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FINITE SUBGROUPS OF AUTOMORPHISMS OF FREE PRODUCTS

IOANNIS PAPAVASILEIOU* AND DIONYSIOS SYRIGOS

Abstract. We study finite subgroups of outer automorphisms of free products. We give upper bounds for the orders of these finite subgroups as well as bounds for the orders of individual torsion outer automorphisms under some (necessary) conditions for the free factors.

1. Introduction

One of the major problems in group theory is to determine the structure of the group of automorphisms $\mathrm{Aut}(G)$ of a group G, especially its finite subgroups as well as the behavior of individual automorphisms. For example, using Nielsen's Realization Theorem one can study finite subgroups of $\mathrm{Aut}(F_p)$ and $\mathrm{Out}(F_p)$ [15, 3, 7]. The maximal order of a finite subgroup of $\mathrm{Out}(F_p)$ is $2^p \cdot p!$ [14] and is realized as the group of automorphisms of the standard rose R_p. It can also been proved [1],[10] that the maximal order of torsion elements in $\mathrm{Aut}(F_p)$ grows asymptotically like $\exp(\sqrt{p \log p})$, i.e., if we denote by $H(p)$ the maximal order of torsion elements in $\mathrm{Aut}(F_p)$, then $\lim\limits_{p \to +\infty} \dfrac{\log H(p)}{\sqrt{p \log p}} = 1$.

In this note we study finite subgroups of the group of outer automorphisms $\mathrm{Out}(G)$ of a group G with a free product decomposition $G = G_1 * \cdots * G_k * F_p$. We denote by $\mathrm{Out}(G, \mathcal{G})$ the subgroup of $\mathrm{Out}(G)$ which consists of those outer automorphisms that permute the conjugacy classes of the free factors $G_1, \ldots, G_k$. We firstly give a (uniform) upper bound for the orders of its finite subgroups.

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* Corresponding author.

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Theorem 3.1. *Let G be a finitely generated group and $G = G_1 * \cdots * G_k * F_p$ a free product decomposition of G such that for the G_i's and $\mathrm{Aut}(G_i)$'s there is a (uniform) bound for the orders of their finite subgroups. Then, there is a (uniform) bound for the orders of finite subgroups of $\mathrm{Out}(G, \mathcal{G})$, too.*

The converse of the above theorem is also true (see Proposition 3.3), thus the assumptions made on the G_i's and $\mathrm{Aut}(G_i)$'s are minimal.

Furthermore, we prove that finite subgroups of $\mathrm{Out}(G)$ have uniformly bounded orders in case where G is either a finite extension of a finitely generated free group (see Corollary 4.4) or a finite extension of a free product as in Theorem 3.1 (see Corollary 4.5).

Finally, we give an upper bound for the orders of individual outer automorphisms.

Theorem 3.2. *Let G be a finitely generated group and $G = G_1 * \cdots * G_k * F_p$ a free product decomposition of G. Let M_i be the maximal order among the elements of finite order in $\mathrm{Aut}(G_i)$ and let N_i be the corresponding maximal order among the elements of finite order in G_i. Then every torsion element of $\mathrm{Out}(G, \mathcal{G})$ has order at most $M = M(M_i, N_i, k, p)$, where*

$$M = \left(\left[e^{\frac{k}{e}} \right] + 1 \right) \cdot \prod_i M_i \cdot \exp\big((1+\theta_p)\sqrt{p \log p}\big) \cdot \big(\max_i \{N_i\} \big)^{k-2+2p}$$

and $\lim\limits_{p \to +\infty} \theta_p = 0$.

2. Preliminaries

2.1. **Free factor systems and automorphisms.** Let G be a finitely generated group and $G = G_1 * \ldots * G_k * F_p$ a free product decomposition of G. If this is the Grushko decompositions of G, then every automorphism of G permutes the conjugacy classes of the G_i's, i.e. if $\phi \in \mathrm{Aut}(G)$, then $\phi(G_i) = gG_jg^{-1}$ for some $g \in G$. Apart from the Grushko decomposition, there are other decompositions which could be useful, too. We let $\mathcal{G} = \{[G_1], \ldots, [G_k]\}$ be the set of G-conjugacy classes of the subgroups G_i; we call it a **free factor system** of G. There is a natural action of the group of outer automorphisms $\mathrm{Out}(G)$ of G on the set of free factor systems of G. If $[\phi] \in \mathrm{Out}(G)$ is an outer automorphism and $\mathcal{G} = \{[G_1], \ldots, [G_k]\}$ is a free factor system, then $[\phi](\mathcal{G}) = \{[\phi(G_1)], \ldots, [\phi(G_k)]\}$ is a free factor system of G, too. In general, it is not true that $[\phi](\mathcal{G}) = \mathcal{G}$. The **relative (to $\mathcal{G}$) outer automorphism group** $\mathrm{Out}(G, \mathcal{G})$ is the subgroup of $\mathrm{Out}(G)$ consisting of those outer automorphisms $[\phi]$ which preserve $\mathcal{G}$, i.e. $[\phi](\mathcal{G}) = \mathcal{G}$.

Let $\mathrm{Out}'(G, \mathcal{G})$ be the finite index subgroup of $\mathrm{Out}(G, \mathcal{G})$ consisting of outer automorphisms fixing each conjugacy class $[G_i]$. Since each G_i equals to its normalizer in G, $\mathrm{Out}'(G, \mathcal{G})$ maps onto $\mathrm{Out}(G_i)$, for every i. There is also a homomorphism $\mathrm{Out}'(G, \mathcal{G}) \to \mathrm{Out}(F_p)$, hence we have a natural map:

$$\pi : \mathrm{Out}'(G, \mathcal{G}) \to \mathrm{Out}(F_p) \times \prod_i \mathrm{Out}(G_i).$$

We will denote by $\mathrm{Out}(G, \mathcal{G}^{(t)})$ the kernel of the epimorphism $\mathrm{Out}'(G, \mathcal{G}) \to \prod_i \mathrm{Out}(G_i)$ and by $\mathrm{Out}_0(G, \mathcal{G}^{(t)})$ the kernel of the epimorphism π.

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2.2. **The relative outer space.** We shall describe a space, which was introduced by Guirardel and Levitt in [4], that depends only on the chosen free factor system $\mathcal{G}$ of G.

We need the following definitions. Let G be a group acting on a metric simplicial tree T, i.e. T is a simplicial tree and each orbit of edges is assigned with a positive length.

Definition 2.1. *We say that a metric simplicial tree T is a G-**tree** if there exists a G-action on T which is:*

1. *isometric, i.e. $d_T(x, y) = d_T(gx, gy)$ for every $x, y \in T$ and $g \in G$,*
2. *simplicial, i.e. vertices are sent to vertices and edges to edges,*
3. *minimal, i.e. there is no proper, non trivial G-invariant subtree of T and*
4. *co-compact, i.e. the quotient space T/G is a finite graph.*

Definition 2.2. *Let $\mathcal{G} = \{[G_1], \ldots, [G_k]\}$ be a free factor system of G. A G-tree T will be said to be a $\mathcal{G}$-**tree** if:*

1. *all edge stabilisers are trivial and*
2. *the free factor system induced by the vertex stabilisers of T is exactly $\mathcal{G}$, i.e. for each i there is exactly one orbit of vertices with stabilizer conjugate to G_i and all other points have trivial stabilizer.*

We are now in position to give the definition of the outer space relative to $\mathcal{G}$.

Definition 2.3. *The **(relative) outer space** $\mathcal{O} = \mathcal{O}(\mathcal{G})$ is defined to be the space of equivalence classes of $\mathcal{G}$-trees where two trees are considered as equivalent if there exists a G-equivariant homothety between them.*

In order to refer to a point of $\mathcal{O}(\mathcal{G})$ we shall write $T \in \mathcal{O}(\mathcal{G})$ instead of its class $[T] \in \mathcal{O}(\mathcal{G})$.

The group $\mathrm{Aut}(G, \mathcal{G})$, consisting of automorphisms which permute the conjugacy classes of the G_i's, acts on the set of $\mathcal{G}$-trees by changing the action, i.e. for $\phi \in \mathrm{Aut}(G, \mathcal{G})$ and a $\mathcal{G}$-tree T, $\phi(T)$ is defined to be the $\mathcal{G}$-tree with the same underlying tree as T and G-action given by $(g, x) \in G \times T \mapsto \phi^{-1}(g)x \in T$. It is clear that inner automorphisms of G act by isometries on the set of $\mathcal{G}$-trees hence there is an induced action of $\mathrm{Out}(G, \mathcal{G})$ on $\mathcal{O}(\mathcal{G})$. Roughly speaking, every element of $\mathcal{O}(\mathcal{G})$ can be viewed as a graph of groups and the action of $\mathrm{Out}(G, \mathcal{G})$ is by changing the marking.

Every $T \in \mathcal{O}$ determines an (open) simplex $\Delta(T) \subset \mathcal{O}$ which is the set of points of $\mathcal{O}$ obtained from T by changing the lengths of (orbits of) edges in such a way so that every edge has positive length and the sum of lengths of edges in the quotient graph T/G is equal to 1. In this way we can embed every open simplex $\Delta(T)$ of $\mathcal{O}$ in a Euclidean space $\mathbb{R}^n$, where n depends on the number of orbits of edges of T. Thus, $\mathcal{O}$ can be thought of as a union of open simplices as described above. It is easy to see that there are only finitely many orbits of simplices under the action of $\mathrm{Out}(G, \mathcal{G})$ on $\mathcal{O}(\mathcal{G})$.

We recall the description of the stabilizer of $\Delta(T)$, i.e. the subgroup $\mathrm{Out}^{\Delta(T)}(G, \mathcal{G}) \leq \mathrm{Out}(G, \mathcal{G})$ sending $\Delta(T)$ to itself, as given in [4]. Let $\mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G}) \leq \mathrm{Out}^{\Delta(T)}(G, \mathcal{G})$ be the finite index subgroup consisting of outer automorphisms acting trivially on the quotient graph T/G. We denote by n_i the degree of every non-free vertex v_i of T/G.

Remark 2.4. *It is well known that for every point $T \in \mathcal{O}$, the sum $\sum_{v_i \in V(T/G)} n_i$ is at most $2(3k-3+2p)$* [13, 11]. *In fact, it is not difficult to see that the above bound can be improved to $2k-2+2p$, since the minimum number of orbits of edges of a point in $\mathcal{O}$ is $k-1+p$.*

The group $\mathrm{Out}_0^{\Delta(T)}(G,\mathcal{G})$ is a direct product $\prod_{i=1}^{k} M_{n_i}(G_i)$ of groups which fit in exact sequences

$$\{1\} \to G_i^{n_i}/Z(G_i) \to M_{n_i}(G_i) \to \mathrm{Out}(G_i) \to \{1\} \tag{2.1}$$

$$\{1\} \to G_i^{n_i-1} \to M_{n_i}(G_i) \to \mathrm{Aut}(G_i) \to \{1\} \tag{2.2}$$

where the center $Z(G_i)$ is embedded diagonally into $G_i^{n_i}$. The latter exact sequence is split hence $M_{n_i}(G_i) = G_i^{n_i-1} \rtimes \mathrm{Aut}(G_i)$ where $\mathrm{Aut}(G_i)$ acts diagonally on $G_i^{n_i-1}$. It can be proved [9, Proposition 4.2] that

$$\mathrm{Out}_0^{\Delta(T)}(G,\mathcal{G}) = \prod_{i=1}^{k} M_{n_i}(G_i) = \prod_{i=1}^{k} \left(G_i^{n_i-1} \rtimes \mathrm{Aut}(G_i)\right). \tag{2.3}$$

In particular, the kernel of the product epimorphism

$$p : \mathrm{Out}_0^{\Delta(T)}(G,\mathcal{G}) = \prod_{i=1}^{k} M_{n_i}(G_i) \to \prod_{i=1}^{k} \mathrm{Aut}(G_i) \tag{2.4}$$

is $\ker p = \prod_{i=1}^{k} G_i^{n_i-1}$.

The following result, which is the Nielsen Realisation for free products, is a key ingredient for the proof of our main theorem.

Theorem 2.1. [6, Corollary 6.1.] *Let G be a finitely generated group and $G = G_1 * \ldots * G_k * F_p$ a free product decomposition of G. Every finite subgroup $H \leq \mathrm{Out}(G,\mathcal{G})$ fixes a point of $\mathcal{O}(\mathcal{G})$.*

2.3. **Finite subgroups.** Let G be a group and g a torsion element of G. We denote by $o(g)$ the order of g in G.

Lemma 2.5. *Let G be a direct product $G = \prod_{1 \leq i \leq n} G_i$ where the G_i's have an upper bound M_i for the orders of their torsion elements. Then, torsion elements of G have order at most*

$$\max_{(g_i)_{1 \leq i \leq n}} \mathrm{lcm}\,\{o(g_1), \ldots, o(g_n)\} \leq \prod_i M_i$$

Proof. Note that if $g = (g_1, \ldots, g_n)$ is a torsion element of G, then $o(g) = \mathrm{lcm}\,\{o(g_1), \ldots, o(g_n)\}$. □

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Remark 2.6. *There are cases where the above inequality is achieved as equality. For example, take $\mathbb{Z}_2 \times \mathbb{Z}_3$. On the other hand, if two of the G_i's are isomorphic, then the inequality is proper. In fact in a direct product of the form $\prod_{1\le i\le n} G_i = A^n$ the torsion elements have order at most*

$$M(M-1)\cdot \ldots \cdot (M-n+1) = (M)_n$$

where M is an upper bound for the order of the torsion elements of A.

Lemma 2.7. *Let G be a direct product $G = \prod_{1\le i\le n} G_i$ where the G_j's have an upper bound M_j for the orders of their finite subgroups. Then, every finite subgroup of G has order at most $\prod_{i=1}^{n} M_i$.*

Proof. We denote by $p_i : G \to G_i$ the natural projections. Let H be a finite subgroup of G. Consider the homomorphism

$$H \to p_1(H) \times \cdots \times p_n(H), \text{ given by } h \mapsto \Big(p_1(h), \ldots, p_n(h)\Big).$$

This map is injective, and each image $p_i(H) \cong H/\mathrm{ker}p_i|_H$ is a finite subgroup of G_i. Hence,

$$|H| \le \prod_{i=1}^{n} |p_i(H)| \le \prod_{i=1}^{n} M_i.$$

□

3. The main theorem

We now give an upper bound on the orders of finite subgroups of $\mathrm{Out}(G, \mathcal{G})$.

Theorem 3.1. *Let G be a finitely generated group and $G = G_1 * \cdots * G_k * F_p$ a free product decomposition of G such that for the G_i's and $\mathrm{Aut}(G_i)$'s there is a (uniform) bound for the orders of their finite subgroups. Then, there is a (uniform) bound for the orders of finite subgroups of $\mathrm{Out}(G, \mathcal{G})$, too.*

Proof. Let H be a finite subgroup of $\mathrm{Out}(G, \mathcal{G})$. By Theorem 2.1, H fixes a point T of $\mathcal{O}(\mathcal{G})$, hence preserves the open simplex $\Delta(T)$, i.e. $H \le \mathrm{Out}^{\Delta(T)}(G, \mathcal{G})$. We will give an upper bound for the cardinality of H in the following four steps.

(1) In order to make use of the strong properties of $\mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G})$ which is of finite index in $\mathrm{Out}^{\Delta(T)}(G, \mathcal{G})$, we first consider the intersection $H_1 = H \cap \mathrm{Out}'(G, \mathcal{G})$. Then H/H_1 is isomorphic to a subgroup of $\prod_{j=1}^{m} \mathrm{Sym}_{k_j}$ (where m is the number of isomorphic classes of the G_j's and k_j is the number of appearances of the j-th class) since

$$H/H_1 = \frac{H}{H \cap \mathrm{Out}'(G, \mathcal{G})} \cong \frac{H \cdot \mathrm{Out}'(G, \mathcal{G})}{\mathrm{Out}'(G, \mathcal{G})} \le \frac{\mathrm{Out}(G, \mathcal{G})}{\mathrm{Out}'(G, \mathcal{G})} \le \prod_{j=1}^{m} \mathrm{Sym}_{k_j}.$$

We denote by A the order of a finite subgroup of maximal order in $\prod_{j=1}^{m} \mathrm{Sym}_{k_j}$. Note that $A \le \prod_{j=1}^{m} k_j!$.

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(2) Let H_0 be the intersection $H_1 \cap \mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G})$. Then, H_0 acts trivially on the quotient graph T/G and has finite index in H_1, since

$$|H_1 : H_0| = |H_1 \cdot \mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G}) : \mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G})| \leq |\mathrm{Out}^{\Delta(T)}(G, \mathcal{G}) : \mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G})| < +\infty.$$

In fact, since H_1 fixes the vertices of T/G with non-trivial vertex group, the index $|H_1 : H_0|$ depends only on $p = \mathrm{rank}(F_p)$.

(3) Since H_0 is a subgroup of $\mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G})$ we can consider the restriction of the homomorphism p to H_0, namely $p|_{H_0} : H_0 \to \prod_{i=1}^{k} \mathrm{Aut}(G_i)$. By the first isomorphism theorem, $H_0/\mathrm{ker}p|_{H_0} \cong \mathrm{Im}p|_{H_0} \leq \prod_{i=1}^{k} \mathrm{Aut}(G_i)$. Since H_0 is finite (being a subgroup of H), the image $\mathrm{Im}p|_{H_0}$ is a finite subgroup of $\prod_{i=1}^{k} \mathrm{Aut}(G_i)$, and therefore, by Lemma 2.7, its cardinality is bounded by the product $\prod_{i=1}^{k} M_i$ where each M_i is an upper bound for the orders of finite subgroups of $\mathrm{Aut}(G_i)$.

(4) Finally we bound the order of the finite group $\mathrm{ker}p|_{H_0}$. By (2.4) $\mathrm{ker}p|_{H_0} = \mathrm{ker}p \cap H_0 \leq \prod_{i=1}^{k} G_i^{n_i-1}$, and thus, once again by Lemma 2.7, the order $|\mathrm{ker}p|_{H_0}|$ is bounded by the product $\prod_{i=1}^{k} N_i^{n_i-1}$, where each N_i is an upper bound for the orders of finite subgroups of G_i. By Remark 2.4, the product $\prod_{i=1}^{k} N_i^{n_i-1}$ is bounded above by $\left(\max_i\{N_i\}\right)^{k-2+2p}$.

Since

$$|H| = |H/H_1| \cdot |H_1/H_0| \cdot |H_0/\mathrm{ker}p|_{H_0}| \cdot |\mathrm{ker}|_{H_0}|$$

the above steps complete the proof. □

Remark 3.1. *Note that the uniform bound M for the orders of the finite subgroups of $\mathrm{Out}(G, \mathcal{G})$ given in the above proof depends only on the G_i's, their automorphism groups, and $p = \mathrm{rank}(F_p)$.*

We also note that the existence of a uniform bound on the orders of finite subgroups of $\mathrm{Out}(G, \mathcal{G})$ can also be proved by imposing certain hypotheses on the virtual cohomological dimension of the G_i's and their automorphism groups.

Remark 3.2. *Let G be a finitely generated group and $G = G_1 * \ldots * G_k * F_p$ a free product decomposition of G. If*

- *either, the G_i's, $G_i/Z(G_i)$'s are torsion-free with finite virtual cohomological dimension and the $\mathrm{Out}(G_i)$'s are virtually torsion-free with finite virtual cohomological dimension,*
- *or, the G_i's, $\mathrm{Aut}(G_i)$'s have finite virtual cohomological dimension and $\mathrm{Out}(G, \mathcal{G})$ is virtually torsion-free,*

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then (see [4, Theorem 5.2]*)* $\mathrm{Out}(G,\mathcal{G})$ *has finite virtual cohomological dimension. In any case, there exists a finite index, torsion-free, normal subgroup* H *of* $\mathrm{Out}(G,\mathcal{G})$*. Thus, every finite subgroup* K *of* $\mathrm{Out}(G,\mathcal{G})$ *embeds into* $\mathrm{Out}(G,\mathcal{G})/H$ *hence it has order at most* $|\mathrm{Out}(G,\mathcal{G}) : H|$*.*

On the other hand, Theorem 3.1 makes the minimal (necessary and sufficient) assumptions required on the G_i's and $\mathrm{Aut}(G_i)$'s as shown by the following proposition.

Proposition 3.3. *Let* G *be a finitely generated group and* $G = G_1 * \ldots * G_k * F_p$ *a free product decomposition of* G*, where* $(k,p) \neq (2,0)$*. There is a uniform bound for the orders of finite subgroups of* $\mathrm{Out}(G,\mathcal{G})$ *if and only if the same holds for the finite subgroups of the* G_i*'s and* $\mathrm{Aut}(G_i)$*'s.*

Proof. We shall prove that there is a uniform upper bound for the orders of the finite subgroups of the G_i's and $\mathrm{Aut}(G_i)$'s, given that the same is true for the finite subgroups of $\mathrm{Out}(G,\mathcal{G})$.

- If $(k,p) = (0,2)$, then $\mathrm{Out}(G,\mathcal{G}) = \mathrm{Out}(F_2)$ hence there are no free factors G_i's and we have nothing to prove.
- If $(k,p) = (1,1)$, then $\mathcal{O}(\mathcal{G})$ consists of only one simplex $\Delta(T)$. The groups G_i and $\mathrm{Aut}(G_i)$ act on $\mathcal{O}(\mathcal{G})$ and stabilise $\Delta(T)$. Hence, G_i and $\mathrm{Aut}(G_i)$ can be embedded into $\mathrm{Out}^{\Delta(T)}(G,\mathcal{G})$ and hence into $\mathrm{Out}(G,\mathcal{G})$.
- If $p + k \geq 3$, then the G_i's and $\mathrm{Aut}(G_i)$'s are isomorphic to subgroups of $\mathrm{Out}(G,\mathcal{G})$ (see [4]), and hence there is an upper bound on the orders of the finite subgroups of the G_i's and $\mathrm{Aut}(G_i)$'s.

□

Remark 3.4. *Note that if* $G = G_1 * G_2$*, then it can be the case that there is a bound for the orders of finite subgroups of* $\mathrm{Out}(G,\mathcal{G})$ *whereas there is no such bound for the orders of finite subgroups of* G_i*'s. Indeed let* G_1 *be the Prüfer* 2*-group* $\mathbb{Z}(2^\infty) = \{z \in \mathbb{C} | z^{2^n} = 1 \text{ for some } n \in \mathbb{Z}\}$ *and* $G_2 = \mathbb{Z}_2$*. It is clear that* $\mathbb{Z}(2^\infty)$ *contains the* 2^n*-th roots of unity as a cyclic subgroup with* 2^n *elements. Furthermore, the endomorphism ring of every finite subgroup of* $\mathbb{Z}(2^\infty)$ *is isomorphic to the ring of* 2*-adic integers, hence* $\mathrm{Aut}\,(\mathbb{Z}(2^\infty))$ *has order at most* 2 *(see* [5, Subsection 5.2]*). Thus, if* H *is a finite subgroup of* $\mathrm{Out}(G,\mathcal{G})$ *it is in fact a subgroup of the stabilizer* $\mathrm{Out}^{\Delta(T)}(G,\mathcal{G})$ *of the unique simplex* $\Delta(T)$ *of* $\mathcal{O}(\mathcal{G})$*, which coincides with* $\mathrm{Out}_0^{\Delta(T)}(G,\mathcal{G})$ *as* G_1 *and* G_2 *are not isomorphic. Therefore, as* $\mathrm{Out}_0^{\Delta(T)}(G,\mathcal{G}) = \mathrm{Aut}(\mathbb{Z}(2^\infty)) \times \mathrm{Aut}(\mathbb{Z}_2) = \mathrm{Aut}(\mathbb{Z}(2^\infty))$*,* H *has order at most* 2*.*

The following theorem gives explicit upper bounds for the order of an outer automorphism of a free product.

Theorem 3.2. *Let* G *be a finitely generated group and* $G = G_1 * \ldots * G_k * F_p$ *a free product decomposition of* G*. Let* M_i *be the maximal order among the elements of finite order in* $\mathrm{Aut}(G_i)$ *and let* N_i *be the corresponding maximal order among the elements of finite order in* G_i*. Then every torsion element* *of* $\mathrm{Out}(G, \mathcal{G})$ *has order at most* $M = M(M_i, N_i, k, p)$*, where*

$$M = \left(\left[e^{\frac{k}{e}} \right] + 1 \right) \cdot \prod_i M_i \cdot \exp\big((1 + \theta_p) \sqrt{p \log p} \big) \cdot \big(\max_i \{N_i\} \big)^{k-2+2p}$$

and $\lim_{p \to +\infty} \theta_p = 0$.

Proof. Let $[\phi] \in \mathrm{Out}(G, \mathcal{G})$ be an outer automorphism of finite order. Since $[\phi]$ permutes the cojugacy classes of the $[G_i]$'s, there is some $\kappa = \kappa(k) \in \mathbb{N}$ such that $[\phi]^\kappa([G_i]) = [G_i]$, i.e. $[\phi]^\kappa \in \mathrm{Out}'(G, \mathcal{G})$. Since $[\phi]^\kappa$ has finite order, by Theorem 2.1, there is a point T of $\mathcal{O}(\mathcal{G})$ such that $[\phi]^\kappa \in \mathrm{Out}^{\Delta(T)}(G, \mathcal{G})$. By denoting by $g(k)$ the largest order of an element of the symmetric group Sym_k, which is given by Landau's function (see [8] for more details), we have that

$$\kappa = \kappa(k) \leq g(k) \leq e^{\frac{k}{e}}.$$

If we denote by $Q(p)$ the maximum order of a torsion element in $\mathrm{Out}(F_p)$, then $[\phi]^{\kappa Q(p)} \in \mathrm{Out}_0^{\Delta(T)}(G, \mathcal{G})$. It is known (see [1]) that $Q(p) = \exp\big((1 + \theta_p)\sqrt{p \log p}\big)$, where $\lim_{p \to +\infty} \theta_p = 0$.

Therefore, in accordance with the proof of Theorem 3.1, we obtain that $[\phi]^{\kappa Q(p) \lambda} = 1$ where $\lambda \leq \prod_i M_i \times \big(\max_i \{N_i\} \big)^{k-2+2p}$. □

4. Automorphisms of finite extensions of free groups and free products

Definition 4.1. *A group theoretic property P will be called a $\mathcal{P}$**-property** if:*

(1) *P is subgroup closed.*
(2) *If H is a subgroup of finite index in a group G and H has P, then G has P.*
(3) *If G has P and N is a finite normal subgroup of G, then G/N has P.*

Examples of $\mathcal{P}$-properties are: the property of being virtually torsion-free, the property of having finite virtual cohomological dimension, the property of being translation discrete for finitely generated groups (see [2]), and the property of having subgroups with uniformly bounded orders (Lemma 4.2 below).

Lemma 4.2. *The property of having finite subgroups with (uniformly) bounded orders is a $\mathcal{P}$-property.*

Proof. It is clear that this property is subgroup closed.

Suppose that the order of every finite subgroup of H is bounded by M. Let G be a group such that $H \leq G$ and $|G : H| = m$. We may assume that G has a normal subgroup of finite index, which we still denote by H (and $|G : H| = m$). If K is a finite subgroup of G, then we have $\left| \frac{HK}{H} \right| \leq |G : H|$ and $|K \cap H| \leq M$. Furthermore, by the second isomorphism theorem, we have

$$\frac{K}{K \cap H} \cong \frac{HK}{H}$$

hence

$$\frac{|K|}{|H \cap K|} = \left|\frac{K}{H \cap K}\right| = \left|\frac{HK}{H}\right| \leq |G : H|$$

which gives

$$|K| \leq |H \cap K| \cdot |G : H| = m \cdot M,$$

i.e. every finite subgroup of G has uniformly bounded order.

Let N be a finite normal subgroup of a group G and assume that every finite subgroup of G does not have order more than M. Let $\overline{K}$ be a finite subgroup of G/N. There is a subgroup $K \leq G$ such that $\overline{K} = \frac{K}{N} \leq \frac{G}{N}$. The groups N and $\overline{K}$ are finite, hence K is also finite with $|K| \leq M$. Therefore, we have

$$|\overline{K}| = \frac{|K|}{|N|} < |K| \leq M.$$

□

The following lemma has been proved useful for the study of $\mathrm{Out}(G)$ in the case where the group G has a finite index subgroup N with trivial center (see for example [12]).

Lemma 4.3. [12, Lemma 3.3.] *Let G be a finitely generated group, N a normal subgroup of G of finite index with trivial center and $\mathcal{P}$ a group theoretic property as above. If $\mathrm{Out}(N)$ satisfies $\mathcal{P}$, then so does $\mathrm{Out}(G)$.*

It is well known that finite subgroups of $\mathrm{Out}(F_n)$ have uniformly bounded orders. More precisely:

Theorem 4.1. [14] *The maximal order of a finite subgroup of $\mathrm{Out}(F_n)$ is $2^n \cdot n!$.*

Combining Lemma 4.3 with Theorem 4.1 we obtain the following.

Corollary 4.4. *Let G be a finitely generated free-by-finite group G. Then every finite subgroup of $\mathrm{Out}(G)$ has uniformly bounded order. In particular, if G is not virtually cyclic, then the order of a finite subgroup of $\mathrm{Out}(G)$ is at most $2^{\mathrm{rank}(F)} \cdot \mathrm{rank}(F)! \cdot |G : F|$, where F is a finite index, normal, free subgroup of G.*

Proof. G contains a finitely generated free (normal) subgroup F of finite index $|G : F|$. If F has rank 1, then $\mathrm{Out}(G)$ is a finite group. If F has rank more than 1, then its center $Z(F)$ is trivial and hence Lemmas 4.2, 4.3 with Theorem 4.1 imply that $\mathrm{Out}(G)$ has finite subgroups of uniformly bounded orders. In fact, from the proof of Lemma 4.2, we see that every finite subgroup of $\mathrm{Out}(G)$ has order no more than $2^{\mathrm{rank}(F)} \cdot \mathrm{rank}(F)! \cdot |G : F|$. □

We also extend Theorem 3.1 to finite extensions of free products.

Corollary 4.5. *Let $G = G_1 * \ldots * G_k * F_p$ be a free product such that the G_i's and $\mathrm{Out}(G_i)$'s have uniformly bounded orders of finite subgroups. If $\widetilde{G}$ contains G as a normal subgroup of finite index, then $\mathrm{Out}(\widetilde{G})$ has uniformly bounded orders of finite subgroups.*

Proof. Since $Z(G) = 1$, appealing to Lemmas 4.2 and 4.3 complete the proof. □

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Ioannis Papavasileiou

Independent Researcher, Athens, Greece

Email: `13giannispapav@hotmail.gr`

Dionysios Syrigos

Independent Researcher, Basingstoke, UK

Email: `Dionisissyrigos@gmail.com`