.

(1) $\Rightarrow$ (2) : Suppose that, $Spec_F(Q)$ is a T_1-space. By Theorem 2.17, for every $P \in Spec_F(Q)$, we have $\overline{\{P\}} = V_F(P) = \{P\}$. Also, by Theorem 2.6, for every $P \in Spec_F(Q)$, $(P : f)$ is a prime ideal for $f \in F$. First, we prove that $(P : f) \cap F = \emptyset$. On the contrary, suppose that $x \in (P : f) \cap F$. Then $(x, f)^\ell \subseteq P$, this yields that, $(x, f_1)^\ell \subseteq P$ or $(f, f_1)^\ell \subseteq P$. But, as F is an ℓ-filter $(f, f_1)^\ell \subseteq P$ is not possible, otherwise $P \cap F \neq \emptyset$, a contradiction. Also, if $(x, f)^\ell \subseteq P$, then as $f \notin P$ implies that $x \in P$. This implies that $P \cap F \neq \emptyset$, again a contradiction. Thus, $(P : f) \cap F = \emptyset$. Now, suppose that P_1 is a prime ideal in Q containing P and disjoint from an ℓ-filter F. Then $P_1 \subseteq P = (P : f)$ implies that $P_1 \in V_F\{P\} = \{P\}$, that is $P_1 = P$. Consecutively, P is a maximal element of the set $\{P' \in Spec(Q) \mid P' \cap F = \emptyset\}$.

(2) $\Rightarrow$ (3) Suppose that, every F-prime ideal P in a poset Q is a maximal element of the set $\{P' \in Spec(Q) \mid P' \cap F = \emptyset\}$. Then every chain of F-prime ideals consists of exactly one F-prime ideal. Therefore, $Dim_F(Q) = 0$.

(3) $\Rightarrow$ (1) Suppose that, $Dim_F(Q) = 0$. Let $P' \in \overline{\{P\}}$. By Theorem, 2.17, $\overline{\{P\}} = V_F(P)$. Therefore, $P' \in V_F(P)$, that is $P \subseteq (P' : f)$ for some $f \in F$. Thus P', $(P' : f) \in Spec_F(Q)$ with $P' \subseteq (P' : f)$. Since, $Dim_F(Q) = 0$, we have $P' = (P' : f)$. Therefore, $P \subseteq (P' : f) = P'$. Again using the fact that $Dim_F(Q) = 0$, we get $P = P'$ and hence $\overline{\{P\}} = \{P\}$. Consecutively, $Spec_F(Q)$ is a T_1-space. $\square$

The subclass of $\mathbb{P}_{MFP}$ consisting of all posets having the smallest element 0 and with the property that every maximal ℓ-filter (maximal among all ℓ-filters) is a maximal filter (maximal among all filters) is denoted by $\mathbb{P}^\ell_{MFP}$.

Therefore, the following lemma of Mundlik, Joshi, and Halaš [12] becomes a corollary of Theorem 4.9.

Corollary 4.10. [12] *For a poset $Q \in \mathbb{P}^\ell_{MFP}$, the following conditions are equivalent.*

(1) *$P(Q)$ is a T_1-space*
(2) *$P = V(P)$ for all $P \in P(Q)$.*
(3) *$(P(Q), \subseteq)$ is an anti-chain.*

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