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ON CHARACTERIZATION OF SIMPLE K_3-GROUPS BY THE NUMBER OF ELEMENTS OF PRIME ORDER

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Abstract. Since the classification theorem of finite simple groups was declared proved in the early 1980s, many group theorists have been attempting to delve deeper into the structure of simple groups from the perspective of group invariants, resulting in a series of research topics on the quantitative characterization of simple groups, such as spectral characterization, two-order characterization, and OD-characterization. In 2018, Moretó proposed a new conjecture for the characterization of finite simple groups by the group order and the number of elements of the largest prime order. A simple group whose order is divisible by exactly three distinct prime numbers is called a simple K_3-group. These groups form a simple class of finite non-abelian simple groups. This paper establishes a characterization of $L_2(8)$ and $L_3(3)$ by combining the group order with the number of elements of the largest prime order, which shows that the conjecture holds for all simple K_3-groups except $L_2(7)$, $U_3(3)$ and $U_4(2)$. In addition, we also characterize $L_2(7)$, $U_3(3)$ and $U_4(2)$ under additional condition of non-solvability. Furthermore, we prove that a conjecture of Li and Shi holds for the alternating groups A_8, A_{10}, and $L_2(7)$. Thus the conjecture of Li and Shi is valid for sporadic simple groups, for alternating groups $A_n (n \geq 5)$, and for all simple K_3-groups except $U_3(3)$ and $U_4(2)$.

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1. Introduction

In this paper, the finite simple groups mentioned are non-abelian. For a finite group G, let $|G|$ be the order of G and $\pi(G)$ denotes the set of prime divisors of the order of G. For a positive integer k, $G(k)$ denotes the set of elements in G with order k. The number of elements in G with order k is denoted by $|G(k)|$.

As is well established, the order of a group, the orders of its elements, and the number of elements of each order are important group invariants. In 2018, Moretó proposed the following Conjecture 1.1 and proved in [9] that it holds for A_p with p a prime that is neither Wilson nor a near Wilson prime of order 2, and for $L_2(p)$ with p a prime that is not Mersenne.

Conjecture 1.1. *Let S be a finite simple group, and p be the largest prime divisor of $|S|$. If a finite group G has the same number of elements of order p and the same order as S, then $G \cong S$.*

In 2021, Li and Shi proved that Conjecture 1.1 holds for all sporadic simple groups and alternating groups $A_n (n \geq 5$ and $n \neq 8, 10)$ in [7]. Meanwhile, Li and Shi provided some counterexamples to the conjecture, such as $L_4(2)$ and $L_3(4)$; $O_7(3)$ and $S_6(3)$; $U_4(2)$ and 2-Frobenius group $((B \times C) \rtimes D) \rtimes A$, where A is a cyclic group of order 4, B is an elementary abelian group of order 2^4, C is an elementary abelian group of order 3^4, and D is a cyclic group of order 5. These pairs of groups have the same number of the largest prime order elements and group order, yet they are distinct.

Although the conjecture proposed by Moretó cannot be universally established, it still inspires us to a certain extent. In this paper, we prove that simple K_3-groups $L_3(3)$ and $L_2(8)$ satisfy the conditions of Conjecture 1.1, and also characterize $L_2(7)$, $U_3(3)$ and $U_4(2)$ with the addition of insoluble constraints.

To address the gaps in Conjecture 1.1, Li and Shi further strengthened its assumptions and proposed the following Conjecture 1.2 in [7].

Conjecture 1.2. *Let G be a finite group and S be a finite simple group. Then $G \cong S$ if and only if $|G| = |S|$ and $npe(G) = npe(S)$, where $npe(G) = \{|G(p)|| \ p \in \pi(G)\}$.*

We prove that Conjecture 1.2 holds for the alternating groups A_8 and A_{10}, and for the simple K_3-group $L_2(7)$, which means that the conjecture is valid for sporadic simple groups, for alternating groups $A_n (n \geq 5)$, and for all simple K_3-groups except $U_3(3)$ and $U_4(2)$.

For the sake of convenience, we shall establish the following terms and notations. For a finite group G, we define its *prime graph* $\Gamma(G)$ as follows: the vertex set of G is the set of all prime divisors of the order of G. Two vertices p and q are connected by an edge if and only if G contains an element of order pq (see [10]). The number of connected components in the prime graph of G is denoted as $t(G)$. The set of connected components in the prime graph of G is denoted by $T(G)$, i.e. $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, t(G)\}$, where $\pi_i(G)$ is the vertex set of the i-th component. When G is a group of even order, we assume that the prime number 2 is included in $\pi_1(G)$. In addition, if a group

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G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that both K and G/H are Frobenius groups with kernels H and K/H, respectively, then we call G a 2-Frobenius group. The remaining undefined notation is standard and can be found in reference [4].

2. Preliminary

In this section, we present some basic lemmas needed in the proofs of our theorems.

Lemma 2.1. *If G is a simple K_3-group, then G is isomorphic to one of the following groups:*

$$A_5, A_6, L_2(7), L_2(17), L_3(3), U_3(3), U_4(2) \text{ or } L_2(8).$$

Proof. For detailed proof, see reference [5]. □

Lemma 2.2. *If G is a finite group and $t(G) > 1$, then one of the following cases holds:*

(i) $t(G) = 2$, G is a Frobenius group or a 2-Frobenius group;

(ii) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Additionally, when $i \geq 2$, $\pi_i(G)$ also belongs to the prime graph of K/H.

Proof. The proof can be directly obtained from [10, Lemmas 1-3], [1, Lemma 1.5], and [3, Lemma 7]. □

Lemma 2.3. *Let G be an even-order Frobenius group with H and K being the Frobenius kernel and Frobenius complement of G respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$, and the structure of G is one of the following:*

(i) If $2 \in \pi(H)$, then all Sylow subgroups of K are cyclic;

(ii) If $2 \in \pi(K)$, then H is abelian. When K is a solvable group, the odd-order Sylow subgroups of K are cyclic, and the Sylow 2-subgroups of K are cyclic or generalized quaternion groups.

Proof. The proof can be directly obtained from [2, Theorem 1]. □

Lemma 2.4. *Suppose G is a 2-Frobenius group of even order. Then $t(G) = 2$, and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Both G/K and K/H are cyclic groups. The order of G/K divides the order of the automorphism group of K/H. In particular, $|G/K| < |K/H|$ and G is solvable.*

Proof. The proof can be directly obtained from [2, Theorem 2]. □

3. Main results and proofs

Theorem 3.1. *Let G be a finite group, S be a simple K_3-group other than $L_2(7)$, $U_3(3)$ and $U_4(2)$, and p be the largest prime divisor of $|S|$. Then $G \cong S$ if and only if $|G| = |S|$ and $|G(p)| = |S(p)|$.*

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Proof. The necessity is obvious, we only need to prove the sufficiency.

Suppose that the groups G and S satisfy the conditions of Theorem 3.1. We can calculate the number of elements of order p denoted as $|G(p)|$, where p is the largest prime in $\pi(G)$ and $\pi(S)$. Let P be a p-Sylow subgroup of G, then the order of P is p and

$$|G : N_G(P)| = \frac{|G(p)|}{p-1}.$$

The order of the normalizer $N_G(P)$ can be easily determined. From [7, 9], we know that when S is A_5, A_6 or $L_2(17)$, and G satisfies the conditions of Theorem 3.1, then $G \cong S$. Next, we will continue our proof by considering the cases of $L_3(3)$ and $L_2(8)$.

(1) If $S = L_3(3)$, then $G \cong S$.

From the hypothesis, we know that $|G| = 2^4 \cdot 3^3 \cdot 13$ and $|G(13)| = 2^6 \cdot 3^3$. Let P be a Sylow 13-subgroup of G. Then $|N_G(P)| = 3 \cdot 13$. Since $C_G(P) \leq N_G(P)$, we have $N_G(P) = C_G(P)$ or $C_G(P)$ is a cyclic group of order 13. Suppose $N_G(P) = C_G(P)$. Then G is a p-nilpotent group and has a normal 13-complement K. Let Q be a Sylow 2-subgroup of K such that PQ is a Hall subgroup of G, and let N be a minimal normal subgroup of PQ. If $|N| = 13$, then P is normal in PQ and $Q \leq N_G(P)$, but 2 does not divide the order of $N_G(P)$, which is a contradiction. If N is an elementary abelian 2-group, assume that the action of P on N is non-trivial, the order of P divides $|\mathrm{Aut}(N)|$, which is obviously not true. Therefore, the action of P on N is trivial and $N \leq C_G(P)$, which is impossible. So we have $C_G(P) = P$.

Consequently, $\Gamma(G)$ is disconnected and $\{13\}$ is an isolated point of $\Gamma(G)$. By Lemma 2.2, G is a Frobenius group, a 2-Frobenius group or satisfies case (ii) in Lemma 2.2.

If G is a Frobenius group, then $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement. By Lemma 2.3, we obtain $T(G) = \{\pi(H), \pi(K)\}$. Suppose $2 \in \pi(H)$, since H is a nilpotent group, we have $H = H_2 \times H_3$, where H_2 and H_3 are Sylow 2-subgroup and Sylow 3-subgroup of H, respectively. Obviously H_i is a normal subgroup of G for $i = 2, 3$. Because of $13 \nmid |\mathrm{Aut}(H_2)|$, P can act trivially on H_2, and so $H_2 \leq C_G(P)$, a contradiction. Suppose $2 \in \pi(K)$, P is a normal subgroup of G, then $N_G(P) = G$, a contradiction.

If G is a 2-Frobenius group, by Lemma 2.4, we know that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) = \pi_1(G)$ and $\pi(K/H) = \pi_2(G)$. Thus, $13 \in \pi_2(G)$. Consider the action of a 13-order element of K on a Sylow 2-subgroup or a Sylow 3-subgroup of H. Since the action is trivial, a contradiction occurs.

Thus, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group, and G/K is a π_1-group such that $|G/K| \mid |\mathrm{Out}(K/H)|$. In addition, when $i \geq 2$, $\pi_i(G)$ is also contained in the prime graph of K/H. Therefore, 13 divides $|K/H|$. By [11, Table 1], $K/H \cong L_3(3)$. G must be isomorphic to $L_3(3)$.

(2) If $S = L_2(8)$, then $G \cong S$.

By [4], we have $|G| = 2^3 \cdot 3^2 \cdot 7$ and $|G(7)| = 2^3 \cdot 3^3$. Let P be a Sylow 7-subgroup of G. Then $|N_G(P)| = 2 \cdot 7$. We have $C_G(P) = P$ as case (1). Therefore, $\{7\}$ is an isolated point of $\Gamma(G)$. By Lemma 2.2, if G is a Frobenius group, then $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement. With Lemma 2.3 we know $T(G) = \{\pi(H), \pi(K)\}$. Suppose $7 \in \pi(H)$, then $|K|$ divides $|H| - 1$, a contradiction. Suppose $7 \in \pi(K)$, then $|H| = 2^3 \cdot 3^2$, and obviously the Sylow 3-subgroup H_3 of H is also a Sylow 3-subgroup of G. Since H is nilpotent, H_3 is a normal subgroup of G. The action of P on H_3 is trivial, which means there exists a Hall subgroup of order $3^2 \cdot 7$ in G, a contradiction. G is not a Frobenius group.

If G is a 2-Frobenius group, by Lemma 2.4, we know that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) = \pi_1(G)$ and $\pi(K/H) = \pi_2(G)$. Thus, $7 \in \pi_2(G)$. Let an element of order 7 in K act on the Sylow 3-subgroup of H. It is easy to see the action is trivial, a contradiction.

Therefore, G is a group as in (ii) of Lemma 2.2, $7 \mid |K/H|$. By [11, Table 1], $K/H \cong L_2(8)$ or $L_2(7)$.

If $K/H \cong L_2(7)$, we know $|K/H| = 2^3 \cdot 3 \cdot 7$ and the order of G/K is a factor of $|\mathrm{Out}(K/H)|$. Then $3 \mid |H|$. Let H_3 be a Sylow 3-subgroup of H. We know that P acts trivially on H_3, so $3 \mid |C_G(P)|$, a contradiction. Hence, $K/H \cong L_2(8)$, therefore $G \cong L_2(8)$. □

By [9], we know that Conjecture 1.1 holds for $L_2(p)$ when p is a non-Mersenne prime. Hence it is interesting to investigate whether Conjecture 1.1 holds for $L_2(7)$ because 7 is a Mersenne prime. The following theorem provides a negative answer.

Theorem 3.2. *Let G be a finite group. If $|G| = 2^3 \cdot 3 \cdot 7$ and $|G(7)| = 2^4 \cdot 3$, then G is isomorphic to one of the following groups: $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \times \mathbb{Z}_3$ or $L_2(7)$.*

Proof. Let P be a Sylow 7-subgroup of G. By $|G| = 2^3 \cdot 3 \cdot 7$ and $|G(7)| = 2^4 \cdot 3$, we know that $|P| = 7$, and so the number of conjugate subgroups of P is $|G : N_G(P)| = \dfrac{|G(7)|}{7-1} = 8$. Thus, G has 8 subgroups of order 7. From [8], G is isomorphic to one of $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \times \mathbb{Z}_3$ or $L_2(7)$. □

Remark 3.3. *Theorem 3.2 gives a complete classification of groups G satisfying $|G| = 2^3 \cdot 3 \cdot 7$ and $|G(7)| = 2^4 \cdot 3$. The classification suggests that adding a non-solvability condition may allow us to break through the obstacles in characterizing $U_3(3)$ and $U_4(2)$. Based on this, we obtain the following conclusion.*

Theorem 3.4. *Let G be a finite non-solvable group, S be one of $L_2(7)$, $U_3(3)$ or $U_4(2)$, p be the largest prime divisor of $|S|$. Then $G \cong S$ if and only if $|G| = |S|$ and $|G(p)| = |S(p)|$.*

Proof. We only need to prove the sufficiency. And the proof will be made through a case-by-case analysis.

(1) If $S = L_2(7)$, then $G \cong S$.

By Theorem 3.2, we know that G is isomorphic to one of $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \times \mathbb{Z}_3$ or $L_2(7)$. Because G is a non-solvable group, G must be isomorphic to $L_2(7)$.

(2) If $S = U_3(3)$, then $G \cong S$.

By hypothesis, we know that $|G| = 2^5 \cdot 3^3 \cdot 7$ and $|G(7)| = 2^6 \cdot 3^3$. Let P be a Sylow 7-subgroup of G. Then $|N_G(P)| = 3 \cdot 7$.

Since $C_G(P) \leq N_G(P)$, we have $C_G(P) = N_G(P)$ or $C_G(P) = P$. If $C_G(P) = N_G(P)$, then G is a solvable group, a contradiction. Hence $C_G(P) = P$. Therefore, $\Gamma(G)$ is not connected and $\{7\}$ is an isolated point of $\Gamma(G)$. And thus G is either a Frobenius group, a 2-Frobenius group or G is a group as in (ii) of Lemma 2.2.

If G is a Frobenius group, then $G = HK$, where H and K are the Frobenius kernel and Frobenius complement of G, respectively. By Lemma 2.3, we know $T(G) = \{\pi(H), \pi(K)\}$. Suppose $7 \in \pi(H)$, then $|K|$ divides $|H|-1$, a contradiction. Suppose $7 \in \pi(K)$, then $|H| = 2^5 \cdot 3^3$, and obviously the Sylow 3-subgroup H_3 of H is also a Sylow 3-subgroup of G. Since H is nilpotent, H_3 is a normal subgroup of G. The action of P on H_3 is trivial, which means that there exists a nilpotent Hall subgroup of order $3^3 \cdot 7$ in G, a contradiction, so G is not a Frobenius group.

If G is a 2-Frobenius group, G is a solvable group by Lemma 2.4, a contradiction.

Thus, by Lemma 2.2, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group, and G/K is a π_1-group such that $|G/K|$ divides the order of $\mathrm{Out}(K/H)$. Besides, when $i \geq 2$, $\pi_i(G)$ is also contained in $\Gamma(K/H)$, so $7 \mid |K/H|$. According to [11, Table 1], K/H can be $L_2(7)$, $L_2(8)$ or $U_3(3)$.

If $K/H \cong L_2(7)$, then $|G/K| \mid 2$ and 2 divides the order of H. Let H_2 be the Sylow 2-subgroup of H. Then $H_2 \leq C_G(P)$, a contradiction. Similarly, it can be proven that K/H is not isomorphic to $L_2(8)$.

Therefore K/H is isomorphic to $U_3(3)$, and so G is isomorphic to $U_3(3)$.

(3) If $S = U_4(2)$, then $G \cong S$.

In this case, we have $|G| = 2^6 \cdot 3^4 \cdot 5$ and $|G(5)| = |S(5)| = 2^6 \cdot 3^4$. Let P be a Sylow 5-subgroup of G. Then $|N_G(P)| = 2^2 \cdot 5$. Let K be a maximal normal solvable subgroup of G. Since G is non-solvable, K is a proper subgroup of G.

Furthermore, we assert that $5 \notin \pi(K)$. If $5 \in \pi(K)$, let P be a Sylow 5-subgroup of K, then $G = N_G(P)K$. By the order of $N_G(P)$, $N_G(P)$ is a solvable group, further G is also solvable, a contradiction.

Let N be a minimal normal subgroup of G/K. Then N is a direct product of nonabelian simple groups. Of course $5 \in \pi(N)$, N is a simple group. According to [11, Table 1], we see that N is isomorphic to one of the following groups: A_5, A_6, $U_4(2)$.

By the order of the group G, N is the unique minimal normal subgroup of G/K.

Let $N = H/K$, then $|G/H|$ divides $|\mathrm{Out}(H/K)|$ from the maximality of K.

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If $N \cong A_5$, then 3 divides the order of K, so one can deduce that 3 divides $|N_G(P)|$, a contradiction.

If $N \cong A_6$, then $3^2 \parallel |G|$, since $|\mathrm{Out}(A_6)| = 4$, a contradiction.

If $N \cong U_4(2)$, then $G \cong U_4(2)$ as well. □

Next, we summarize the existing results on Conjecture 1.2 and prove that it holds for the alternating groups A_8 and A_{10}, and for the simple K_3-group $L_2(7)$. Thus, Conjecture 1.2 is valid for sporadic simple groups, alternating groups $A_n (n \geq 5)$, and all simple K_3-groups except $U_3(3)$ and $U_4(2)$.

Theorem 3.5. *Let G be a finite group and S be one of sporadic simple groups, alternating groups $A_n(n \geq 5)$ and simple K_3-groups other than $U_3(3)$ and $U_4(2)$. Then $G \cong S$ if and only if $|G| = |S|$ and $npe(G) = npe(S)$.*

Proof. From [6, Theorem 3.2], [7, Theorem 1.6] and Theorem 3.1, it follows that Conjecture 1.2 obviously holds for sporadic simple groups, for alternating groups A_n ($n \geq 5$ and $n \neq 8, 10$), and for all simple K_3-groups except $L_2(7)$, $U_3(3)$ and $U_4(2)$. Therefore, what we need is to prove Theorem 3.5 follows for A_8, A_{10} and $L_2(7)$.

If $S = L_2(7)$, then by Theorem 3.2, G is isomorphic to $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \times \mathbb{Z}_3$ or $L_2(7)$. From $|G(3)| = |S(3)|$, we get $G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_3$ or $L_2(7)$. Furthermore, it follows that $G \cong L_2(7)$ by $|G(2)| = |S(2)|$.

If $S = A_8$, then $G \cong A_8$ or $L_3(4)$ by [7]. We compare the number of 5-order elements of A_8 and $L_3(4)$. According to [4], the number of 5-order elements in A_8 is $2^6 \cdot 3 \cdot 7$, while the number of 5-order elements in $L_3(4)$ is $2^7 \cdot 3^2 \cdot 7$. From this, we can conclude that $G \cong A_8$.

If $S = A_{10}$, then $G \cong A_{10}$ or $G \cong J_2 \times Z(G)$, where $Z(G)$ is a cyclic group of order 3 by [7]. Similarly, we can get $G \cong A_{10}$ by comparing the number of 5-order elements of A_{10} and $J_2 \times Z(G)$. □

Remark 3.6. *We note that this paper does not confirm whether Conjecture 1.2 holds for the simple K_3-groups $U_3(3)$ and $U_4(2)$. Therefore, its validity for these groups remains an open question and will be discussed further.*

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References

[1] G. Y. Chen, On Thompson's conjecture for sporadic simple groups, in: First Academic Annual Meeting of Youths, *Proc. China Assoc. Sci. and Tech.*, First Academic Annual Meeting of Youths. China Sci. & Technol. Press, Beijing; 1992 1–6. [in Chinese]

[2] G. Y. Chen, On structure of Frobenius group and 2-Frobenius group, *J. Southwest China Norm. Univ.*, **20** no. 5 (1995) 485–487. [in Chinese]

[3] G. Y. Chen, Further reflections on Thompson's conjecture, *J. Algebra*, **218** no. 1 (1999) 276–285.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, $\mathbb{ATLAS}$ *of finite groups*, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray, Oxford University Press, Eynsham, 1985.

[5] M. Herzog, On finite simple groups of order divisible by three primes only, *J. Algebra*, **10** (1968) 383–388.

[6] A. Khosravi and B. Khosravi, Two new characterizations for sporadic simple groups, *Pure Math. Appl.*, **16** no. 3 (2005) 287–293.

[7] J. B. Li and W. J. Shi, On some conjectures related to quantitative characterizations of finite nonabelian simple groups, *Tr. Inst. Math. Mekh.*, **27** no. 4 (2021) 269–275.

[8] G. A. Miller, Determination of all the groups of order 168, *Amer. Math. Monthly*, **9** no. 1 (1902) 1–5.

[9] A. Moretó, The number of elements of prime order, *Monatsh. Math.*, **186** no. 1 (2018) 189–195.

[10] J. S. Williams, Prime graph components of finite groups, *J. Algebra*, **69** no. 2 (1981) 487–513.

[11] A. V. Zavarnitsine, Finite simple groups with narrow prime spectrum, *Sib. Elektron. Mat. Izv.*, **6** (2009) 1–12.

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