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HAMILTONIAN DEGREE OF FINITE GROUPS

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Abstract. This paper examines the probability of finite groups being Hamiltonian,a property defined by all subgroups being normal, and its implications for group structure analysis. To this end, we introduce the Hamiltonian degree, a novel extension of commutativity degrees in finite groups, and propose a comprehensive framework for its evaluation. Explicit formulas are derived for the Hamiltonian degree of dihedral groups, alongside general bounds applicable to broader group classes. Additionally, we explore its relationship to conjugacy class subgroups, shedding light on new structural connections within group theory.

1. Introduction

The study of commutativity degree of a finite group provides significant insights into the structure of finite groups by quantifying the proportion of commuting pairs of elements [8]. Over time, this concept has been extended to address various structural aspects of groups. This foundational concept has evolved into various generalizations, including the relative commutativity degree [5], which focuses on pairs of elements from a subgroup and the whole group, and the subgroup commutativity degree, which examines commutative interactions within subgroups. These developments have linked commutativity degrees to solvability, nilpotency, and modular group properties. The subgroup commutativity degree, introduced in works such as Tărnăuceanu's studies on lattice properties [9], focuses on the commuting behavior of subgroups rather than individual elements, offering a lattice-theoretic perspective. Similarly, the relative

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commutativity degree, examined in contexts involving extensions and automorphisms, incorporates additional group-theoretic constraints. Furthermore, the normality degree and the cyclicity degree were studied [10, 11]. The commutativity degrees were extended to include other classes [1, 2, 3]. Additional related work can be found in [4, 6, 7].

This paper explores the probability that a finite group is Hamiltonian.The idea of measuring how "Hamiltonian" a finite group can be was first discussed by Sherman, Tucker, and Walker in [12], where they introduced a probabilistic measure to study Hamiltonian properties of groups. Our work builds on this concept by defining the *Hamiltonian degree* of a finite group, deriving new bounds, and providing explicit formulas, particularly for dihedral groups. In this sense, our results can be viewed as a natural extension and refinement of the ideas presented in [12].

The notion of Hamiltonian degree is also inspired by the concept of the *commutativity degree* of a group, which measures the probability that two randomly chosen elements commute. This topic has been extensively studied in the literature, and for a detailed overview and various generalizations, we refer the reader to the survey paper [13]. Our results can therefore be seen as part of the broader effort to explore structural probabilities within finite groups.

By defining and analyzing this probability, the paper provides new insights into the structure and behavior of finite groups. We present a formula for the dihedral group and discuss its significance in the context of group theory. The rest of the paper is organized as follows: In Section 2, we introduce a new commutativity degree on finite groups, and call it *Hamiltonian degree*. We provide some non-trivial examples, and study some of its properties. In Section 3, we find some lower and upper bounds for the Hamiltonian degree of finite groups. Furthermore, we consider a detailed study on simple groups. In Section 4, we use the conjugacy classes subgroups to derive an equivalent formula for the Hamiltonian degree of a finite group. In Section 5, we derive explicit formulas for the Hamiltonian degree of dihedral groups. The paper concludes with a summary of findings and suggestions for future research.

2. Introducing the Hamiltonian degree of a finite group: Definitions and properties

In this section, we investigate the probability $P(G)$ that a finite group G is Hamiltonian. A group is defined as *Hamiltonian* if every subgroup of it is normal, which is a characteristic of Dedekind groups. We introduce the concept of the commutativity degree of an arbitrary element in a group with an arbitrary subgroup and use this to define the probability that a group is Hamiltonian. Through various examples and lemmas, we illustrate how this probability can be calculated and its implications for different types of groups, including symmetric and dihedral groups.

Definition 2.1. *Let G be a finite group and $L(G)$ be the set of all subgroups of G. Then the commutativity degree of an arbitrary element in G with any subgroup of G is defined as follows.*

$$P(G) = \frac{|\{(x, H) \in G \times L(G) : xH = Hx\}|}{|G||L(G)|}.$$

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It is clear that $P(G) = 1$ for all finite abelian or cyclic groups.

A group is *Dedekind* if every subgroup of it is normal, and a non-abelian Dedekind group is Hamiltonian. It is clear that $xH = Hx$ for every $x \in G$ if and only if H is a normal subgroup of G. Thus, G is a Hamiltonian group if and only if $P(G) = 1$. We can consider $P(G)$ as the probability that G is Hamiltonian or shortly, Hamiltonian degree of G.

Example 2.2. The quaternion group Q_8 is Hamiltonian, and hence, $P(Q_8) = 1$.

Remark 2.3. Let G be a finite non-Hamiltonian group. Then $0 < P(G) < 1$.

Definition 2.4. *Let k be a positive integer and G be a finite group with distinct subgroups $H_1, \ldots, H_k$. Then the probability table of G is defined as follows.*

	H_1	H_2	$\cdots$	H_k
x_1	H_{11}	H_{12}	$\cdots$	H_{1k}
x_2	H_{21}	H_{22}	$\cdots$	H_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	H_{n1}	H_{n2}	$\cdots$	H_{nk}

Here for all $1 \leq i \leq n$, $1 \leq j \leq k$, $H_{ij} = \begin{cases} 1 & \textit{if } x_iH_j = H_jx_i, \\ 0 & \textit{otherwise.} \end{cases}$

Using the above table, we get the $n \times k$ matrix $H(G) = (H_{ij})$.

Lemma 2.5. *Let $(G, \cdot)$ be a finite group with subgroup commutativity table (H_{ij}). Then*

$$P(G) = \frac{1}{|G||L(G)|} \sum_{j=1}^{k} \sum_{i=1}^{n} H_{ij}.$$

Example 2.6. Let S_3 be the symmetric group on three letters. Then the probability table of S_3 is given as follows.

	$\{(1)\}$	$\{(1), (12)\}$	$\{(1), (13)\}$	$\{(1), (23)\}$	$\{(1), (123), (132)\}$	S_3
(1)	1	1	1	1	1	1
(12)	1	1	0	0	1	1
(13)	1	0	1	0	1	1
(23)	1	0	0	1	1	1
(123)	1	0	0	0	1	1
(132)	1	0	0	0	1	1

It is clear that $P(S_3) = \frac{24}{36} = \frac{2}{3}$.

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Definition 2.7. *Let G be a finite group and H a subgroup of G. Then the probability that H is a normal subgroup of G is defined as follows:*

$$P_n(H,G) = \frac{|\{x \in G : xH = Hx\}|}{|G|}.$$

Remark 2.8. Let G be a finite group and H a subgroup of G. Then $P_n(H,G) \geq \frac{|H|}{|G|}$.

It is clear that $P(G) = \frac{1}{|L(G)|} \sum_{H \in L(G)} P_n(H,G)$.

Lemma 2.9. *Let G be a finite group, $L_n(G)$ the set of all normal subgroups of G, and $L'_n(G) = L(G) - L_n(G)$ the set of all subgroups of G that are not normal. Then the following are true.*

(i) $P(G) \geq \frac{1}{|L(G)|}(|L_n(G)| + \frac{1}{|G|} \sum_{H \in L'_n(G)} |H|)$.

(ii) $Rank(H(G)) \leq |L(G)| - |L_n(G)| + 1$.

(iii) *If $H(G)$ is a square matrix, then it's a singular matrix.*

Proof. The proof of (i) follows from Remark 2.8. For (ii), it is clear that an entire column of 1's corresponds to a normal subgroup in which we have at least two such columns. This implies that columns corresponding to normal subgroups are linearly dependent. Hence, the minimum number of linearly dependent columns is equal to $|L_n(G)|$. The latter implies that $Rank(H(G)) \leq |L(G)| - |L_n(G)| + 1$. For (iii), we have $Rank(H(G)) \leq |L(G)| - 1$ (by (ii)). Thus, the determinant of $H(G)$ is equal to 0. $\square$

As an illustration of Lemma 2.9, we consider Example 2.6. $L_n(S_3) = \{H_1, H_5.H_6\}, L'_n(S_3) = \{H_2, H_3, H_4\}$. We have $P(S_3) \geq \frac{1}{6}(3+\frac{6}{6}) = \frac{2}{3}$. Moreover, $H(S_3)$ is a singular matrix with $Rank(H(S_3)) = 4$, and eigenvalues $0, 1, 4$ of multiplicities $2, 2, 1$ respectively.

Lemma 2.10. *Let G be a finite group, $L(G) = \{H_1, H_2, \ldots, H_k\}$ the set of all subgroups of G, and $N_G(H_i)$ the normalizer of H_i in G for $i = 1, 2, \ldots, n$. Then*

$$P(G) = \frac{\sum_{i=1}^{k} |N_G(H_i)|}{|G||L(G)|}.$$

Example 2.11. Let $G = GL_2(\mathbb{Z}_2) = \{I_2, A_1, A_2, A_3, A_4, A_5\}$ be the group of two-by-two invertible matrices over $\mathbb{Z}_2$ with $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $A_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then $L(GL_2(\mathbb{Z}_2)) = \{H_1, H_2, H_3, H_4, H_5, H_6\}$ where: $H_1 = \{I_2\}$, $H_2 = \{I_2, A_1\}$, $H_3 = \{I_2, A_2\}$, $H_4 = \{I_2, A_3\}$, $H_5 = \{I_2, A_4, A_5\}$, $H_6 = G$. Having $N_G(H_1) = N_G(H_5) = N_G(H_6) = G$, $N_G(H_2) = H_2$, $N_G(H_3) = H_3$, and $N_G(H_4) = H_4$ implies that

$$P(G) = \frac{|N_G(H_1)| + |N_G(H_2)| + |N_G(H_3)| + |N_G(H_4)| + |N_G(H_5)| + |N_G(H_6)|}{36} = \frac{6+2+2+2+6+6}{36} = \frac{2}{3}.$$

One can easily notice that $P(S_3) = P(GL_2(\mathbb{Z}_2))$. This is clear as $GL_2(\mathbb{Z}_2)) \cong S_3$.

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Example 2.12. Let A_4 be the alternating group on four letters. Then $|A_4| = 12$ and $|L(A_4)| = 10$. The following table represents all subgroups of A_4, their normalizers in A_4, and the number of elements in each normalizer. One can easily see that $P(A_4) = \frac{1}{2}$.

H_i	$N(H_i)$	$\lvert N(H_i)\rvert$
$\{(1)\}$	A_4	12
$\{(1),(12)(34)\}$	$\{(1),(12)(34),(13)(24),(14)(23)\}$	4
$\{(1),(13)(24)\}$	$\{(1),(12)(34),(13)(24),(14)(23)\}$	4
$\{(1),(14)(23)\}$	$\{(1),(12)(34),(13)(24),(14)(23)\}$	4
$\{(1),(123),(132)\}$	$\{(1),(123),(132)\}$	3
$\{(1),(124),(142)\}$	$\{(1),(124),(142)\}$	3
$\{(1),(134),(143)\}$	$\{(1),(134),(143)\}$	3
$\{(1),(234),(243)\}$	$\{(1),(234),(243)\}$	3
$\{(1),(12)(34),(13)(24),(14)(23)\}$	A_4	12
A_4	A_4	12

Definition 2.13. *Let G be a finite group and $x \in G$ a fixed element.Then $P(x,G)$ is the probability that $xH = Hx$ for arbitrary subgroup H of G. i.e.,*

$$P(x,G) = \frac{|\{H \in L(G) : xH = Hx\}|}{|L(G)|}.$$

It is clear that $P(x,G) \geq \frac{|L_n(G)|}{|L(G)|}$.

Lemma 2.14. *Let G be a finite group. Then $P(G) = \frac{1}{|G|}\sum_{x\in G} P(x,G)$.*

Lemma 2.15. *Let G be a finite group with center $Z(G)$. Then $P(x,G) = P(y,G)$ if $x \in yZ(G)$.*

Lemma 2.16. *Let G be a finite group of center $Z(G)$ and $[G : Z(G)] = k$ with $\{x_iZ(G) = Z(G), x_2Z(G), \ldots, x_kZ(G)\}$ are the k distinct left cosets of G. Then*

$$P(G) = \frac{|Z(G)|}{|G|}(1 + \sum_{i=2}^{k} P(x_i,G)).$$

Example 2.17. Let $D_8 = \langle a, b : a^4 = b^2 = e, bab = a^{-1}\rangle$ be the Dihedral group of 8 elements. Then the ten subgroups of D_8 are: $H_1 = \{e\}$, $H_2 = \{e, a^2\} = Z(D_8)$, $H_3 = \{e, b\}$, $H_4 = \{e, ab\}$, $H_5 = \{e, a^2b\}$, $H_6 = \{e, a^3b\}$, $H_7 = \{e, a, a^2, a^3\}$, $H_8 = \{e, a^2, b, a^2b\}$, $H_9 = \{e, a^2, ab, a^3b\}$, $H_{10} = D_8$.
Having $[D_8 : Z(D_8)] = 4$ implies that $D_8 = Z(D_8) \cup aZ(D_8) \cup bZ(D_8) \cup abZ(D_8)$

We have:

$$P(a, D_8) = \frac{|\{(a,H_1),(a,H_2),(a,H_7),(a,H_8),(a,H_9),(a,H_{10})\}|}{10} = \frac{6}{10}$$

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and

$$P(b, D_8) = \frac{|\{(b,H_1),(b,H_2),(b,H_3),(b,H_5),(b,H_7),(a,H_8),(a,H_9),(a,H_{10})\}|}{10} = \frac{8}{10}.$$

Similarly, $P(ab, D_8) = \frac{8}{10}$. Thus, $P(D_8) = \frac{2}{8}(1 + P(a, D_8) + P(b, D_8) + P(ab, D_8)) = \frac{4}{5}$.

In group theory, it is well known that if G_1 and G_2 are finite groups with co-prime orders and A is a subgroup of $G_1 \times G_2$ then there exist subgroups H_1, H_2 of G_1, G_2 respectively such that $A = H_1 \times H_2$. The following two lemmas follow from this fact.

Lemma 2.18. *Let G_1 and G_2 be finite groups with co-prime orders. Then $L(G_1 \times G_2) = L(G_1) \times L(G_2)$.*

Lemma 2.19. *Let G_1 and G_2 be finite groups and H_1, H_2 be subgroups of G_1, G_2 respectively. Then $N_{G_1 \times G_2}(H_1 \times H_2) = N_{G_1}(H_1) \times N_{G_2}(H_2)$.*

Theorem 2.20. *Let G_1 and G_2 be finite groups with co-prime orders. Then $P(G_1 \times G_2) = P(G_1)P(G_2)$.*

Proof. Let G_1 and G_2 be finite groups with co-prime orders, $L(G) = \{H_1, H_2, \dots, H_m\}$, $L(G_2) = \{K_1, K_2, \dots, K_n\}$. Then

$$\sum_{1 \le i \le m, 1 \le j \le n} |N_{G_1}(H_i)||N_{G_2}(K_j)| = |N_{G_1}(H_1)|(\sum_{1 \le j \le n} |N_{G_2}(K_j)|) + \cdots + |N_G(H_m)|(\sum_{1 \le j \le n} |N_{G_2}(K_j)|).$$

Thus,

$$\sum_{1 \le i \le m, 1 \le j \le n} |N_{G_1}(H_i)||N_{G_2}(K_j)| = \sum_{1 \le i \le m} |N_{G_1}(H_i)| \sum_{1 \le j \le n} |N_{G_2}(K_j)|.$$

We get

$$P(G_1 \times G_2) = \frac{\sum\limits_{1 \le i \le m, 1 \le j \le n} |N_{G_1}(H_i)||N_{G_2}(K_j)|}{|G_1 \times G_2||L(G \times G_2)|} = \frac{\sum\limits_{1 \le i \le m} |N_{G_1}(H_i)|}{|G_1||L(G_1)|} \frac{\sum\limits_{1 \le j \le n} |N_{G_2}(K_j)|}{|G_2||L(G_2)|}.$$

Therefore, $P(G_1 \times G_2) = P(G_1)P(G_2)$. □

Example 2.21. Let S_3 be the symmetric group on three letters and $\mathbb{Z}_p$ be the group of integers modulo the prime number $p \ge 5$. Then $P(S_3 \times \mathbb{Z}_p) = P(S_3) = \frac{2}{3}$.

We present a generalization of Theorem 2.20.

Theorem 2.22. *Let $G_1, G_2, \dots, G_k$ be finite groups with pairwise co-prime orders. Then $P(G_1 \times G_2 \times \cdots \times G_k) = P(G_1)P(G_2)\cdots P(G_k)$.*

Example 2.23. Let D_8 be the Dihedral group with eight elements and $\mathbb{Z}_n$ be the group of integers modulo the integer number $n \ge 2$. Then $P(\mathbb{Z}_9 \times D_8 \times \mathbb{Z}_{11}) = P(D_8) = \frac{4}{5}$.

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3. Bounds for the Hamiltonian degree of a finite group

In this section, we find some lower and upper bounds for $P(G)$. Moreover, we consider finite simple groups as a special class with a focus on the alternating group.

Theorem 3.1. *Let G be a non-Hamiltonian group of finite even order. Then*

$$\frac{2}{|L(G)|}+\frac{2}{|G|}-\frac{4}{|G||L(G)|}\leq P(G)\leq 1-\frac{1}{2|L(G)|}.$$

Proof. Let H_i be a subgroup of G for $i=1,\dots,n=|L(G)|$ with $H_1=\{e\}$ and $H_n=G$. Then $P(G)=\dfrac{\sum_{j=1}^{k}|N_G(H_i)|}{|G||L(G)|}=\dfrac{|G|+|G|+|N_G(H_2)|+\cdots+|N_G(H_{n-1})|}{|G||L(G)|}$. Having $H_i\neq\{e\}$ a subgroup of $N_G(H_i)$ implies that $|N_G(H_i)|\geq|H_i|\geq 2$. We get that $P(G)\geq\frac{2|G|+2(|L(G)|-2)}{|G||L(G)|}$.

Since G is a non-Hamiltonian group, it follows that there exists a subgroup H_i for some $i=i_0$ of G with $N_G(H_i)\neq G$. Having $N_G(H_{i_0})$ a subgroup of G implies that $|N_G(H_{i_0})|$ is a divisor of $|G|$ and hence, $|N_G(H_{i_0})|\leq\frac{|G|}{2}$. The latter implies that $P(G)=\frac{|N_G(H_{i_0})|+\sum_{i\neq i_0}|N_G(H_i)|}{|G||L(G)|}\leq\frac{\frac{|G|}{2}+(|L(G)|-1)|G|}{|G||L(G)|}\leq 1-\frac{1}{2|L(G)|}$. □

Example 3.2. As an illustration of Theorem 3.1, we consider the groups A_4, S_3 and D_8.
We consider first A_4. We have: $\frac{1}{3}\leq P(A_4)=\frac{1}{2}\leq\frac{19}{20}$. Similarly, $\frac{5}{9}\leq P(S_3)=\frac{2}{3}\leq\frac{11}{12}$ and $\frac{2}{5}\leq P(D_8)=\frac{4}{5}\leq\frac{19}{20}$.

Lemma 3.3. *Let G be a non-Hamiltonian group of finite even order that is not simple. Then*

$$\frac{3}{|L(G)|}+\frac{2}{|G|}-\frac{6}{|G||L(G)|}\leq P(G)\leq 1-\frac{1}{2|L(G)|}.$$

Proof. Since G is not simple, it follows that there exists a non-trivial and proper normal subgroup H of G. Let H_i be a subgroup of G for $i=1,\dots,n=|L(G)|$ with $H_1=\{e\}$, $H_2=H$, and $H_n=G$. Then $P(G)=\dfrac{\sum_{i=1}^{n}|N_G(H_i)|}{|G||L(G)|}=\dfrac{|G|+|G|+|G|+|N_G(H_3)|+\cdots+|N_G(H_{n-1})|}{|G||L(G)|}$. Having $H_i\neq\{e\}$ a subgroup of $N_G(H_i)$ implies that $|N_G(H_i)|\geq|H_i|\geq 2$. We get that $P(G)\geq\frac{3|G|+2(|L(G)|-3)}{|G||L(G)|}$. □

Example 3.4. As an illustration of Lemma 3.3, we consider the groups S_3 and D_8.
We consider first $\frac{2}{3}\leq P(S_3)=\frac{2}{3}\leq\frac{11}{12}$. Similarly, $\frac{19}{40}\leq P(D_8)=\frac{4}{5}\leq\frac{19}{20}$.

We present a generalization of Theorem 3.1 and Lemma 3.3.

Theorem 3.5. *Let G be a non-Hamiltonian group of finite order $n=p_1\cdots p_k$ with $p_1<p_2<\cdots<p_k$ be prime numbers. Then*

$$\frac{2}{|L(G)|}+\frac{p_1}{|G|}-\frac{2p_1}{|G||L(G)|}\leq P(G)\leq 1-\frac{p_1-1}{p_1|L(G)|}.$$

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Lemma 3.6. *Let G be a non-Hamiltonian group of finite order $n = p_1 \cdots p_k$ with $p_1 < p_2 < \cdots < p_k$ be prime numbers. If G is a non-simple group, then*

$$\frac{3}{|L(G)|} + \frac{p_1}{|G|} - \frac{3p_1}{|G||L(G)|} \leq P(G) \leq 1 - \frac{p_1 - 1}{p_1|L(G)|}.$$

We consider the simple non-Hamiltonian group, and we find upper bound for its probability to be Hamiltonian.

Theorem 3.7. *Let G be a non-Hamiltonian finite group with even order. If G is simple, then*

$$P(G) \leq \frac{1}{2} + \frac{1}{|L(G)|}.$$

Proof. Let G be a simple group with $L(G) = \{\{e\}, H_2, H_2, \ldots, H_{m-1}, G\}$. It is clear that $|H_i| \leq \frac{|G|}{2}$ for all $i = 2, \ldots, m-1$. The latter and having $N_G(H_i) \neq G$ a subgroup of G implies that $|N_G(H_i)| \leq \frac{|G|}{2}$. Having $P(G) = \frac{|N_G(\{e\})| + |N_G(H_2)| + \cdots + |N_G(H_{m-1})| + |N_G(G)|}{m|G|}$ implies that $P(G) \leq \frac{2|G| + \frac{(m-2)|G|}{2}}{m|G|} = \frac{1}{2} + \frac{1}{|L(G)|}$. ☐

Corollary 3.8. *Let G be a non-Hamiltonian finite group with even order. If G is simple and $|L(G)| \to \infty$, then $P(G) \leq \frac{1}{2}$.*

Corollary 3.9. *Let A_n be the alternating group. If $n \to \infty$, then $P(A_n) \leq \frac{1}{2}$.*

Theorem 3.10. *Let G be a non-Hamiltonian finite group with the prime number p as the smallest divisor of $|G|$. If G is simple, then $P(G) \leq \frac{1}{p} + \frac{2}{|L(G)|}(1 - \frac{1}{p})$.*

Proof. The proof is similar to That of Theorem 3.7. ☐

Corollary 3.11. *Let G be a non-Hamiltonian finite group with the prime number p as the smallest divisor of $|G|$. If G is simple and $|L(G)| \to \infty$, then $P(G) \leq \frac{1}{p}$.*

The alternating group A_n is a simple non-Hamiltonian group for $n \geq 5$. In what follows, we find an upper bound for the probability of A_n to be Hamiltonian.

Theorem 3.12. *Let A_n be the alternating group with $n \geq 4$. Then*

$$P(A_n) = \begin{cases} 1 & \textit{if } n = 1,\ 2,\ \textit{or } 3; \\ \frac{1}{2} & \textit{if } n = 4. \end{cases}$$

Furthermore, $P(A_n) \leq \frac{19}{70}$ for $n \geq 5$.

Proof. For $n \in \{1, 2, 3\}$, A_n is a cyclic group and hence $P(A_n) = 1$. For $n = 4$, $P(A_n) = \frac{1}{2}$ by Example 2.12. Let $n \geq 5$. Then A_n has at least 15 subgroups of order 2, 10 subgroups of order 3, 1 subgroup of order 5. As a result, $|L(A_n)| = m \geq 28$. For every proper subgroup H of A_n, H has an index of at least n, i.e., $|H| \leq \frac{|A_n|}{2}$. We have

$$P(A_n) \leq \frac{2|A_n| + (m-2)\frac{|A_n|}{n}}{m|A_n|} = \frac{2}{m}(1 - \frac{1}{n}) + \frac{1}{n} \leq \frac{2}{m} + \frac{1}{n}.$$

The latter and having $n \geq 5, m \geq 28$ implies that $P(A_n) - \frac{19}{70} \leq \frac{2}{28} + \frac{1}{5} - \frac{19}{70} = 0$. □

Since $|L(A_n)|$ increases rapidly with large n, it follows from the proof of Theorem 3.12 that $\lim_{n\to\infty} P(A_n) = 0$.

4. Relationship between Hamiltonian degree of a finite group and its conjugacy classes subgroups

In this section, we use the conjugacy classes subgroups to derive an equivalent formula for the Hamiltonian degree of a finite group. Furthermore, we provide examples on the Hamiltonian degree of S_n and A_n for small numbers of n.

Theorem 4.1. *Let G be a finite group and $L = \{H_1^G, H_2^G, \dots, H_k^G\}$ be the set of conjugacy classes subgroups of G. Here, $H_i^G = \{H_i^g : g \in G\}$ is the set of all conjugate subgroups of H_i, $i = 1, 2, \dots, k$. Then*

$$P(G) = \frac{1}{|G||L(G)|}\sum_{j=1}^{k} |H_i^G||N_G(H_i)|.$$

Proof. First, we note that $L(G)$ can be partitioned into distinct conjugacy classes subgroups of G. In other words, if $L = \{H_1^G, H_2^G, \dots, H_k^G\}$ is set of all distinct conjugacy classes subgroups of G, then $|L(G)| = \sum_{i=1}^{k} |H_i^G|$. Moreover, if $G = \{H_i, {H_i}^{g_2}, \dots, {H_i}^{g_t}\}$, then it is easy to see that $N({H_i}^{g_i}) = N(H_i)^{g_i}$ for $i = 2, \dots, t$. Thus, $N({H_i}^{g_i}) = N(H_i)$. Therefore,

$$P(G) = \frac{1}{|G||L(G)|}\sum_{H\in L(G)} |N_G(H)| = \frac{1}{|G||L(G)|}\sum_{j=1}^{k} |H_i^G||N_G(H_i)|. \quad \square$$

Corollary 4.2. *Let G be a finite group and k the number of distinct conjugacy classes subgroups of G. Then $P(G) = \frac{k}{|L(G)|}$.*

Proof. We have that the number of conjugate subgroups of H in G is equal to $[G : N_G(H)]$. Theorem 4.1 asserts that

$$P(G) = \frac{1}{|G||L(G)|}\sum_{j=1}^{k} |H_i^G||N_G(H_i)| = \frac{1}{|G||L(G)|}\sum_{j=1}^{k} \frac{|G|}{|N_G(H_i)|}|N_G(H_i)|.$$

Therefore, $P(G) = \frac{k}{|L(G)|}$. □

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Example 4.3. The conjugacy classes subgroups of S_3 are $L = \{\{e\}^{S_3}, \langle(12)\rangle^{S_3}, \langle(123)\rangle^{S_3}, S_3^{S_3}\}$. Here, $\{e\}^{S_3} = \{e\}$, $\langle(12)\rangle^{S_3} = \{e,(12),(13),(23)\}$, $\langle(123)\rangle^{S_3} = \{\langle(123)\rangle\}$, and $S_3^{S_3} = \{S_3\}$. Thus, we have 4 distinct conjugacy classes subgroups for S_3, and hence, $P(S_3) = \frac{4}{6} = \frac{2}{3}$.

Next, we apply Corollary 4.2 to compute the Hamiltonian degrees for A_n and S_n with $n = 3, 4, \ldots, 9$. We use the group theory package GAP for computational purposes.

Example 4.4. The below table represents the Hamiltonian degrees for A_n for $3 \leq n \leq 9$. Here, k represents the number of distinct conjugacy classes of A_n.

n	$\lvert A_n\rvert$	$\lvert L(A_n)\rvert$	k	$P(A_n)$
3	3	2	2	1
4	12	10	5	$\frac{1}{2}$
5	60	59	9	$\frac{9}{59}$
6	360	501	22	$\frac{22}{501}$
7	2520	3786	40	$\frac{20}{1893}$
8	20160	48337	137	$\frac{137}{48337}$
9	181440	508402	223	$\frac{223}{508402}$

Example 4.5. The below table represents the Hamiltonian degrees for S_n for $3 \leq n \leq 9$. Here, k represents the number of distinct conjugacy classes of S_n.

n	$\lvert S_n\rvert$	$\lvert L(S_n)\rvert$	k	$P(S_n)$
3	6	6	4	$\frac{2}{3}$
4	24	30	11	$\frac{11}{30}$
5	120	156	19	$\frac{19}{156}$
6	720	1455	56	$\frac{56}{1455}$
7	5040	11300	96	$\frac{24}{2825}$
8	40320	151221	296	$\frac{296}{151221}$
9	362880	164723	554	$\frac{554}{164723}$

5. Formula for the Hamiltonian degree of Dihedral groups

This section focuses on deriving a specific formula for the probability that a Dihedral group is Hamiltonian. Dihedral groups, which represent the symmetries of regular polygons, have unique structural properties that make them an interesting subject of study in group theory. We present a detailed analysis of the subgroups of dihedral groups and their normalizers, leading to the formulation of a probability expression for these groups.

The dihedral group $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1}\rangle$ is the symmetry group of a regular polygon with n sides and it has the order $2n$. $L(D_{2n}) = \tau(n) + \sigma(n)$. For odd n, the only normal subgroups of D_{2n} are $\langle a^d\rangle$ for all divisors d of n. Subgroups of D_{2n} have the form $\langle a^d\rangle$ and $\langle a^d, a^i b\rangle$ where d is a divisor of n and $i = 0, 1, \ldots, d-1$.

Theorem 5.1. *Let p be a prime number and D_{2p} the Dihedral group with $2p$ elements. Then*

$$P(D_{2p}) = \begin{cases} 1 & \text{if } p = 2; \\ \frac{4}{p+3} & \text{otherwise.} \end{cases}$$

Proof. For $p = 2$, D_{2p} is an abelian group (Klein four-group), and hence, $P(D_{2p}) = 1$. Let p be an odd prime. We have $D_{2p} = \langle a, b : a^p = b^2 = e, bab = a^{-1}\rangle$. The elements of D_{2p} are given as follows: $\{e, a, \ldots, a^{p-1}, b, ab, \ldots, a^{p-1}b\}$. D_{2p} has p subgroups of order 2 given by $H_i = \langle a^i b\rangle$ for $i = 0, \ldots, p-1$, one subgroup of order p given by $K = \langle a\rangle$, in addition to the trivial subgroup $\{e\}$ and D_{2p}. We have $N_{D_{2p}}(D_{2p}) = N_{D_{2p}}(K) = N_{D_{2p}}(\{e\}) = D_{2p}$ and $N_{D_{2p}}(H_i) = H_i$ for $i = 0, 1, \ldots, p-1$. We get that the total number of elements in all normalizers of D_{2p} is $2p + 2p + 2p + 2p = 8p$. The latter and having $|L(D_{2p})| = p + 3$ implies that $P(D_{2p}) = \frac{8p}{2p(p+3)} = \frac{4}{p+3}$. ☐

Corollary 5.2. *Let p be a prime number, $k > 1$ an integer, and D_{2p} the Dihedral group with $2p$ elements. Then there exist a prime number p with $P(D_{2p}) = \frac{1}{k}$ if and only if $4k - 3$ is a prime number.*

Proof. The proof follows from Theorem 5.1. ☐

Example 5.3. Let D_6 be the Dihedral group of 6 elements. Theorem 5.1 implies that $P(D_{2(3)}) = \frac{4}{3+3} = \frac{2}{3} = P(S_3)$.

Theorem 5.4. *Let p be a prime number and D_{2p^2} the Dihedral group with $2p^2$ elements. Then*

$$P(D_{2p^2}) = \begin{cases} \frac{4}{5} & \text{if } p = 2; \\ \frac{6}{p^2+p+4} & \text{otherwise.} \end{cases}$$

Proof. For $p = 2$, $D_{2p^2} = D_8$. Example 2.17 implies that $P(D_8) = \frac{4}{5}$. Let p be an odd prime. D_{2p^2} has p^2 subgroups of order 2 given by $H_i = \langle a^i b\rangle$ for $i = 0, 1, \ldots, p^2 - 1$, one subgroup of order p^2 given by $K = \langle a\rangle$, one subgroup of order p given by $K' = \langle a^p\rangle$, p subgroups of order $2p$ given by $K_i = \langle a^p, a^i b\rangle$ for $i = 0, 1, \ldots, p-1$ besides the trivial subgroup $\{e\}$ and D_{2p}. Thus, $|L(D_{2p^2})| = p^2 + p + 4$. We have $N_{D_{2p^2}}(D_{2p^2}) = N_{D_{2p^2}}(K) = N_{D_{2p^2}}(K') = N_{D_{2p^2}}(\{e\}) = D_{2p^2}$, $N_{D_{2p^2}}(H_i) = H_i$ for $i = 0, 1, \ldots, p^2 - 1$, and $N_{D_{2p^2}}(K_i) = K_i$ for $i = 0, 1, \ldots, p-1$. We get that the total number of elements in all normalizers of D_{2p^2} is $4(2p^2) + 2p^2 + 2p^2 = 12p^2$. The latter implies that $P(D_{2p^2}) = \frac{12p^2}{2p^2(p^2+p+4)} = \frac{6}{p^2+p+4}$. ☐

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Example 5.5. Let $D_{18} = D_{2(3^2)}$ be the Dihedral group of 18 elements. Theorem 5.4 implies that $P(D_{18}) = \frac{6}{3^2+3+4} = \frac{3}{8}$.

Theorem 5.6. *Let p be an odd prime number, $n \geq 1$ an integer, and D_{2p^n} the Dihedral group with $2p^n$ elements. Then $P(D_{2p^n}) = \frac{2n+2}{n+2+p+p^2+\cdots+p^n}$.*

Proof. $L(D_{2p^n}) = \tau(2p^n) + \sigma(2p^n) = n + 2 + p + \cdots + p^n$. There are n normal subgroups of D_{2p^n} besides D_{2p^n} and the trivial subgroup. These subgroups are given by $\langle a^i \rangle$ with $i = 1, p, p^2, \ldots, p^{n-1}$. The remaining subgroups are characterized as follows:
$H_{i,p^k} = \langle a^{p^k}, a^i b \rangle$ with $2p^{n-k}$ elements and $i = 0, 1, \ldots, p^k - 1$. Having $N_{D_{2p^n}}(H_{i,p^k}) = H_{i,p^k}$ implies that $|N_{D_{2p^n}}(H_{i,p^k})| = 2p^{n-k}$ for every $i = 0, 1, \ldots, p^k - 1$. Thus, for every k, we have p^k distinct subgroups of order $2p^{n-k}$. The latter and having $N_{D_{2p^n}}(H_{i,p^k}) = H_{i,p^k}$ implies that for every k, we have $\sum_{i=1}^{p^k-1} |N_{D_{2p^n}}(H_{i,p^k})| = 2p^n$. Hence, $P(D_{2p^n}) = \frac{2p^n(n+2)+n2p^n}{2p^n(n+2+p+p^2+\cdots+p^n)} = \frac{2n+2}{n+2+p+p^2+\cdots+p^n}$. □

Example 5.7. Let $D_{54} = D_{2(3^3)}$ be the Dihedral group of 54 elements. Theorem 5.6 implies that $P(D_{54}) = \frac{2(3)+2}{3+2+3+3^2+3^3} = \frac{2}{11}$.

Corollary 5.8. *Let p be an odd prime, $n \geq 1$ an integer, and D_{2p^n} be the dihedral group of $2p^n$ elements. Then $\lim_{n\to\infty} P(D_{2p^n}) = 0$.*

Proof. The proof follows from Theorem 5.6. □

Corollary 5.9. *Let p be an odd prime, $n \geq 1$ an integer, and D_{2p^n} be the dihedral group of $2p^n$ elements. Then $\lim_{p\to\infty} P(D_{2p^n}) = 0$.*

Proof. The proof follows from Theorem 5.6. □

Theorem 5.10. *Let p be an odd prime, $n \geq 1$ an integer, and D_{2p^n} be the dihedral group of $2p^n$ elements. Then $P(D_{2p^n}) = \frac{1}{2}$ if and only if $p = 5$ and $n = 1$.*

Proof. Theorem 5.1 implies that, for $p = 5$ and $n = 1$, $P(D_{2p^n}) = \frac{4}{p+3} = \frac{1}{2}$.

We prove that $P(D_{2p^{n-1}}) > P(D_{2p^n})$ for every $n \geq 1$. One can easily see that $P(D_{2p}) > P(D_{2p^2})$. Let $n \geq 2$, Some computations lead to $P(D_{2p^{n-1}}) - P(D_{2p^n}) > 0$. The latter implies that $P(D_{2p^n}) < P(D_{2p^2}) \leq P(D_{2(3^2)}) = \frac{3}{8} < \frac{1}{2}$. □

6. Conclusion

This paper introduced the concept of Hamiltonian degree $P(G)$ as a measure to determine the likelihood of a finite group G being Hamiltonian. By examining commutativity degrees and subgroup normality, explicit formulas for $P(G)$ were derived for various finite groups, with a particular focus on dihedral groups. General bounds for $P(G)$ were also established, creating a framework for analyzing Hamiltonian degree in both simple and non-simple groups, including a detailed study of the alternating

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group A_n. Additionally, the Hamiltonian degree of the direct product of finite groups with coprime orders was computed. These findings offer theoretical insights and practical tools for analyzing complex group behaviors.

For future research, we propose exploring the Hamiltonian degree in other group categories, and investigating further applications in different mathematical contexts. Additionally, extending our methods to infinite groups and other algebraic structures could provide greater understanding of the nature of commutativity and normality in group theory.

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