https://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. xx No. x (202x), pp. xx–xx.

https://www.ui.ac.ir

SOME STAR-CRITICAL CONNECTED RAMSEY NUMBERS

XIAOQIAN SUN AND YANBO ZHANG*

Abstract. The star-critical connected Ramsey number $r_c^*(G, H)$, introduced by Moun, Jakhar, and Budden (*TOC*, 2025), is a more deeply studied variant of the connected Ramsey number. For paths versus complete graphs, they obtained the exact values of $r_c^*(P_m, K_n)$ when $m = 5$ and when $n = 3$. For stars versus complete graphs, they determined the exact values of $r_c^*(K_{1,m}, K_n)$ when $m = 3$ and when $n = 3$. We complete the generalization of both cases by establishing the exact values of $r_c^*(P_m, K_n)$ for $m \geq 6$ and $n \geq 4$, and $r_c^*(K_{1,m}, K_n)$ for $m \geq 4$ and $n \geq 4$. The former result disproves a conjecture proposed by Moun et al. (*TOC*, 2025).

1. Introduction

Ramsey theory is one of the central topics in combinatorics. A (red-blue) 2-edge-coloring of a graph G refers to an assignment of each edge of G with either red or blue. Given two graphs G_1 and G_2, we use the notation $G \to (G_1, G_2)$ to denote that for any 2-edge-coloring of G, there exists either a red copy of G_1 or a blue copy of G_2. The *Ramsey number* $r(G_1, G_2)$ is defined as the smallest positive integer r such that the complete graph K_r satisfies $K_r \to (G_1, G_2)$. The existence of Ramsey numbers for any pair of graphs is guaranteed by Ramsey's theorem [13]. For a comprehensive and regularly updated survey of Ramsey numbers for various classes of graphs, we refer the reader to [12].

MSC(2010): Primary: 05C55; Secondary: 05D10.

Keywords: Ramsey number, connected Ramsey number, star-critical Ramsey number.

Article Type: Research Paper.

Communicated by Gholamreza Omidi.

*Corresponding author.

Received: 08 April 2025, Accepted: 14 February 2026, Published Online: 21 February 2026.

Cite this article: X. Sun and Y. Zhang, Some star-critical connected Ramsey numbers, *Trans. Comb.*, **xx** no. x (202x) xx–xx. https://dx.doi.org/10.22108/toc.2026.144868.2264 .

When $r = r(G_1, G_2)$, it is clear that $K_r \to (G_1, G_2)$ and $K_{r-1} \not\to (G_1, G_2)$. A natural question arises: if a graph G is a proper subgraph of K_r and contains K_{r-1} as a proper subgraph, is it necessarily true that $G \to (G_1, G_2)$? To study this problem, Hook [9] introduced a variant known as the star-critical Ramsey number. Let $K_{r-1} \sqcup K_{1,k}$ denote the graph obtained by joining an additional vertex to k vertices of K_{r-1}. The *star-critical Ramsey number* $r^*(G_1, G_2)$ is defined as the smallest positive integer k such that $K_{r-1} \sqcup K_{1,k} \to (G_1, G_2)$, where $r = r(G_1, G_2)$. Clearly, if neither G_1 nor G_2 contains an isolated vertex, then $1 \leq k \leq r-1$; otherwise, the exceptional case $k = 0$ may occur (see [**?**]). When $r^*(G_1, G_2) = r(G_1, G_2) - 1$, i.e., $K_r - e \not\to (G_1, G_2)$, the pair (G_1, G_2) is said to be *Ramsey-full* [16]. For further results on star-critical Ramsey numbers, see the monograph [1].

In 1978, Sumner [15] introduced connectivity constraints into the concept of Ramsey numbers and defined the connected Ramsey number. A connected 2-coloring of a graph G refers to a 2-edge-coloring of G such that the red edges induce a connected spanning subgraph of G, and so do the blue edges. Given two graphs G_1 and G_2, we write $G \xrightarrow{conn.} (G_1, G_2)$ to indicate that in every connected 2-coloring of G, there exists either a red copy of G_1 or a blue copy of G_2. The *connected Ramsey number* $r_c(G_1, G_2)$ is defined as the smallest positive integer r such that $K_r \xrightarrow{conn.} (G_1, G_2)$. Sumner [15] proved that if both G_1 and G_2 are 2-edge-connected graphs with at least four vertices, then $r_c(G_1, G_2) = r(G_1, G_2)$. As a result, subsequent research on connected Ramsey numbers has focused on the case where either G_1 or G_2 contains a bridge. For results on connected Ramsey numbers, see [2, 3, 5, 6, 8, 11, 14, 15].

By comparing the three aforementioned definitions, Moun, Jakhar, and Budden [11] introduced in 2025 the concept of the star-critical connected Ramsey number, which further refines the connected Ramsey number. Given two graphs G_1 and G_2, their *star-critical connected Ramsey number* $r_c^*(G_1, G_2)$ is defined as the smallest positive integer k such that $K_{r-1} \sqcup K_{1,k} \xrightarrow{conn.} (G_1, G_2)$, where $r = r_c(G_1, G_2)$. It is easy to see that $r_c^*(G_1, G_2)$ always exists and lies between 2 and $r_c(G_1, G_2)$. Moreover, if $r_c^*(G_1, G_2) = r_c(G_1, G_2) - 1$, then the pair (G_1, G_2) is called *connected Ramsey-full.*

For the reader's convenience, we summarize the above four definitions in set notation as follows:

$$\begin{aligned}
r(G_1, G_2) &= \min\{r \mid K_r \to (G_1, G_2)\},\\
r^*(G_1, G_2) &= \min\{k \mid K_r \sqcup K_{1,k} \to (G_1, G_2), r = r(G_1, G_2)\},\\
r_c(G_1, G_2) &= \min\{r \mid K_r \xrightarrow{conn.} (G_1, G_2)\},\\
r_c^*(G_1, G_2) &= \min\{k \mid K_r \sqcup K_{1,k} \xrightarrow{conn.} (G_1, G_2), r = r_c(G_1, G_2)\}.
\end{aligned}$$

This paper focuses on the study of two graph pairs (P_m, K_n) and $(K_{1,m-1}, K_n)$, where P_m denotes a path on m vertices and $K_{1,m-1}$ denotes a star on m vertices. These two graphs represent the two extremal cases of a tree T_m, with the path and star corresponding to the maximum and minimum possible diameters, respectively.

https://dx.doi.org/10.22108/toc.2026.144868.2264

A classical result in graph Ramsey theory, due to Chvátal [4], states that for any tree with m vertices,

$$r(T_m, K_n) = (m-1)(n-1) + 1.$$

Hook and Isaak [10] further established that the star-critical Ramsey number satisfies

$$r^*(T_m, K_n) = (m-1)(n-2) + 1.$$

However, when considering the connected Ramsey number, the situation becomes more intricate. In particular, a notable discrepancy arises between $r_c(P_m, K_n)$ and $r_c(K_{1,m-1}, K_n)$. For the former, Faudree and Schelp [6] proved that when $m \geq 5$ and $n \geq 3$,

$$r_c(P_m, K_n) = (n-2)(\lfloor m/2 \rfloor - 1) + \lceil m/2 \rceil + 1. \tag{1}$$

For the latter, they showed that

$$r_c(K_{1,m}, K_n) = \begin{cases} (n-1)(m-1)+1, & m \geq 4,\ n \geq 3, \\ (n-1)(m-1)+2, & m = 3,\ n \geq 3. \end{cases}$$

We now shift our focus to the main topic of this paper: the star-critical connected Ramsey number. In the trivial case, it is easy to observe that any connected 2-coloring of K_4 contains both a red P_4 and a blue P_4 [15]. On the other hand, the graph $K_3 \sqcup K_{1,2}$ admits no connected 2-coloring. Therefore, for any graph G and $2 \leq i \leq 4$, we have $r_c(P_i, G) = 4$ and $r_c^*(P_i, G) = 3$.

In the nontrivial cases, when one of the two graphs has small order, Moun, Jakhar, and Budden [11] obtained some exact results. Specifically, they proved that

$$\begin{aligned} &r_c^*(P_m, K_3) = m - 1, \quad \text{for all } m \geq 4, \\ &r_c^*(P_5, K_n) = n + 1, \quad \text{for all } n \geq 3, \\ &r_c^*(K_{1,m}, K_3) = 2m - 2, \quad \text{for all } m \geq 4, \text{ and} \\ &r_c^*(K_{1,3}, K_n) = 2, \quad \text{for all } n \geq 3. \end{aligned}$$

They further conjectured that $r_c^*(P_m, K_n) = m + n - 4$ for all $m \geq 4$ and $n \geq 3$.

In this paper, we determine the exact values of $r_c^*(P_m, K_n)$ and $r_c^*(K_{1,m}, K_n)$ in all remaining cases, and the former result disproves the aforementioned conjecture. Our two main theorems are stated as follows.

Theorem 1. *For $m \geq 6$ and $n \geq 4$, we have*

$$r_c^*(P_m, K_n) = (n-3)(\lfloor m/2 \rfloor - 1) + \lceil m/2 \rceil + 2.$$

Theorem 2. *For $m \geq 4$ and $n \geq 4$, we have*

$$r_c^*(K_{1,m}, K_n) = (n-1)(m-1).$$

https://dx.doi.org/10.22108/toc.2026.144868.2264

By comparing the values of $r_c^*(K_{1,m}, K_n)$ and $r_c(K_{1,m}, K_n)$, it follows that $(K_{1,m}, K_n)$ is connected Ramsey-full for $m \geq 4$ and $n \geq 3$.

The proofs of Theorems 1 and 2 will be presented in Sections 2 and 3, respectively.

2. Paths versus complete graphs

We need the following result on Ramsey numbers as a lemma, where $C_{\geq m}$ denotes a cycle with at least m vertices.

Lemma 1 (Faudree and Schelp [7]). *For $m \geq 3$ and $n \geq 1$, we have*

$$r(C_{\geq m}, K_n) = (m-1)(n-1) + 1.$$

For $m \geq 6$ and $n \geq 4$, let

$$N = (n-2)(\lfloor m/2 \rfloor - 1) + \lceil m/2 \rceil + 1.$$

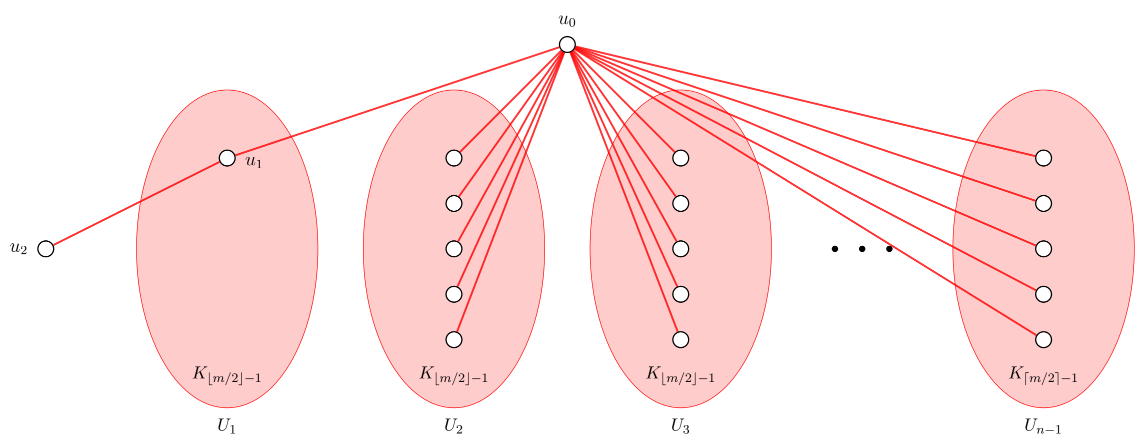

FIGURE 1. The graph G for the lower bound of $r_c^*(P_m, K_n)$

We begin by establishing the lower bound. Consider the graph H consisting of $n-2$ pairwise disjoint copies of $K_{\lfloor m/2 \rfloor -1}$, denoted by $U_1, \ldots, U_{n-2}$, together with another disjoint copy of $K_{\lceil m/2 \rceil -1}$, denoted by U_{n-1}. Select a vertex u_1 from U_1, and add two new vertices u_0 and u_2. Define the neighborhood of u_0 to be $(V(H) \setminus U_1) \cup \{u_1\}$, and let u_2 be adjacent only to u_1. Denote the resulting graph by G (see Figure 1). It is easy to verify that G has N vertices, is connected, and contains no P_m as a subgraph.

We now color all edges of G red. In the complement graph $\overline{G}$, color all edges blue except those between u_2 and U_1. It is straightforward to check that the blue edges induce a connected spanning subgraph, and the largest blue clique containing u_0 is a K_2. Suppose, for the sake of contradiction, that a blue K_n exists. Then, by the pigeonhole principle, this K_n must contain at least two vertices from one of the sets $U_1 \cup \{u_2\}, U_2, \ldots, U_{n-1}$, which is impossible.

https://dx.doi.org/10.22108/toc.2026.144868.2264

Hence, we obtain a connected 2-coloring in which there is neither a red P_m nor a blue K_n. Moreover, since the edges between u_2 and $U_1 \setminus \{u_1\}$ are neither red nor blue, we conclude that $K_{N-1} \sqcup K_{1,N-\lfloor m/2\rfloor+1} \overset{conn.}{\not\longrightarrow} (P_m, K_n)$. That is,

$$r_c^*(P_m, K_n) \geq (n-3)(\lfloor m/2\rfloor - 1) + \lceil m/2\rceil + 2.$$

We now proceed to prove the upper bound. Let G be the graph obtained by adding a new vertex v_0 to K_{N-1}, such that v_0 is adjacent to exactly $(n-3)(\lfloor m/2\rfloor - 1) + \lceil m/2\rceil + 2$ vertices in K_{N-1}. We aim to show that, for any connected 2-coloring of G, there exists either a red copy of P_m or a blue copy of K_n. We split the argument into two cases.

Case 1. v_0 is incident to at most one red edge.

We argue by contradiction. Suppose that G contains neither a red copy of P_m nor a blue copy of K_n. For the subgraph induced by $V(K_{N-1})$, let H_R denote the spanning subgraph whose edge set consists of all red edges, and let H_B denote the spanning subgraph whose edge set consists of all blue edges. Since the coloring of G is connected 2-coloring and v_0 is incident with at most one red edge, it follows that H_R is connected.

Let P_ℓ be a longest path in H_R. If $\ell \geq m$, then H_R already contains a red P_m, and we are done. Hence, we may assume that $\ell \leq m-1$ in what follows. Let x and y denote the two end vertices of P_ℓ, and let Γ be the complete graph induced by $V(H_R) \setminus V(P_\ell)$ with the inherited 2-edge-coloring. We state the following claims.

Claim 1. $\ell = m-1$ and the edge xy is blue.

Proof. If xy were red, then together with P_ℓ it would form a red cycle C_ℓ. As H_R is connected, any vertex outside C_ℓ can be joined to C_ℓ by a shortest red path, yielding a red path with at least $\ell+1$ vertices, again contradicting the maximality of P_ℓ. Thus, the edge xy is blue.

We argue by contradiction. Suppose that $\ell \leq m-2$. Then

$$\begin{aligned}
|\Gamma| &= N - 1 - \ell \\
&\geq (n-2)\left(\left\lfloor \tfrac{m}{2}\right\rfloor - 1\right) + \left\lceil \tfrac{m}{2}\right\rceil - (m-2) \\
&= (n-3)\left(\left\lfloor \tfrac{m}{2}\right\rfloor - 1\right) + m - 1 - (m-2) \\
&= (n-3)\left(\left\lfloor \tfrac{m}{2}\right\rfloor - 1\right) + 1.
\end{aligned}$$

By Lemma 1, the graph Γ contains either a red cycle $C_{\geq\lfloor m/2\rfloor}$ of length at least $\lfloor m/2\rfloor$, or a blue copy of K_{n-2}.

In the former case, we can construct a red path longer than P_ℓ. Since H_R is connected, there exists a shortest red path wQz joining P_ℓ and $C_{\geq\lfloor m/2\rfloor}$, where $w \in V(P_\ell)$ and $z \in V(C_{\geq\lfloor m/2\rfloor})$. The vertex w splits P_ℓ into two paths, namely $xP_\ell w$ and $yP_\ell w$. The total number of vertices in these two paths is $\ell+1$ (with w counted twice), so one of them contains at least $\lceil(\ell+1)/2\rceil$ vertices. Traversing this longer subpath from its endpoint to w, then following Q to z, and finally proceeding along the cycle $C_{\geq\lfloor m/2\rfloor}$, we obtain a red path of length at least

$$\left\lceil\frac{\ell+1}{2}\right\rceil+\left\lfloor\frac{m}{2}\right\rfloor\geq\left\lceil\frac{\ell+1}{2}\right\rceil+\left\lfloor\frac{\ell+1}{2}\right\rfloor=\ell+1>\ell,$$

contradicting the maximality of P_ℓ.

In the latter case, if Γ contains a blue copy of K_{n-2}, then we can find a blue K_n in H_B. Indeed, since x and y are the two end vertices of P_ℓ, all edges between $\{x,y\}$ and the vertices of K_{n-2} must be blue. Consequently, the blue K_{n-2} together with x and y forms a blue K_n, contrary to our assumption. □

Let V_1 denote the set consisting of the first $\left\lfloor\frac{m}{2}\right\rfloor-1$ vertices of P_ℓ starting from the end vertex x, and let V_2 denote the set consisting of the first $\left\lfloor\frac{m}{2}\right\rfloor-1$ vertices of P_ℓ starting from the other end vertex y.

Recall that Claim 1 says that $\ell=m-1$. If m is even, then there is exactly one vertex of P_ℓ that belongs to neither V_1 nor V_2; we denote this vertex by v_{c}. If m is odd, then there are exactly two vertices of P_ℓ that belong to neither V_1 nor V_2; we denote them by v_{c} and v_{c}'. We proceed with the following claims.

Claim 2. Γ contains no red cycle of length at least $\lfloor m/2\rfloor$.

Proof. Suppose, to the contrary, that Γ contains a red cycle C with $|C|\geq\lfloor m/2\rfloor$. Since H_R is connected, there exists a red path wQz joining C to the red path P_ℓ, where $w\in V(P_\ell)$ and $z\in V(C)$. Note that the longer of the two subpaths $xP_\ell w$ and $yP_\ell w$ contains at least $\lceil m/2\rceil$ vertices (with w counted twice when considering both subpaths). Extending this longer subpath through wQz and then along C, we obtain a red path P_m, contradicting the assumption. □

Claim 3. If $6\leq m\leq 7$, then Γ contains $n-3$ pairwise vertex-disjoint red edges, that is, $n-3$ copies of P_2. Denote the corresponding vertex sets of these copies of P_2 by $V_3,\ldots,V_{n-1}$. Then each of $V_1,V_2,\ldots,V_{n-1}$ induces a red clique, and every edge between any two distinct sets among them is blue.

Proof. Suppose that Γ contains a red P_4. Since H_R is connected, there exists a red path wQz joining P_ℓ to this P_4, where $w\in V(P_\ell)$ and $z\in V(P_4)$. The longer of the two subpaths $xP_\ell w$ and $yP_\ell w$ contains at least $\lceil m/2\rceil$ vertices, and clearly P_4 contains a red P_3 with z as an endpoint. Hence we can extend along wQz and then through this P_3 to obtain a red $P_{\ell+1}$, contradicting the maximality of P_ℓ. Therefore, Γ contains no red path on at least four vertices. Furthermore, by Claim 2, Γ contains no red cycle. Consequently, every nontrivial red component of Γ is a star.

https://dx.doi.org/10.22108/toc.2026.144868.2264

Assume that Γ contains t nontrivial red components. Delete the t centers of these stars from Γ. The remaining vertices then form an independent set in H_R, and thus a clique in H_B, of size

$$|\Gamma| - t = 2(n-3) - t.$$

On the other hand, the maximum blue clique in Γ has size at most $n-3$; otherwise, together with x and y it would form a blue K_n, contrary to our assumption. Therefore,

$$n-3 \geq 2(n-3) - t,$$

which implies $t \geq n-3$. In particular, Γ has exactly $n-3$ nontrivial red components, so the red subgraph induced by Γ is a perfect matching consisting of $n-3$ vertex-disjoint copies of P_2. Recall that V_1 and V_2 correspond to the two end-edges of P_ℓ. By the maximality of P_ℓ, every edge between $V_1 \cup V_2$ and $V_3 \cup \cdots \cup V_{n-1}$ must be blue.

It remains to show that all edges between V_1 and V_2 are also blue. Since H_R is connected, there is at least one red edge between P_ℓ and Γ. By the discussion above, the endpoint of such an edge on P_ℓ must be either v_c or v_c'; assume it is v_c. If there were a red edge between V_1 and V_2, then within the subgraph induced by $V(P_\ell)$ we could always find a red path of order $\ell - 1 = m-2$ with endpoint v_c. Because v_c has at least one further red neighbor in Γ, and the red edges in Γ form a perfect matching, we could then extend this path through Γ to obtain a red path on at least m vertices, a final contradiction. □

Claim 4. Let $m \geq 8$. If Γ contains a red cycle $C_{\lfloor m/2 \rfloor -1}$, then any red path between $C_{\lfloor m/2 \rfloor -1}$ and P_ℓ consists of exactly one red edge. Denote this edge by wz, where $w \in V(P_\ell)$ and $z \in V\big(C_{\lfloor m/2 \rfloor -1}\big)$. Then $w \in \{v_c, v_c'\}$, and the successor z^+ of z on the cycle is joined by blue edges to every vertex outside $V\big(C_{\lfloor m/2 \rfloor -1}\big) \cup \{v_c, v_c'\}$.

Proof. Suppose that there is a red path wQz between the red cycle $C_{\lfloor m/2 \rfloor -1}$ and the red path P_ℓ, where $w \in V(P_\ell)$ and $z \in V\big(C_{\lfloor m/2 \rfloor -1}\big)$. Since $\ell = m-1$, the two subpaths $xP_\ell w$ and $yP_\ell w$ together contain m vertices, counting w twice. Hence the longer one has at least $\lceil m/2 \rceil$ vertices. Extending this subpath along wQz to z and then continuing along the cycle $C_{\lfloor m/2 \rfloor -1}$ yields a red path with at least $m-1$ vertices; denote it by P'. By the maximality of P_ℓ, the path P' must have exactly $m-1$ vertices. Consequently, wQz must be a single red edge.

Since P' has exactly $m-1$ vertices, the longer of the two subpaths $xP_\ell w$ and $yP_\ell w$ has exactly $\lceil m/2 \rceil$ vertices. If $w \in V_1$, then $yP_\ell w$ has at least $\lceil m/2 \rceil + 1$ vertices; if $w \in V_2$, then $xP_\ell w$ has at least $\lceil m/2 \rceil + 1$ vertices. Therefore, $w \notin V_1 \cup V_2$, and hence $w \in \{v_c, v_c'\}$.

Let z^+ be the successor of z on $C_{\lfloor m/2 \rfloor -1}$. By choosing an appropriate direction along the cycle, we may assume that z^+ is an endpoint of P'. From the above, every edge between z^+ and $V(P_\ell) \setminus \{v_c, v_c'\}$ is blue. Moreover, since P' is also a longest red path, z^+ is joined by blue edges to every vertex outside $V\big(C_{\lfloor m/2 \rfloor -1}\big) \cup \{v_c, v_c'\}$. □

https://dx.doi.org/10.22108/toc.2026.144868.2264

Claim 5. If $m \geq 8$, then Γ contains $n-3$ pairwise vertex-disjoint red cycles $C_{\lfloor m/2\rfloor-1}$. Denote the corresponding vertex sets of these copies of $C_{\lfloor m/2\rfloor-1}$ by $V_3, \ldots, V_{n-1}$. Then each of $V_1, V_2, \ldots, V_{n-1}$ induces a red clique, and every edge between any two distinct sets among them is blue.

Proof. Note that

$$\begin{aligned}|\Gamma| &= N-1-\ell \\ &= N-1-(m-1) \\ &= (n-3)\big(\lfloor \tfrac{m}{2} \rfloor - 1\big) \\ &\geq (n-3)\big(\lfloor \tfrac{m}{2} \rfloor - 2\big) + 1.\end{aligned}$$

By Lemma 1, Γ contains either a red cycle $C_{\geq\lfloor m/2\rfloor-1}$ or a blue K_{n-2}. In the latter case, together with x and y this yields a blue K_n by Claim 1, a contradiction. In the former case, by Claim 2, Γ must contain a red $C_{\lfloor m/2\rfloor-1}$ as a subgraph.

Let V_3 denote the vertex set of this red cycle $C_{\lfloor m/2\rfloor-1}$, and let $\Gamma_1 = \Gamma - V_3$. By Claim 4, every edge between V_3 and $V_1 \cup V_2$ is blue. Moreover, there exists a vertex in V_3, denoted by z_3^+, such that all edges between z_3^+ and Γ_1 are blue.

Note that $|\Gamma_1| \geq (n-4)\big(\lfloor m/2\rfloor - 2\big) + 1$. Applying Lemma 1 again, Γ_1 contains either a red cycle $C_{\geq\lfloor m/2\rfloor-1}$ or a blue K_{n-2}. In the latter case, together with x, y, and z_3^+ this yields a blue K_n, a contradiction. In the former case, by Claim 2, Γ_1 must contain a red $C_{\lfloor m/2\rfloor-1}$ as a subgraph.

Let V_4 be the vertex set of this red cycle $C_{\lfloor m/2\rfloor-1}$, and let $\Gamma_2 = \Gamma_1 - V_4$. By Claim 4, every edge between V_4 and $V_1 \cup V_2$ is blue. Furthermore, there exists a vertex in V_4, denoted by z_4^+, such that all edges between z_4^+ and Γ_2 are blue. Repeating the above procedure inductively, we obtain $n-3$ pairwise vertex-disjoint red cycles $C_{\lfloor m/2\rfloor-1}$.

Next we show that for $3 \leq i < j \leq n-1$, every edge between V_i and V_j is blue. Suppose, to the contrary, that there exists a red edge between V_i and V_j. By Claim 4, there is a red edge between V_i and P_ℓ. Consequently, starting from the red cycle in V_j we can reach P_ℓ by a red path of length at least 2, which contradicts Claim 4. This proves that every edge between V_i and V_j is blue.

Next, we further show that all edges between V_1 and V_2 are also blue. Since H_R is connected, there is at least one red edge between P_ℓ and Γ. By the discussion above, the endpoint of such an edge on P_ℓ must be either v_c or v_c'; assume it is v_c.

If there were a red edge between V_1 and V_2, without loss of generality let v_jv_k be a red edge with $v_j \in V_1$ and $v_k \in V_2$. Starting from v_c, go along the red path P_ℓ to v_j, then traverse the edge v_jv_k, and finally follow P_ℓ to y; this yields a red path, denoted by $v_cP_\ell v_jv_kP_\ell y$. Similarly, we obtain another red path $v_cP_\ell v_kv_jP_\ell x$. It is easy to see that at least one of these two paths contains at least $\lceil m/2\rceil + 1$ vertices. From its endpoint v_c, we can further extend the longer path by a red edge into Γ. Note that every vertex of Γ lies on a red cycle $C_{\lfloor m/2\rfloor-1}$; hence we can continue along such a cycle to obtain a red path P_m, contradicting our assumption. This proves that every edge between V_1 and V_2 is blue.

For each $i \in [n-1]$, we regard V_i as a partite set; together these sets form a blue complete $(n-1)$-partite graph. If some V_i contains a blue edge, then this edge together with one arbitrary vertex from each of the other V_j yields a blue copy of K_n, contradicting our assumption. Therefore, every V_i induces a red clique. $\square$

Now we consider the whole graph G. Recall that v_0 is adjacent to exactly $(n-3)\big(\lfloor m/2\rfloor - 1\big) + \lceil m/2\rceil + 2$ vertices in K_{N-1}. Moreover, in the present case, v_0 is incident with at most one red edge. Hence there are at least $(n-3)\big(\lfloor m/2\rfloor - 1\big) + \lceil m/2\rceil + 1$ blue edges between v_0 and K_{N-1}. Note that if m is even, then v_c is the only vertex of K_{N-1} that does not belong to $V_1 \cup \cdots \cup V_{n-1}$; if m is odd, then v_c and v_c' are the only such vertices. Therefore, the number of blue edges between v_0 and $V_1 \cup \cdots \cup V_{n-1}$ is at least $(n-2)\big(\lfloor m/2\rfloor - 1\big) + 1$. By the pigeonhole principle, for each $i \in [n-1]$, the vertex v_0 has at least one blue neighbor in V_i. By Claims 3 and 5, choosing one such blue neighbor from each V_i and together with v_0 yields a blue copy of K_n, a final contradiction.

Case 2. v_0 is incident to at least two red edges.

In this case, there exists a red path P_3 with v_0 as an internal vertex. Let P_ℓ be the longest blue path with v_0 as an internal vertex, and denote its endpoints by v_1 and v_2. If $\ell \geq m$, then we already obtain a red P_m. Hence, we may assume that $\ell \leq m-1$.

We claim that v_1v_2 is a blue edge. Otherwise, the path $v_1P_\ell v_2v_1$ forms a red cycle C_ℓ. Since the subgraph induced by the red edges is connected, there must exist a vertex $w \notin V(C_\ell)$ and a vertex $v \in V(C_\ell)$ such that wv is a red edge. Let v^- and v^+ denote the predecessor and successor of v on the cycle C_ℓ, respectively. Note that at least one of v^- and v^+ is not v_0. If $v^- \neq v_0$, then the path $v^-\overleftarrow{C_\ell}vw$ is a longer red path with v_0 as an internal vertex. If $v^+ \neq v_0$, then the path $v^+\overrightarrow{C_\ell}vw$ is a longer red path with v_0 as an internal vertex. In either case, this contradicts the maximality of P_ℓ. Therefore, v_1v_2 must be a blue edge.

Let G' denote the subgraph $G - V(P_\ell)$. Then

$$|G'| = N - \ell \geq (n-2)(\left\lfloor \frac{m}{2} \right\rfloor - 1) + \left\lceil \frac{m}{2} \right\rceil + 1 - (m-1) = (n-3)(\left\lfloor \frac{m}{2} \right\rfloor - 1) + 1.$$

According to Lemma 1, the graph G' either contains a red cycle of length at least $\lfloor m/2\rfloor$ or a blue clique K_{n-2}. In the latter case, by the maximality of the red path P_ℓ, all edges between $\{v_1, v_2\}$ and $V(K_{n-2})$ must be blue, which yields a blue K_n. Hence, we may assume the former holds, that is, G' contains a red cycle of length at least $\lfloor m/2\rfloor$, denoted by $C_{\geq\lfloor m/2\rfloor}$.

Since the subgraph induced by red edges is connected, there exists a shortest red path between the red path P_ℓ and the cycle $C_{\geq\lfloor m/2\rfloor}$, denoted by Q. Let v_3 be the common vertex of Q and P_ℓ. Then the vertex v_0 lies either on the segment $v_1P_\ell v_3$ or on the segment $v_2P_\ell v_3$. Without loss of generality, we assume the former holds.

If the segment $v_2P_\ell v_3$ contains at least $\lceil m/2\rceil$ vertices, then there exists a red path that includes this segment together with all vertices on the cycle $C_{\geq\lfloor m/2\rfloor}$ (i.e., starting from v_2, following P_ℓ to v_3, then along Q to the cycle $C_{\geq\lfloor m/2\rfloor}$, and finally traversing the entire cycle). The total number of vertices in this red path is at least $\lceil m/2\rceil + \lfloor m/2\rfloor = m$.

If the segment $v_2P_\ell v_3$ contains at most $\lceil m/2\rceil - 1$ vertices, and noting that the vertex v_3 belongs to both segments, then the segment $v_1P_\ell v_3$ contains at least $\ell - \lceil m/2\rceil + 2$ vertices. Then there exists a red path that includes the segment $v_1P_\ell v_3$ and all vertices on the cycle $C_{\geq\lfloor m/2\rfloor}$ (i.e., starting from v_1, following P_ℓ to v_3, then along Q to the cycle $C_{\geq\lfloor m/2\rfloor}$, and finally traversing the entire cycle). This red path contains v_0 as an internal vertex and has at least $\ell - \lceil m/2\rceil + 2 + \lfloor m/2\rfloor \geq \ell + 1$ vertices, which contradicts the maximality of P_ℓ. This contradiction completes the proof.

3. Stars versus complete graphs

Recall that when $m \geq 4$ and $n \geq 4$, we have $r_c(K_{1,m}, K_n) = (n-1)(m-1)+1$. Let $N = (n-1)(m-1)+1$, and denote by $K_N - e$ the graph obtained from the complete graph K_N by deleting an arbitrary edge. To show that $r_c^*(K_{1,m}, K_n) = (n-1)(m-1)$, it suffices to establish that

$$K_N - e \overset{conn.}{\not\longrightarrow} (K_{1,m}, K_n).$$

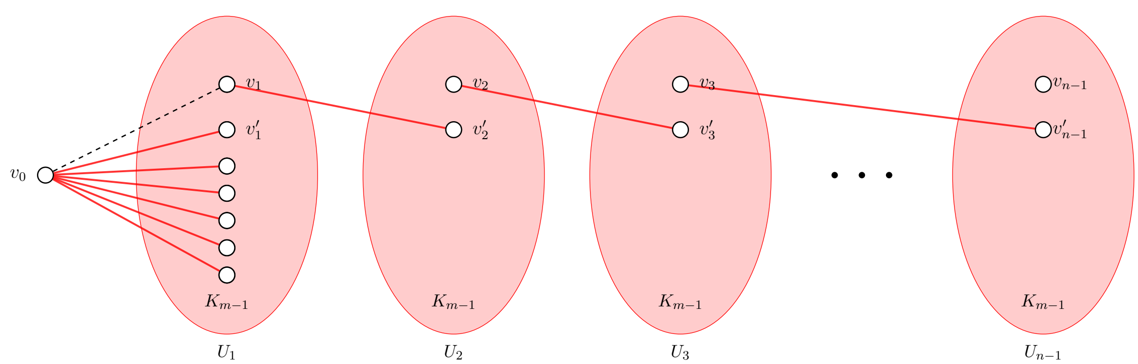

FIGURE 2. The graph G for the lower bound of $r_c^*(K_{1,m}, K_n)$

Consider $n-1$ pairwise disjoint copies of the complete graph K_{m-1}, with their vertex sets denoted by $U_1, \ldots, U_{n-1}$. Let $v_i, v_i' \in U_i$ for each $i \in [n-1]$. For each $i \in [n-2]$, we add an edge between v_i and v_{i+1}'. Additionally, we introduce a new vertex v_0 and connect it to every vertex in $U_1 \setminus \{v_1\}$. Let G denote the resulting graph (see Figure 2). It is easy to verify that G is connected and $\Delta(G) \leq m-1$.

We color all the edges of G red. In its complement $\overline{G}$, we color all edges blue except for the edge v_0v_1. Clearly, the blue edges induce a connected spanning subgraph. Suppose there exists a blue copy of K_n. By the pigeonhole principle, this K_n must contain at least two vertices from one of the

https://dx.doi.org/10.22108/toc.2026.144868.2264

sets among $U_1 \cup \{v_0\}, U_2, \ldots, U_{n-1}$, which is impossible. Hence, we have constructed a connected 2-coloring of $K_N - e$ that contains neither a red $K_{1,m}$ nor a blue K_n. Therefore,

$$K_N - e \overset{conn.}{\not\longrightarrow} (K_{1,m}, K_n).$$

Acknowledgments

The authors would like to thank the anonymous referees for their careful reading and helpful comments. Y. Zhang was partially supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11601527 and by the Natural Science Foundation of Hebei Province under Grant No. A2023205045.

References

[1] M. R. Budden, *Star-critical Ramsey numbers for graphs*, SpringerBriefs in Mathematics, Springer, Cham, 2023.

[2] M. Budden, Ramsey numbers for connected 2-colorings of complete graphs, *Theory Appl. Graphs*, **10** no. 1 Art. 7 (2023) 10 pp.

[3] M. Budden and T. Wimbish, Connected Ramsey numbers involving trees, *Electron. J. Math.* **9** (2025) 31–37.

[4] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** no. 1 (1977) 93–93.

[5] R. J. Faudree and R. H. Schelp, Connected Ramsey numbers for paths and stars, *Colloq. Math. Soc. János Bolyai*, **18** (1976) 307–313.

[6] R. J. Faudree and R. H. Schelp, Some connected Ramsey numbers, *J. Graph Theory*, **2 2** (1978) 119–128.

[7] R. J. Faudree and R. H. Schelp, Some problems in Ramsey theory, *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*, Lecture Notes in Math., Springer, Berlin-New York, (1978) 500–515.

[8] R. Häggkvist, R. J. Faudree and R. H. Schelp, Pancyclic graphs-connected Ramsey number, *Ars Combin.*, **11** (1981) 37–49.

[9] J. Hook, *The classification of critical graphs and star-critical Ramsey numbers*, (Ph.D.)–Lehigh University, ProQuest LLC, Ann Arbor, MI, 2010, 110 pp.

[10] J. Hook and G. Isaak, Star-critical Ramsey numbers, *Discrete Appl. Math.*, **159** no. 5 (2011) 328–334.

[11] M. Moun, J. Jakhar and M. Budden, Star-critical connected Ramsey numbers for 2-colorings of complete graphs, *Trans. Combin.*, **14** no. 4 (2025) 211–222.

[12] S. P. Radziszowski, Small Ramsey numbers – revision #17, *Electron. J. Combin.*, **DS1.17** (2024) 133 pp.

[13] F. Ramsey, On a problem of formal logic, *Proc. London Math. Soc. (2)*, **30** no. 4 (1929) 264–286.

[14] J. S. Sherr, *General and connected Ramsey theory for graphs*, Thesis (Ph.D.)–Emory University. ProQuest LLC, Ann Arbor, MI, 1987, 89 pp.

[15] D. P. Sumner, The connected Ramsey number, *Discrete Math.*, **22** no. 1 (1978) 49–55.

https://dx.doi.org/10.22108/toc.2026.144868.2264

[16] Y. Zhang, H. Broersma and Y. Chen, On star-critical and upper size Ramsey numbers, *Discrete Appl. Math.*, **202** (2016) 174–180.

Xiaoqian Sun
School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, China
Hebei Research Center of the Basic Discipline Pure Mathematics, Shijiazhuang, China
Email: xqsun.edu@outlook.com

Yanbo Zhang
School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, China
Hebei Research Center of the Basic Discipline Pure Mathematics, Shijiazhuang, China
Email: ybzhang@hebtu.edu.cn

https://dx.doi.org/10.22108/toc.2026.144868.2264