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FAILED ZERO FORCING NUMBERS OF GRASSMANN GRAPHS

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Abstract. For a graph G with vertices colored either black or white, consider the following rule to change the colors: the color of a vertex which is the only white neighbor of a black vertex, changes from white to black. A proper subset S of the vertex set of G is called a failed zero forcing set if, regardless of how many times this rule is applied to a graph with the initial black vertices S, at least one white vertex always remains. The maximum size of such a subset is called the failed zero forcing number of G and is denoted by $F(G)$. In this paper, we look at the failed zero forcing numbers of Grassmann graphs $J_q(n,2)$ and prove that $F(J_q(n,2)) = {n \brack 2}_q - b_2'(q)$, for $n \geq 5$, where $b_2'(q)$ is the maximum number of points in the affine or projective plane of order q such that there is no line that passes through exactly one of these points. Moreover, using maximum arcs in the projective planes, we show that if q is a power of two, then $F(J_q(n,2)) = {n \brack 2}_q - (q+2)$.

1. Introduction

Let G be a graph (which is always assumed to be finite and simple) where each vertex is assigned one of two possible colors: black or white. Consider the following rule for spreading the black vertices throughout the graph: if u is a black vertex and v is the only white vertex adjacent to u, then u is a *forcing vertex* and v is *forced* by u to become black. This situation is depicted in Figure 1.

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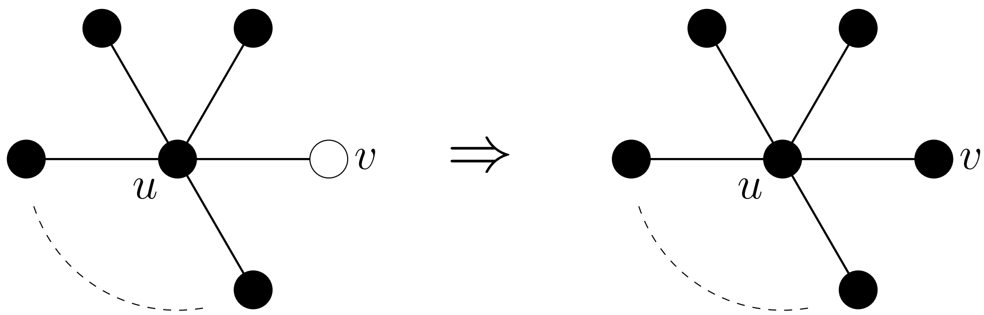

FIGURE 1. u forces v to become black.

To understand the usefulness of this definition, consider a homogeneous system of linear equations where each vertex u of G corresponds to a variable x_u. The equations take the form $c_u x_u + \sum_{w \in N(u)} c_w x_w = 0$, where c_u and all c_w (for $w \in N(u)$) are nonzero scalars and $N(u)$ denotes the set of vertices adjacent to u. Now, if $x_u = 0$ and $x_w = 0$ for all $w \in N(u) \setminus \{v\}$, then $x_v = 0$. In other words, if the black vertices represent variables with a value of zero, then the above rule for changing colors provides a simple method to identify additional zeros among the variables. A *zero forcing* set of G is a proper subset S of its vertex set such that by initially coloring the vertices in S with black and repeatedly applying the aforementioned rule to the resulting graph, all vertices eventually are forced to turn black. A proper subset of the vertex set which is not a zero forcing set is called a *failed zero forcing set*. The *zero forcing number* of G is the minimum cardinality of a zero forcing set of G and is denoted by $Z(G)$. To see the original motivation for defining this graph parameter and its linear algebraic aspects the reader may consult [1]. Further results about zero forcing numbers of graphs can be found in [5, 3, 6].

A closely related parameter is the *failed zero forcing number* of G, which is defined as the maximum cardinality of a failed zero forcing set of G and is denoted by $F(G)$. This graph parameter was first defined in [7]. In that paper, the value of this parameter was determined for complete graphs, complete bipartite graphs, path graphs, cycle graphs, and wheel graphs. In [8] it is proved that if the graph G has at least 7 vertices, then $F(G) \geq 3$ and using this, the complete list of graphs for which $F(G) = 2$ is obtained. In [13] a much stronger inequality was proved: $F(G) \geq \lfloor \frac{n-1}{2} \rfloor$, where n is the size of the vertex set of G. On the other hand, it is proved in [12] that computing the failed zero forcing number of an arbitrary graph is NP-complete, so finding an efficient algorithm for computing this parameter for a general graph is unlikely.

Motivated by this, the authors in [2] investigated the failed zero forcing numbers of some graphs with specific combinatorial structures, namely Kneser graphs, Johnson graphs and hypercubes. Investigating the failed zero forcing numbers of these graphs proves to be an intriguing combinatorial problem, resulting in relatively simple formulas for this parameter. A natural next step in advancing the study of failed zero forcing sets and failed zero forcing numbers in combinatorial graphs is to examine Grassmann graphs. The *Grassmann graph* $J_q(n, k)$, where q is a prime power and $k \leq n$ are positive integers, is a graph

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whose vertex set is the collection of all k-dimensional subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements, and two vertices are adjacent if their intersection is $(k-1)$-dimensional. Note the analogy between $J_q(n,k)$ and the Johnson graph $J(n,q)$ whose vertex set is the collection of all k-subsets of $\{1,\ldots,n\}$ and two vertices are adjacent if their intersection has $k-1$ elements. In [2] it is proved that $F(J(n,k)) = \binom{n}{k} - (k+2)$, when $k \geq 2$ and $n \geq 2k+1$. It turns out that the determination of the failed zero forcing number of $J_q(n,k)$ is a more challenging problem that may not be tackled with the same techniques. So, we confine ourselves to the case $k=2$. In the following sections, we show that the problem of determining the failed zero forcing number of $J_q(n,2)$ is intimately connected to the problem of finding some type of blocking sets in the affine and projective planes. Thus, we begin by reviewing some key concepts regarding these structures.

2. Affine and Projective Planes

Throughout this paper, q denotes a prime power. The *affine plane* of order q or $\mathrm{AG}(2,\mathrm{q})$ is an incidence structure whose point set and blocks are points and lines of $\mathbb{F}_q^2$. So, $\mathrm{AG}(2,\mathrm{q})$ is a 2-design with q^2 points and q^2+q lines in which each line contains q points and each point lies on $q+1$ lines. The *projective plane* of order q or $\mathrm{PG}(2,\mathrm{q})$ is an incidence structure whose points and blocks (or lines) are one and two-dimensional subspaces of $\mathbb{F}_q^3$, respectively. So, $\mathrm{PG}(2,\mathrm{q})$ is a symmetric 2-design, that is, every two lines intersect in exactly one point and every two points lie on exactly one line. Also, $\mathrm{PG}(2,\mathrm{q})$ has q^2+q+1 points and q^2+q+1 lines, each line contains $q+1$ points and each point lies on $q+1$ lines. For further information on the affine and projective planes, the reader may consult [10, sections 12.4, 12.4.1 and 12.5.1].

A set S of points in the projective plane $\mathrm{PG}(2,\mathrm{q})$ such that no three points in it are collinear is called an *arc* and the maximum size of an arc in $\mathrm{PG}(2,\mathrm{q})$ is denoted by $m(2,q)$. The following theorem, establishes the value of $m(2,q)$ for all prime powers q.

Theorem 2.1. [9, Theorem 8. 1. 3]

$$m(2,q) = \begin{cases} q+2 & q \text{ even} \\ q+1 & q \text{ odd.} \end{cases}$$

Using this theorem, we could prove the next lemma, which is an essential tool for determining the failed zero forcing number of $J_q(n,2)$ for even q and $n \geq 5$.

Lemma 2.2. *Let q be even and S be a maximum arc in the projective plane* $\mathrm{PG}(2,\mathrm{q})$. *Then every line that meets S, meets it at least twice.*

Proof. By Theorem 2.1, $|S| = m(2,q) = q+2$. Let $S = \{x_1,\ldots,x_{q+2}\}$ and fix a point of S, namely x_1. We show that each line that meets S in x_1, contains another point of S. For any x_i, $2 \leq i \leq q+2$, the points x_1 and x_i lie on a unique line l_i. Since no three points of S are collinear, $l_2,\ldots,l_{q+2}$ are distinct.

Thus, $l_2, \ldots, l_{q+2}$ are all of the $q+1$ lines in $\mathrm{PG}(2,\mathrm{q})$ that pass through x_1, and since each of these lines passes through another point of S, the assertion is proved. $\square$

3. Grassmann graphs and their failed zero forcing numbers

As mentioned in the introduction, our purpose is to study the failed zero forcing number of $J_q(n,k)$. The number of vertices of this graph can be found using the Gaussian binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

If $k=1$, then $J_q(n,k)$ is a complete graph. Therefore, suppose that $k \geq 2$. Since the function that sends each k-dimensional subspace of $\mathbb{F}_q^n$ to its orthogonal complement is an isomorphism, $J_q(n,k) \cong J_q(n,n-k)$. So, it suffices to consider $k \leq \frac{n}{2}$. Here, we confine ourselves to the special case $k=2$ and investigate the value of $F(J_q(n,2))$. Note that to prove $F(J_q(n,2)) = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q - t$, we could introduce a set of t vertices in $J_q(n,2)$ such that there is no vertex outside this set with exactly one neighbor in it, and also show that every set with this property has at least t elements.

Let V be an n-dimensional vector space over $\mathbb{F}_q$ and $\mathcal{A} = \{V_1, \ldots, V_t\}$ be a set of two-dimensional subspaces of V. In the sequel, we use the following two definitions:

(i) The set $\mathcal{A}$ is called a *stationary white* set if there is no two-dimensional subspace of V outside $\mathcal{A}$ that intersects exactly one element of $\mathcal{A}$ in a one-dimensional subspace.

(ii) The set $\mathcal{A}$ is called a *self-covering* set if any subspace in $\mathcal{A}$ is covered by other subspaces in this set. In other words, for every $1 \leq i \leq t$, $V_i \subseteq \bigcup_{1 \leq j \leq t, j \neq i} V_j$.

Note that computing the failed zero forcing number of $J_q(n,2)$ is equivalent to finding the minimum size of a stationary white set.

The first lemma of this section establishes the equivalence of the above two definitions for certain values of n and t. In the proof, $\langle \alpha_1, \ldots, \alpha_m \rangle$ denotes the subspace spanned by the vectors $\alpha_1, \ldots, \alpha_m$.

Lemma 3.1. *Let $n \geq 5$ and $t \leq 2q$. A collection $\mathcal{A} = \{V_1, \ldots, V_t\}$ of two-dimensional subspaces of an n-dimensional vector space V over $\mathbb{F}_q$ is a stationary white set if and only if it is self-covering.*

Proof. Let $\mathcal{A}$ be self-covering and suppose that U is a two-dimensional subspace of V outside $\mathcal{A}$ such that U has a one-dimensional intersection with exactly one element of $\mathcal{A}$, say V_i. If α is a non-zero vector in $U \cap V_i$, then $\alpha \notin \bigcup_{1 \leq j \leq t, j \neq i} V_j$ and this contradicts the assumption of $\mathcal{A}$ being self-covering. Therefore, there is no such subspace and $\mathcal{A}$ is a stationary white set.

Now, suppose that $\mathcal{A}$ is not self-covering. So, there are $1 \leq i \leq t$ and $\alpha \in V_i \setminus \{0\}$ such that $\alpha \notin \bigcup_{1 \leq j \leq t, j \neq i} V_j$. To show that $\mathcal{A}$ is not a stationary white set, we find a vector β such that $U = \langle \alpha, \beta \rangle$ is a two-dimensional subspace of V outside $\mathcal{A}$ that has a non-zero intersection with only one element of $\mathcal{A}$, namely V_i. To do so, let

$$\beta \in V \setminus \bigcup_{c \in \mathbb{F}_q} (V_1 \cup \cdots \cup V_t - c\alpha).$$

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Note that for every $c \in \mathbb{F}_q$,

$$|V_1 \cup \cdots \cup V_t - c\alpha| = |V_1 \cup \cdots \cup V_t| \leq t(q^2-1)+1,$$

so

$$\left| \bigcup_{c \in \mathbb{F}_q} (V_1 \cup \cdots \cup V_t - c\alpha) \right| \leq tq(q^2-1)+q$$

which implies that $V \setminus \bigcup_{c\in\mathbb{F}_q} (V_1 \cup \cdots \cup V_t - c\alpha)$ has at least

$$q^n - tq(q^2-1) - q \geq q^n - 2q^2(q^2-1) - q = q^4(q^{n-4}-2) + 2q^2 - q$$

elements. Since $n \geq 5$, $q^4(q^{n-4}-2)+2q^2-q > 0$, so the choice of β is possible. Now, since β is not in any of $V_1, \ldots, V_t$, $U = \langle \alpha, \beta \rangle$ is outside $\mathcal{A}$. Suppose that there is a non-zero element of U, say $a\alpha + b\beta$ in V_j, $j \neq i$. If $b = 0$, then $\alpha \in V_j$, which is not the case. So, $b \neq 0$ and $\frac{a}{b}\alpha + \beta \in V_j$ and $\beta \in V_j - \frac{a}{b}\alpha$, contrary to the choice of β. This contradiction completes the proof. □

The next lemma shows that a self-covering collection of size at most $2q+1$ span a subspace of dimension at most 3. This lemma enables us to reduce the problem of finding $F(J_q(n,2))$ to a simpler one regarding projective or affine planes.

Lemma 3.2. *Let $t \leq 2q+1$. If distinct two-dimensional subspaces $V_1, \ldots, V_t$ of a vector space V over $\mathbb{F}_q$ constitute a self-covering collection, then they are contained in a three-dimensional subspace of V.*

Proof. If $t = 1$ then there is nothing to prove. So, suppose that $t \geq 2$. Let m be the maximum number of subspaces among $V_1, \ldots, V_t$ that are contained in a three-dimensional subspace U of V, and without the loss of generality, suppose that $V_1, \ldots, V_m$ are these subspaces. Note that $m \geq 2$ (choose a subspace V_i, $i \geq 2$, such that $V_1 \cap V_i \neq \{0\}$ and note that $V_1 + V_i$ is three-dimensional). We prove that $m \geq q+2$. Suppose on the contrary that $m \leq q+1$. Let $1 \leq i \leq m$ and divide the one-dimensional subspaces of V_i into two sets:

- Subspaces that are not contained in any other V_j, $1 \leq j \leq m, j \neq i$. Call the set of these subspaces $\mathcal{A}_i$.
- The collection of other one-dimensional subspaces of V_i, which is denoted by $\mathcal{B}_i$.

Since $V_1, \ldots, V_m$ are distinct two-dimensional subspaces, each V_j, $1 \leq j \leq m, j \neq i$, can contain at most one of the one-dimensional subspaces of V_i. Thus, $|\mathcal{B}_i| \leq m-1$, and since V_i has $q+1$ one-dimensional subspaces,

$$|\mathcal{A}_i| \geq (q+1) - (m-1) = q - m + 2.$$

Now, since $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are disjoint, $|\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m| \geq m(q-m+2)$. On the other hand, being self-covering implies that any subspace in $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m$ is contained in one of $V_{m+1}, \ldots, V_t$. But if two of these subspaces, say $\langle\alpha\rangle$ and $\langle\beta\rangle$ are contained in one V_j, $m+1 \leq j \leq t$, then $V_j = \langle \alpha, \beta \rangle \subseteq U$, and this contradicts the maximality of m. So, each subspace in $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m$ is contained in exactly one of $V_{m+1}, \ldots, V_t$ which implies that $|\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m| \leq t - m$. Therefore,

$$t - m \geq |\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m| \geq m(q - m + 2),$$

or $t \geq m(q - m + 3)$. Now, $m(q - m + 3)$ is a quadratic polynomial in m, and its minimum value for $2 \leq m \leq q + 1$ is $2q + 2$ which is attained at $m = 2$ or $m = q + 1$. Thus, $t \geq 2q + 2$, a contradiction. This contradiction completes the proof of the claim.

Now, contrary to the statement of the lemma, suppose that $m < t$ and let $m + 1 \leq j \leq t$. If two one-dimensional subspaces of V_j are contained in one of $V_1, \ldots, V_m$, then $V_j \subseteq U$, which contradicts the maximality of m. So, at most one of the $q + 1$ one-dimensional subspaces of V_j are contained in $V_1 \cup \cdots \cup V_m$. On the other hand, by the self-covering property, each of the remaining q one-dimensional subspaces of V_j are contained in one of the $V_{m+1}, \ldots, V_{j-1}, V_{j+1}, \ldots, V_t$. Since no two one-dimensional subspaces of V_j could be in one of $V_{m+1}, \ldots, V_{j-1}, V_{j+1}, \ldots, V_t$, we conclude that $q \leq t - m - 1$. Therefore, $t \geq q + m + 1 \geq 2q + 3$, a contradiction. This contradiction finishes the proof of the lemma. ☐

The following lemma establishes the connection between the problem of finding $F(J_q(n, 2))$ and special type of point sets in projective and affine planes.

Lemma 3.3. *Let $n \geq 3, t \leq 2q$ be positive integers. The following statements are equivalent:*

(i) *There exists a self-covering set of t two-dimensional subspaces of an n-dimensional vector space V over $\mathbb{F}_q$.*

(ii) *There exist t lines in the projective plane $\mathrm{PG}(2, \mathrm{q})$ such that there is no point which lies on exactly one of these lines.*

(iii) *There exist t points in the projective plane $\mathrm{PG}(2, \mathrm{q})$ such that there is no line which passes through exactly one of these points.*

(iv) *There exist t points in the affine plane $\mathrm{AG}(2, \mathrm{q})$ such that there is no line which passes through exactly one of these points.*

Proof. First, we prove the equivalence of (i) and (ii). Suppose that (i) holds and let $\{V_1, \ldots, V_t\}$ be a self-covering collection of t two-dimensional subspaces of V. By Lemma 3.1, there is a three-dimensional subspace U of V that contains $V_1, \ldots, V_t$. Now, $V_1, \ldots, V_t$ are lines in the projective space $\mathbb{P}(U) \cong \mathrm{PG}(2, \mathrm{q})$ and being self-covering implies that every one-dimensional subspace of any V_i is a subspace of at least another V_j, $j \neq i$. In other words, every projective point that lies on some V_i, should lie on another V_j, $j \neq i$. Therefore, (ii) holds. Now, suppose that (ii) holds. So, there are t two-dimensional subspaces $V_1, \ldots, V_t$ of $\mathbb{F}_q^3$ such that there is no one-dimensional subspace of $\mathbb{F}_q^3$ that is contained in exactly one of these subspaces, which means that $\{V_1, \ldots, V_t\}$ is self-covering. By embedding $\mathbb{F}_q^3$ into V, we find a self-covering collection of size t in V.

The equivalence of (ii) and (iii) follows from the duality between points and lines in the projective plane.

By embedding $\mathrm{AG}(2,\mathrm{q})$ into $\mathrm{PG}(2,\mathrm{q})$, one can see that (iv) implies (iii). On the other hand, suppose that (iii) holds and $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$ are one-dimensional subspaces of $\mathbb{F}_q^3$ such that there is no two-dimensional subspace that contains exactly one of $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$. First, we show that there is a two-dimensional subspace of $\mathbb{F}_q^3$ that contains none of $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$. Each $\langle\alpha_i\rangle$ is contained in exactly $q+1$ two-dimensional subspace of $\mathbb{F}_q^3$. If we list the two-dimensional subspaces that contain at least one of $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$, each subspace appears at least twice. So, there are at most

$$\frac{t(q+1)}{2} \le \frac{2q(q+1)}{2} = q^2+q$$

two-dimensional subspaces of $\mathbb{F}_q^3$ that contain at least one of $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$. But, $\mathbb{F}_q^3$ has q^2+q+1 two-dimensional subspaces, so there is a two-dimensional subspace U of $\mathbb{F}_q^3$ such that none of $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$ is contained in U. Now, let $\gamma\in\mathbb{F}_q^3\setminus U$. For each i, since $\langle\alpha_i\rangle\nsubseteq U$, $U+\langle\alpha_i\rangle=\mathbb{F}_q^3$, so there are $u\in U$ and $c\in\mathbb{F}_q$ such that $u+c\alpha_i=\gamma$. Since $\gamma\notin U$, $c\neq 0$ and $c\alpha_i=-u+\gamma$ is a non-zero vector in the intersection of $\langle\alpha_i\rangle$ and $U+\gamma$. To complete the proof, consider the affine plane $U+\gamma$ and t intersection points of $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$ with this plane. By the assumption about $\langle\alpha_1\rangle,\dots,\langle\alpha_t\rangle$, there is no line in $U+\gamma$ that passes through exactly one of these points and therefore, (iv) holds. □

A *double blocking set* in the projective plane $\mathrm{PG}(2,\mathrm{q})$ is a subset of points that intersects every line in at least two points. To address the problem of determining the failed zero forcing number of $J_q(n,2)$, we pose a similar concept with the following definition. A *pseudo double blocking set* in $\mathrm{PG}(2,\mathrm{q})$ is a non-empty subset of S of points such that every line that meets S, meets it at least twice. The minimum size of a pseudo double blocking set in $\mathrm{PG}(2,\mathrm{q})$ is denoted by $b_2'(q)$. Note that pseudo double blocking sets of $\mathrm{PG}(2,\mathrm{q})$ and self-covering sets in a 3-dimensional vector space are the same objects, merely described using different terms.

Lemma 3.4. *The following statements hold:*

- (i) $q+2\le b_2'(q)\le 2q$.
- (ii) *The minimum size of a non-empty set of points in the affine plane* $\mathrm{AG}(2,\mathrm{q})$ *such that every line that meets this set, meets it at least two points, is* $b_2'(q)$.

Proof. To prove (i), note that every line in $\mathrm{PG}(2,\mathrm{q})$ contains $q+1$ points. So, if L_1 and L_2 are two distinct lines in $\mathrm{PG}(2,\mathrm{q})$ that intersect in P, then $(L_1\cup L_2)\setminus\{P\}$ is a pseudo double blocking set with $2q$ points. Therefore, $b_2'(q)\le 2q$. On the other hand, if $\{p_1,\dots,p_t\}$ is a pseudo double blocking set, then each of $q+1$ lines passing through p_1 should contain one of $p_2,\dots,p_t$. Since no two lines of this collection could contain one of $p_2,\dots,p_t$, $t-1\ge q+1$, or $t\ge q+2$, which implies that $b_2'(q)\ge q+2$.

Part (ii) follows from Part (i) and the equivalence of Parts (iii) and (iv) in Lemma 3.3. □

Now, we are ready to prove the main result of this section.

Theorem 3.5. *Let $n \geq 5$ and G be the Grassmann graph $J_q(n, 2)$. Then,*

$$F(G) = \begin{bmatrix} n \\ 2 \end{bmatrix}_q - b_2'(q).$$

Proof. By definition, there is $b_2'(q)$ points in $\mathrm{PG}(2,\mathrm{q})$ such that there is no line that passes through exactly one of these points. By Part (i) of Lemma 3.4, $b_2'(q) \leq 2q$, so Lemma 3.3 implies that there is a self-covering collection $\mathcal{A}$ of $b_2'(q)$ two-dimensional subspaces of $\mathbb{F}_q^n$. Since $n \geq 5$, Lemma 3.1 implies that $\mathcal{A}$ is a stationary white set. Thus, if we color the vertices in A white and all other vertices of $J_q(n, 2)$ black, there will be no forcing vertex. Therefore, $F(G) \geq \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q - b_2'(q)$. If $F(G) > \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q - b_2'(q)$, then there would be a non-empty stationary white set $\mathcal{A}$ such that $|\mathcal{A}| < b_2'(q)$. By reversing the above argument, we find a pseudo double blocking set of size $|\mathcal{A}| < b_2'(q)$ in $\mathrm{PG}(2,\mathrm{q})$, a contradiction. Thus, $F(G) = \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q - b_2'(q)$. ☐

The problem of determining $b_2'(q)$ seems to be hard. In the following theorem, we find the exact value of this parameter when q is even.

Theorem 3.6. *If q is a power of 2, then $b_2'(q) = q + 2$, and therefore $F(J_q(n, 2)) = \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q - (q + 2)$, for $n \geq 5$.*

Proof. By Theorem 2.1, there is an arc S in $\mathrm{PG}(2,\mathrm{q})$ of size $q+2$. By Lemma 2.2, every line that meets S, meets it at least twice. So S is a pseudo double blocking set and $b_2'(q) \leq |S| \leq q + 2$. On the other hand, by Part (i) in Lemma 3.4, $q + 2 \leq b_2'(q)$, so $b_2'(q) = q + 2$. The second assertion follows from Theorem 3.5. ☐

4. **Computational Results**

By Theorem 3.5, the problem of finding $F(J_q(n, 2))$ for $n \geq 5$, reduces to the problem of computing $b_2'(q)$. Note that by the second part of Lemma 3.4, $b_2'(q)$ equals the minimum size of a non-empty set of points in the affine plane $\mathrm{AG}(2,\mathrm{q})$ such that every line that meets this set, meets it in at least two points. In Theorem 3.6, we have determined the value of $b_2'(q)$ for all powers of 2. Here, we propose an integer linear programming formulation of this problem for an arbitrary prime power that helps us determine the value of $b_2'(q)$ for small values of q. An *integer linear programming* (ILP) problem is an optimization problem where the objective function is linear, the constraints are linear equations or inequalities, and the variables are restricted to integer values. A special type of ILP is a *binary* ILP, in which the only possible values for the variables are zero and one. Binary variables are well-suited for modeling item selection, where one indicates selection and zero indicates non-selection. Since the problem of determining $b_2'(q)$ can be framed as selecting the minimum number of points in the projective or affine plane of order q to satisfy certain constraints, reformulating it as a binary ILP is relatively straightforward. Since the affine plane of order q is smaller than the projective plane of the same order, we prefer to work with the affine plane.

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Let $m = q^2$, $n = q^2 + q$, and suppose that $\mathcal{P} = \{p_1, \ldots, p_m\}$ and $\mathcal{L} = \{l_1, \ldots, l_n\}$ are the sets of points and lines of $\mathrm{AG}(2, \mathrm{q})$, respectively. We aim to select a subset of $\mathcal{P}$ with the smallest possible cardinality that meets the above requirement.

Variables: We use $m + n$ binary variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ in such a way that $x_i = 1$ means the inclusion of p_i in the resulting set, and $y_j = 1$ means that at least one point of l_j is selected.

Objective: Since we want a set with the minimum size, the objective function would be $\sum_{i=1}^{m} x_i$ and our goal is to minimize this function.

Constraints: Since the point set should be non-empty, we require that $x_1 = 1$. The condition that there is no line that passes through exactly one of the selected points could be written as,

$$\sum_{1 \leq i \leq m, p_i \in l_j} x_i \neq 1, \qquad \text{for } 1 \leq j \leq n.$$

But this condition is, of course, nonlinear. To formulate the problem using only linear constraints, we use $y_1, \ldots, y_n$ and impose the following constraints:

$$2y_j \leq \sum_{1 \leq i \leq m, p_i \in l_j} x_i \leq q y_j, \qquad \text{for } 1 \leq j \leq n.$$

So, when $y_j = 0$, there should be no selected point on l_j, and when $y_j = 1$, there should be at least two points on that line.

The above ILP could be solved using an ILP solver. We used SageMath [11] to translate the above model into a standard ILP and employed the SCIP Optimization Suite [4] to solve this problem. The results are presented in Table (1).

TABLE 1. The values of $b'_2(q)$ for odd prime powers q less than 13.

q	3	5	7	9	11
$b'_2(q)$	6	10	12	15	18

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