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THE HAMILTONIAN (s,t)-PATH PROBLEM IN ODD-SIZED H-ALPHABET GRID GRAPHS

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Abstract. The Hamiltonian path problem is a well-known problem in graph theory with numerous applications in many fields such as routing, robotics, and parallel processing. In general, this problem is NP-complete for general grid graphs; however, efficient solutions can be found for specific classes of graphs. This paper investigates the Hamiltonian (s,t)-path problem in odd-sized H-alphabet grid graphs, a sub-class of solid grid graphs. We begin by establishing the conditions under which a Hamiltonian path between two given vertices s and t does not exist. For the cases where a Hamiltonian path exists, we propose an efficient linear-time algorithm to find a Hamiltonian path between the two given vertices.

1. Introduction

The Hamiltonian path (cycle) problem is a fundamental problem in graph theory with numerous applications in many fields such as computational geometry, robotics, and parallel processing [4, 5, 21]. A Hamiltonian (s,t)-path is a simple path that visits each vertex of a graph exactly once, starting from a source vertex s and ending at a target vertex t. While this problem is NP-complete in general, efficient algorithms have been developed for some special classes of graphs, though the problem remains open for many other classes [3, 10, 16, 11].

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One class of graphs for which this problem has been studied is grid graphs. A *grid graph* is a subgraph of the two-dimensional infinite integer lattice. If all the interior faces of grid graph are unit squares, it is called a *solid grid graph*. A solid grid graph that is defined within an $m \times n$ rectangular region is called a *rectangular grid graph (mesh)*.

Itai et al. [6] proved that the Hamiltonian path problem is NP-complete for general grid graphs. They also provided a solution for the Hamiltonian (s,t)-path problem in rectangular grid graphs. Umans and Lenhart [19] presented a polynomial-time algorithm to solve the Hamiltonian cycle problem in solid grid graphs. Chen et al. [2] presented an efficient linear-time algorithm for constructing Hamiltonian (s,t)-paths in meshes, improving the previous algorithm [6] by reducing the partition steps. Salman et al. [17] introduced alphabet grid graphs, which correspond to the letters of the English alphabet. They provided a characterization of which of these alphabet grid graphs are Hamiltonian.

Chang et al. [1] addressed the longest (s,t)-path problem on meshes with a missing vertex and proposed a linear-time solution for it. Keshavarz-Kohjerdi and Bagheri [7] studied the Hamiltonian (s,t)-path problem for rectangular grid graphs with a rectangular hole and proposed a linear-time algorithm for solving it. They also solved this problem for solid supergrid graphs in polynomial time [9]. Nishat et al. [12, 13] studied the reconfiguration of simple Hamiltonian (s,t)-paths in rectangular grid graphs, providing an algorithm to transform one such path to another using only local operations in unit grid cells, with a linear time-complexity. In [14, 15] Nishat and Whitesides studied reconfiguration of Hamiltonian cycles in L-shaped and rectangular grid graphs, respectively. Selcuk and Tankul [18] presented an efficient algorithm for finding Hamiltonian paths in Connected Square Network Graphs (CSNG). Xie [20] proposed a sufficient condition for constructing a Hamiltonian cycle in grid graphs and developed an algorithm based on this condition.

In this paper, we address the Hamiltonian (s,t)-path problem in odd-sized H-alphabet grid graphs and propose a linear-time algorithm for solving this problem. The structure of this paper is as follows: Section 2 introduces the definitions, notations, and preliminary results necessary for the subsequent sections. In Section 3, we derive the necessary conditions for the existence of Hamiltonian paths in H-alphabet grid graphs. Section 4 presents a linear-time algorithm for constructing a Hamiltonian (s,t)-path. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section, we introduce several key definitions and results that are crucial for the development of the concepts explored in this paper. Some of these definitions have been previously established in [6, 7, 8].

The *two-dimensional integer grid* G^∞ is an infinite undirected graph where each vertex corresponds to a point on the plane with integer coordinates. Two vertices in G^∞ are connected by an edge if their Euclidean distance equals 1. A *grid graph* G_g is a finite subgraph induced by a subset of the vertices

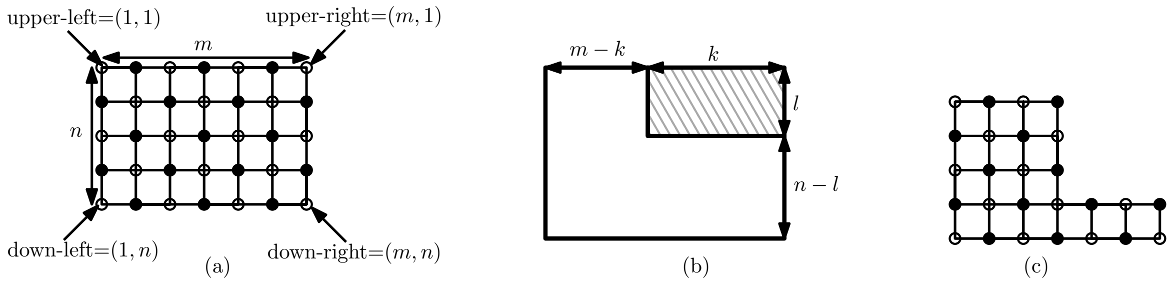

FIGURE 1. (a) An illustration of a rectangular grid graph, (b) the defining parameters of an L-shaped grid graph, and (c) an example of an L-shaped grid graph.

from the two-dimensional integer grid G^∞. Let $G_g = (V, E)$ be a grid graph. Here, V represents the set of vertices of G_g, also is denoted as $V(G_g)$, and E represents the set of edges of G_g, also is denoted as $E(G_g)$. For a vertex $v \in V(G_g)$, its coordinates are denoted as v_x and v_y, representing the x and y coordinates of the corresponding point on the plane (sometimes is referred to as (v_x, v_y)). If an edge $(u, v) \in E(G_g)$ exists, then u and v are said to be *adjacent*, denoted by $u \sim v$. The *degree* of a vertex $v \in V(G_g)$ refers to the number of vertices adjacent to v and is represented as degree(v). In G_g, each vertex has a degree of at most four, indicating that a vertex v in the grid graph can have adjacent vertices located at $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, and $(v_x, v_y + 1)$.

The vertices of the grid graph G_g are assigned one of two colors black or white. A vertex v is colored *white* if the sum of its coordinates, i.e. $v_x + v_y$ is even; otherwise, it is colored *black*. Since no edges connect vertices of the same color, the grid graph is bipartite. In bipartite graphs, every cycle or path alternates between black and white vertices. The number of black vertices in G_g may differ from the number of white vertices. The color with more vertices is referred to as the *majority color*, while the other is called the *minority color*. Let V_B and V_W represent the sets of black and white vertices, respectively. If $|V_B| > |V_W|$, black is the majority color; if $|V_B| < |V_W|$, white is the majority. When $|V_B| = |V_W|$, either color can be considered as the majority. A grid graph G_g is classified as *even-sized* if $|V_B| = |V_W|$, or as *odd-sized* if $||V_B| - |V_W|| = 1$. If $||V_B| - |V_W|| > 1$, the graph is neither even-sized nor odd-sized.

Let $R(m, n)$ or G_R represent a *rectangular grid graph*, which is a subgraph of the two-dimensional infinite integer grid graph G^∞. The vertex set of $R(m, n)$ is defined as $V(R) = \{v \mid 1 \leq v_x \leq m, 1 \leq v_y \leq n\}$. Thus, $R(m, n)$ consists of m columns and n rows of vertices from the infinite grid G^∞. For example, the graph $R(7, 5)$ is shown in Figure 1(a). The size of an rectangular grid graph is defined as $m \times n$. Additionally, we say that $R(m, n)$ is a *k-rectangle* if either $k = m$ or $k = n$. Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. We define v as a specific corner of $R(m, n)$ based on its position relative to the other vertices in the graph. Specifically, v is called the *upper-right corner* if, for every vertex $u = (u_x, u_y) \in R(m, n)$, it holds that $u_y \geq v_y$ and $u_x \leq v_x$. Similarly, v is the *upper-left corner* if $u_y \geq v_y$ and $u_x \geq v_x$. Additionally, v is the *down-right corner* if $u_y \leq v_y$ and $u_x \leq v_x$, and v is the *down-left corner* if $u_y \leq v_y$ and $u_x \geq v_x$.

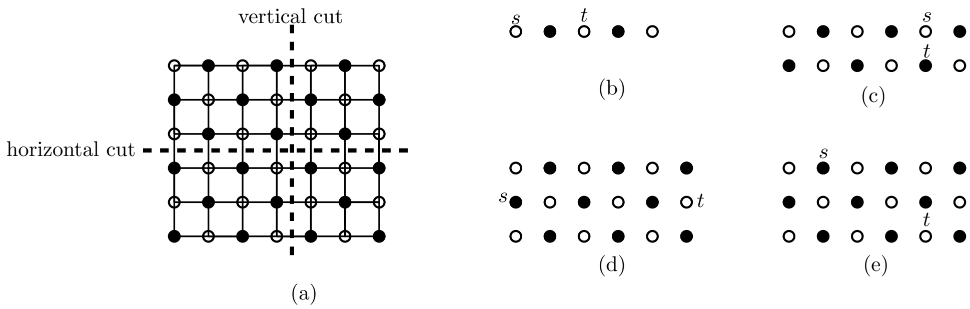

Figure 2. (a) A rectangular grid graph $R(7,6)$, with bold dashed lines representing vertical and horizontal cuts, and (b)–(e) rectangular grid graphs where no Hamiltonian path exists between the vertices s and t.

For any edge $e=(u,v)$ in $R(m,n)$, we classify the edge as *horizontal* if $u_y=v_y$ (i.e., the vertices share the same y-coordinate), or *vertical* if $u_x=v_x$ (i.e., the vertices share the same x-coordinate). In the figures throughout this paper, the vertices at coordinates $(1,1)$ and $(m,1)$ are assumed to be located at the upper-left and upper-right corners, respectively, unless explicitly stated otherwise.

An *L-shaped grid graph*, denoted as $L(m,n;k,l)$ or G_L, is derived from a rectangular grid graph $R(m,n)$ by removing a rectangular subgraph $R(k,l)$ from its upper-right corner, where $m,n>1$ and $k,l\geq 1$. This implies that $m-k\geq 1$ and $n-l\geq 1$. The structures of $L(m,n;k,l)$ is shown in Figures 1(b) and 1(c). The size of an L-shaped grid graph is defined as $m\times n-k\times l$.

Let $e_1=(u_1,v_1)$ and $e_2=(u_2,v_2)$ be two edges in G_g. If $u_1\sim u_2$ and $v_1\sim v_2$, we say that the edges e_1 and e_2 are *parallel*, denoted by $e_1\approx e_2$. The number of vertices in G_g is denoted by $|G_g|$. Consider a grid graph G_g with two distinct vertices s and t. We denote this as (G_g,s,t). Without loss of generality, assume that $s_x\leqslant t_x$. A Hamiltonian path from s to t is referred to as a Hamiltonian (s,t)-path, and we denote such a path in G_g as $HP(G_g,s,t)$. If a Hamiltonian (s,t)-path exists in G_g, we say that $HP(G_g,s,t)$ exists.

A *cut operation* on G_g involves partitioning G_g into κ vertex-disjoint subgraphs $G_1,G_2,\ldots,G_\kappa$, where each G_i is a rectangular or an L-shaped grid graph. This implies that $V(G)=V(G_1)\cup V(G_2)\cup\cdots\cup V(G_\kappa)$ and $V(G_i)\cap V(G_j)=\emptyset$, where $1\leq i,j\leq\kappa$. If the cut intersects only horizontal (resp. vertical) edges, it is referred to as a *vertical* (resp. *horizontal*) cut. In Figure 2(a), the bold dashed horizontal (respectively vertical) line illustrates a horizontal (respectively vertical) cut of the grid graph $R(7,6)$, partitioning it into $R(7,3)$ and $R(7,3)$ (respectively $R(4,6)$ and $R(3,6)$).

Definition 2.1. *Assume that G_g is a grid graph. (G_g,s,t) is said to be color-compatible if one of the following conditions holds:*

(1) $||V_B|-|V_W||=1$ *and vertices s and t have the majority color, or*
(2) $|V_B|=|V_W|$ *and vertices s and t have different colors.*

In an odd-sized grid graph, the number of vertices with the minority color is exactly one less than the number of vertices with the majority color. As a result, the two endpoints of any Hamiltonian path in such graphs must be colored with the majority color. On the other hand, in an even-sized grid graph, the number of black and white vertices are equal. Therefore, the two endpoints of any Hamiltonian path in such graphs must have different colors. From this, we can conclude that for existence of the Hamiltonian path $HP(G_g, s, t)$, the color-compatibility of the vertices s and t is a necessary condition.

Definition 2.2. *Let G_g be a connected graph, and let V_1 be a subset of the vertex set $V(G_g)$. The set V_1 is called a vertex cut of G_g if the graph obtained by removing the vertices of V_1, denoted as $G_g - V_1$, becomes disconnected. Additionally, a vertex v in G_g is referred to as a cut vertex if the set $\{v\}$ is a vertex cut of G_g. For instance, in Figure 2(b), the vertex t is a cut vertex, and in Figure 2(c), the set $\{s, t\}$ forms a vertex cut.*

Definition 2.3. [6] *Assume that $(R(m, n), s, t)$ is a rectangular grid graph with two given vertices s and t. $(R(m, n), s, t)$ is considered acceptable if it satisfies the following two conditions:*

(1) *It is color-compatible, and*
(2) *Neither condition* (F1) *nor* (F2) *holds for $(R(m, n), s, t)$, where the conditions* (F1) *and* (F2) *are defined as follows.*

(F1) Either s or t is a cut vertex, or the set $\{s, t\}$ forms a vertex cut of $R(m, n)$ (see Figures 2(b) and 2(c)).
(F2) All the cases that are isomorphic to the following case:
  (1) m is even, $n = 3$, s is black, t is white, and
  (2) $s_y = 2$ and $s_x < t_x$ (Figure 2(d)) or $s_y \neq 2$ and $s_x < t_x - 1$ (Figure 2(e)).

Theorem 2.4. [6] *Consider $R(m, n)$, a rectangular grid graph with $m, n \geqslant 1$, and two given vertices s and t. The Hamiltonian (s, t)-path $HP(R(m, n), s, t)$ exists if and only if $(R(m, n), s, t)$ is acceptable.*

In the following, note that isomorphic cases have been excluded.

Definition 2.5. [7] *Assume that $(L(m, n; k, l), s, t)$ is an L-shaped grid graph with two given vertices s and t. $(L(m, n; k, l), s, t)$ is considered acceptable if it satisfies the following two conditions:*

(1) *It is color-compatible, and*
(2) *None of the conditions* (F1) *(see Figures 3(a) and 3(b)) and* (F3) *to* (F9) *hold for $(R(m, n), s, t)$, where conditions* (F3) *to* (F9) *are defined as follows.*

(F3) There exists a vertex $w \in V(G)$ such that degree$(w) = 1$, and neither w is equal to t nor s (see Figure 3(c)).
(F4) $m > 3$, $n > 3$, $L(m, n; 1, 1)$ is even-sized, and $[(s = (m-1, 2)$ and $t \neq (m-1, 1)$ or $t \neq (m, 2))$ or $(t = (m-1, 2)$ and $s \neq (m-1, 1))]$ (see Figure 3(d)).

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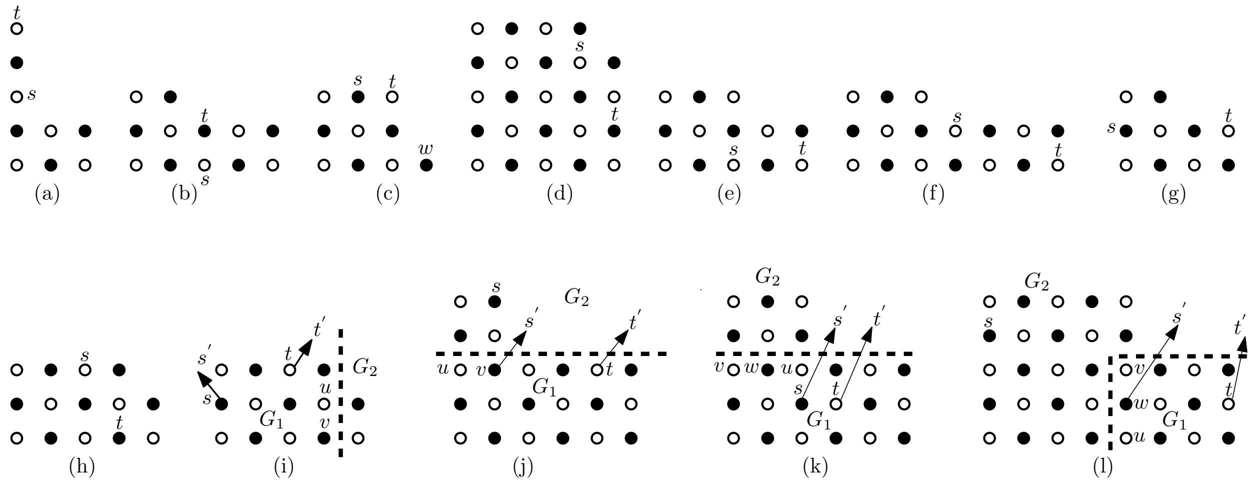

FIGURE 3. Examples of L-shaped grid graphs where no Hamiltonian (s, t)-path exists.

(F5) $L(m, n; k, l)$ is odd-sized, $n - l = 2$, $m - k \geq 3$ is odd, $s_x, t_x \geqslant m - k$, and $s_y, t_y \geqslant n - 1$ (see Figures 3(e) and 3(f)).

(F6) $n - l = 2$, $m - k = 2$, $s = (1, n - l)$, and $t_x > 2$ (see Figure 3(g))

(F7) $L(m, n; k, l)$ is even-sized, $n = 3$, $l = 1$, $m - k > 2$ is even, $s = (m - k - 1, 1)$, and $t = (m - k, 3)$ (see Figure 3(h))

(F8) Assume that $L(m, n; k, l)$ is even-sized and satisfies $[(m - k = 2$ and $n - l > 2)$ or $(n - l = 2$ and $m - k > 2)]$. There exists a vertical (or horizontal) cut on $L(m, n; k, l)$ such that G_1 is a 3-rectangle grid graph, G_2 is a 2-rectangle grid graph, and exactly two vertices, u and v, in G_1 are connected to G_2. Additionally, (G_1, s', t') satisfies condition (F2) (see Figures 3(i) and 3(j)), where $(s', t') = (s, t)$, unless $s \notin V(G_1)$, in which $s' = u$, or $t \notin V(G_1)$, in which $t' = u$.

(F9) Assume that $L(m, n; k, l)$ is even-sized and satisfies $[(m - k = 3$ and $n - l \geqslant 3)$ or $(m - k > 3$ and $n - l = 3)]$. There exists a vertical (or horizontal, or both vertical and horizontal) cut(s) on $L(m, n; k, l)$ such that G_1 and G_2 are even-sized, G_1 is a 3-rectangle grid graph, G_2 is either:
  - a rectangular grid graph (see Figure 3(k)), or
  - an L-shaped grid graph, where m is even, n is odd, k is odd, l is even, $n - l = 3$, $m - k \geqslant 5$, and $V(G_1) = \{(v_x, v_y) \mid m - k \leq v_x \leq m,\ l + 1 \leq v_y \leq n\}$ (see Figure 3(l)).

  There are exactly three vertices v, w, and u in G_1 that are connected to G_2, (G_1, s', t') satisfies condition (F2), where $(s', t') = (s, t)$, unless $s \notin V(G_1)$, in which $s' = w$, or $t \notin V(G_1)$, in which $t' = w$.

Theorem 2.6. [7] *Consider an L-shaped grid graph $L(m, n; k, l)$ with $m, n \geqslant 1$, and two distinct vertices s and t. A Hamiltonian (s, t)-path $HP(L(m, n; k, l), s, t)$ exists if and only if $(L(m, n; k, l), s, t)$ is acceptable.*

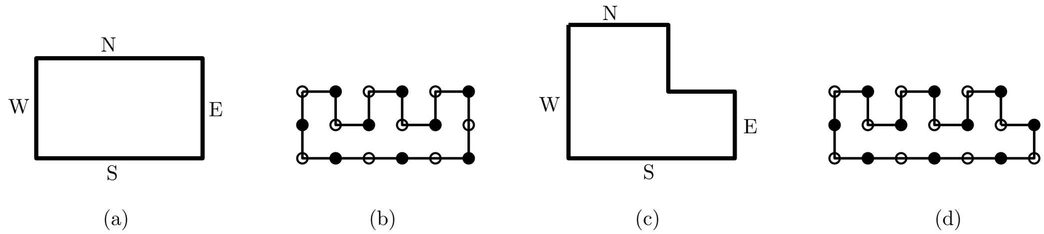

FIGURE 4. (a) and (c) A grid graph with four sides: North (N), West (W), South (S), and East (E), showing both a rectangular and an L-shaped structure; (b) the corresponding Hamiltonian cycle pattern for the rectangular grid graph, and (d) the Hamiltonian cycle pattern for the L-shaped grid graph.

Theorem 2.7. [6, 7] *For an acceptable $(R(m, n), s, t)$ and $(L(m, n; k, l), s, t)$, a Hamiltonian (s, t)-path can be computed in linear time.*

The following lemma provides the necessary and sufficient conditions for the existence of a Hamiltonian cycle in a rectangular or an L-shaped grid graph.

Lemma 2.8. [6, 8] *Given an L-shaped or a rectangular grid graph G_g, it is Hamiltonian if and only if it does not satisfy conditions $\mathcal{FC}1$ and $\mathcal{FC}2$ as defined below, and its Hamiltonian cycle can be constructed in linear time.*

$\mathcal{FC}1$ **::** In G_g, $|V_B| \neq |V_W|$.

$\mathcal{FC}2$ **::** G_g has a cut vertex.

Lemma 2.9. [8] *Given a rectangular grid graph $R(m, n)$, it is always possible to construct a Hamiltonian cycle that includes all the edges on the four sides N, W, S, and E of $R(m, n)$ as illustrated in Figures 4(a) and 4(b), except at most one side of $R(m, n)$ which includes boundary edges every other one.*

Lemma 2.10. [8] *Given an L-shaped grid graph $L(m, n; k, l)$, it is always possible to construct a Hamiltonian cycle that includes all the edges on at least three of four sides N, W, S, and E of $L(m, n; k, l)$ as illustrated in Figures 4(c) and 4(d).*

An *H-alphabet grid graph*, denoted as $H(m, n; k_u, l_u; k_d, l_d; x_1, x_2)$ or G_H, is derived from a rectangular grid graph $R(m, n)$ by removing two rectangular subgraphs $R_U(k_u, l_u)$ and $R_D(k_d, l_d)$. These subgraphs are positioned such that $R_U(k_u, l_u)$ shares only its upper boundary with $R(m, n)$, while $R_D(k_d, l_d)$ shares only its lower boundary with $R(m, n)$. Figure 5 illustrates an example of an H-alphabet grid graph. In H-alphabet grid graphs, it holds that $x_1 = x_2 = x_3 = x_4$, $k_u = k_d$, and $l_u = l_d$, where $x_3 = m - x_1 - k_u$ and $x_4 = m - x_2 - k_d$. Since $||V_B(G_H)| - |V_W(G_H)|| = 1$, $x_1 = x_2 = x_3 = x_4$, $l_u = l_d$, and $k_u = k_d$, the only possible scenario is when m, n, and k_u are odd, and one of the following conditions holds:

- l_u is even, in which $|V_W(G_H)| - |V_B(G_H)| = 1$ (i.e., the colors of vertices s and t are white), or

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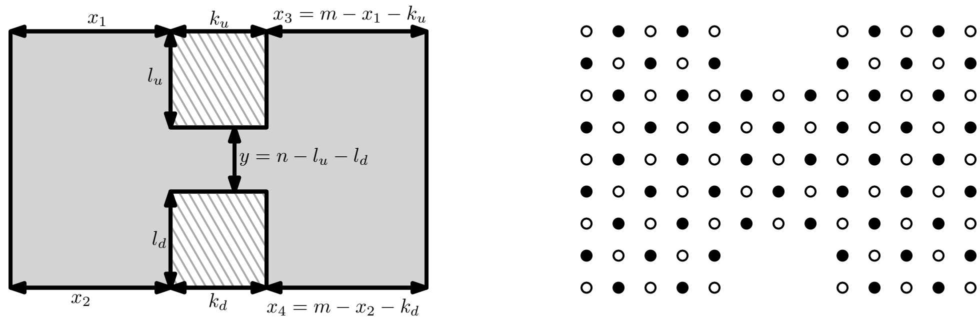

(a) (b)

FIGURE 5. (a) The defining parameters of an H-alphabet grid graph and (b) an example of an H-alphabet grid graph.

- l_u is odd and x_1 is even, in which $|V_B(G_H)| - |V_W(G_H)| = 1$ (i.e., the colors of vertices s and t are black).

3. Necessary Conditions for the Existence of Hamiltonian (s,t)-Paths in H-alphabet Grid Graphs

In this section, we will discuss the conditions under which a Hamiltonian (s,t)-path cannot exist in an H-alphabet grid graph. A necessary condition for the existence of such a path is that (G_H, s, t) must be color-compatible. If this condition is not satisfied, a Hamiltonian (s,t)-path cannot exist. However, even if (G_H, s, t) is color-compatible, there are still specific cases in which a Hamiltonian (s,t)-path is impossible. This section focuses on identifying and analyzing these cases.

We will analyze possible positions of vertices s and t within G_H, which can be categorized into the following four cases:

Case I: $s_x, t_x \leq x_1 + k_u$. Case I has three subcases:

Case I.a: $s_x, t_x \leq x_1$.

Case I.b: $s_x \leq x_1$ and $x_1 < t_x \leq x_1 + k_u$.

Case I.c: $x_1 < s_x, t_x \leq x_1 + k_u$.

Case II: $s_x \leq x_1$ and $t_x > x_1 + k_u$.

Case III: $x_1 < s_x \leq x_1 + k_u$ and $t_x > x_1 + k_u$.

Case IV: $s_x, t_x > x_1 + k_u$.

Due to the symmetry in the structure of G_H, it is sufficient to consider only Cases I and II. In the following, we present the forbidden conditions that prevent the existence of a Hamiltonian (s,t)-path in G_H. In addition to forbidden conditions (F1) and (F3) (see Figure 6), if any of the following conditions, i.e. (FH1), (FH2), (FH3), or (FH4) are satisfied, then a Hamiltonian (s,t)-path in G_H does not exist. Note that isomorphic cases are excluded.

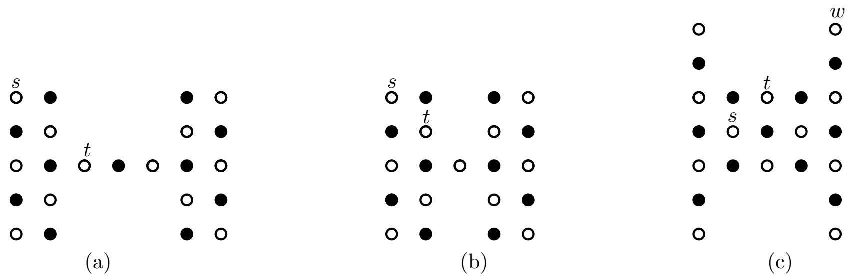

FIGURE 6. Forbidden conditions (F1) and (F3) for G_H.

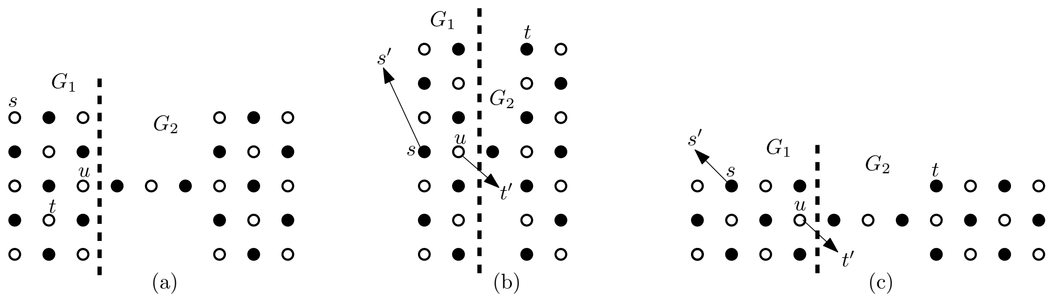

FIGURE 7. Examples where $y = n - l_u - l_d = 1$ and there is no Hamiltonian path between s and t in G_H.

(FH1) $y = n - l_u - l_d = 1$ and $s_x, t_x \leq x_1$

Lemma 3.1. *If forbidden condition* (FH1) *is satisfied, then a Hamiltonian* (s,t)*-path does not exist.*

Proof. We first partition G_H into two subgraphs $G_1 = R(x_1, n)$ and $G_2 = G_H \setminus G_1$ by applying a vertical cut, as shown in Figure 7(a). Let $u = (x_1, l_u + 1)$ be the vertex at the boundary between G_1 and G_2. Any Hamiltonian path starting at vertex s must traverse some vertices of G_1, and then exit G_1 through u to enter G_2. Then, after traversing the vertices of G_2, it must re-enter G_1 through the vertex u. It is clear that u must be visited more than once in any Hamiltonian path. However, this violates the definition of a Hamiltonian path, which requires each vertex to be visited exactly once. Therefore, under conditions of (FH1), it is impossible to construct a Hamiltonian (s,t)-path in G_H. □

(FH2) Let $y = n - l_u - l_d = 1$, x_1 is even, $s_x < x_1$, $t_x > x_1 + k_u$, and there exists a vertical cut on G_H such that $G_1 = R(x_1, n)$ and $G_2 = G_H \setminus G_1$. Furthermore, there is exactly one vertex u in G_1 that is connected to G_2. In addition, (G_1, s', t') satisfies condition (F1) or (F2), where $s' = s$ and $t' = u$ as shown in Figures 7(b) and 7(c).

Lemma 3.2. *If forbidden condition* (FH2) *is satisfied, then a Hamiltonian* (s,t)*-path does not exist.*

Proof. Since two subgraphs G_1 and G_2 are connected exactly through vertex u, any Hamiltonian path starting from vertex s must traverse all the vertices of G_1 and then exit from vertex u. Therefore, if

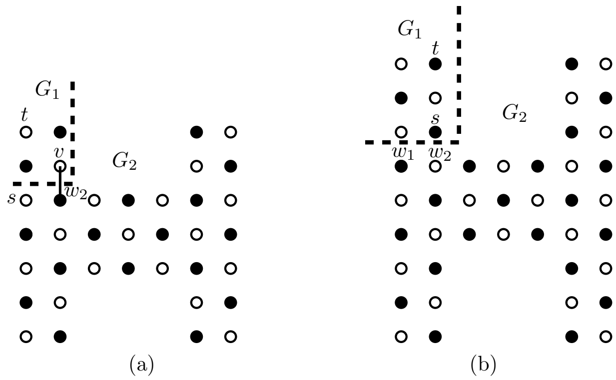

FIGURE 8. An example for forbidden condition (FH3).

(G_1, s, u) satisfies one of forbidden conditions (F1) or (F2), by Theorem 2.4, no Hamiltonian path exists between s and u. As a result, there is no Hamiltonian path between s and t in G_H. □

(FH3) $x_1 = 2$, $y = n - l_u - l_d > 1$, $s_x, t_x \leq x_1$, and $s_y, t_y \leq l_u + 1$.

Lemma 3.3. *If forbidden condition* (FH3) *is satisfied, then a Hamiltonian* (s,t)*-path does not exist.*

Proof. We first partition G_H into two subgraphs $R(x_1, l_u)$ and $G_H \setminus G_1$ by applying a vertical cut and a horizontal cut, as shown in Figure 8. Let $v = (2, l_u)$, $w_1 = (1, l_u + 1)$ and $w_2 = (2, l_u + 1)$. Since x_1 is even, we have $|V_B(G_1)| = |V_W(G_1)|$. On the other hand, since $||V_B(G_H)| - |V_W(G_H)|| = 1$, it follows that $||V_B(G_2)| - |V_W(G_2)|| = 1$. Now, consider the following two cases:
Case 1: s_y(or t_y) $= l_u + 1$ and t_y(or s_y) $\leq l_u$. Without loss of generality, we assume that $s_y = l_u + 1$. In this case, $s = (1, l_u + 1)$, as shown in Figure 8(a). Since subgraphs G_1 and G_2 are connected through the two vertices w_1 and w_2, any Hamiltonian path starting at vertex s must traverse all the vertices of G_2 and enter G_1 through vertex v, and then reach vertex t. However, since $||V_B(G_2)| - |V_W(G_2)|| = 1$ and the colors of the vertices s and w_2 are different, it is clear that the color compatibility condition does not hold in (G_2, s, w_2). Consequently, it is impossible to find a Hamiltonian path between s and w_2 in G_2. Therefore, constructing a Hamiltonian path between s and t in G_H is also impossible.
Case 2: $s_y, t_y \leq l_u$. A Hamiltonian path starting from vertex s must traverse several vertices inside G_1 and then enter G_2 via either vertex w_1 or w_2, as shown in Figure 8(b). The path will then traverse G_2 and exit through either w_2 or w_1, re-entering G_1 and reaching vertex t. Since $||V_B(G_2)| - |V_W(G_2)|| = 1$ and the colors of the vertices w_1 and w_2 are different, the color compatibility condition does not hold in (G_2, w_1, w_2). Therefore, there is no Hamiltonian path between w_1 and w_2 in G_2. As a result, it is impossible to construct a Hamiltonian path between s and t in G_H. □

(FH4) $y = n - l_u - l_d = 3$, x_1 is even, $s_x, t_x < x_1$, and $[(x_1 = 2$ and $l_u < s_y, t_y \leq n - l_d)$ or $(x_1 > 2$ and $n = 5)]$.

Lemma 3.4. *If forbidden condition* (FH4) *is satisfied, then a Hamiltonian* (s,t)*-path does not exist.*

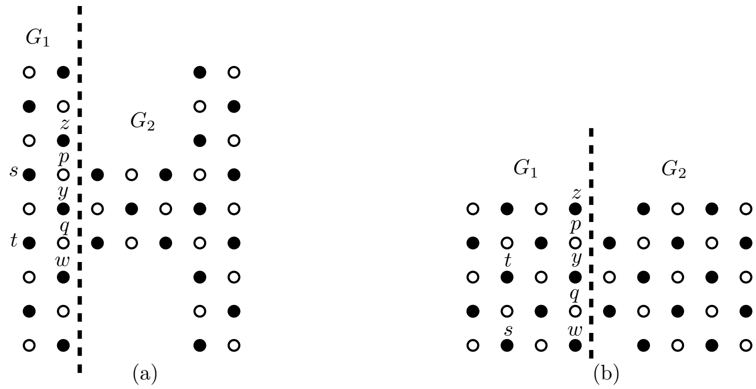

FIGURE 9. An example illustrating forbidden condition (FH4).

Proof. We partition G_H into two subgraphs $G_1 = R(x_1, n)$ and $G_2 = G_H \setminus G_1$ by applying a vertical cut, as shown in Figure 9. Let $z = (x_1, l_u)$, $p = (x_1, l_u + 1)$, $y = (x_1, l_u + 2)$, $q = (x_1, l_u + 3)$, and $w = (x_1, l_u + 4)$. Since x_1 is even, it is evident that $|V_B(G_1)| = |V_W(G_1)|$. On the other hand, since $||V_B(G_H)| - |V_W(G_H)|| = 1$, we have $||V_B(G_2)| - |V_W(G_2)|| = 1$. Since $||V_B(G_2)| - |V_W(G_2)|| = 1$, any Hamiltonian path starting from vertex s must traverse some vertices of G_1 and exit from G_1 through either vertex p or q to enter G_2. After traversing all vertices of G_2, the path must re-enter G_1 through one of the vertices q or p and then reach vertex t. A simple check shows that one of the vertices z, y, or w will be excluded from the path. Therefore, there is no Hamiltonian path between s and t in G_H. $\square$

Definition 3.5. *Assume that (G_H, s, t) is an H-alphabet grid graph with two given vertices s and t. (G_H, s, t) is considered acceptable if it satisfies the following two conditions:*

(1) *It is color-compatible, and*
(2) *None of the conditions* (F1), (F3), *and* (FH1) *to* (FH4) *hold for (G_H, s, t).*

Theorem 3.6. ***(Necessary Condition)*** *Let G_H be an odd-sized H-alphabet grid graph, and let s and t be two given vertices in G_H. If (G_H, s, t) is not acceptable, then a Hamiltonian (s, t)-path does not exist.*

Proof. It directly follows from Definition 2.1 and Lemmas 3.1–3.4. $\square$

4. Construction of Hamiltonian (s, t)-Paths in H-alphabet Grid Graphs

In this section, we present the construction of a Hamiltonian path between two vertices s and t in scenarios where (G_H, s, t) is color-compatible and does not fall into any of forbidden conditions (F1), (F3), and (FH1) to (FH4). To construct the path between s and t, we decompose the H-alphabet grid graph G_H into several smaller subgraphs in which the Hamiltonian path or cycle problem has been already solved. The partitioning of G_H is a crucial step, as it ensures that each subgraph possesses a Hamiltonian path or cycle. This decomposition directly impacts the existence and construction of the

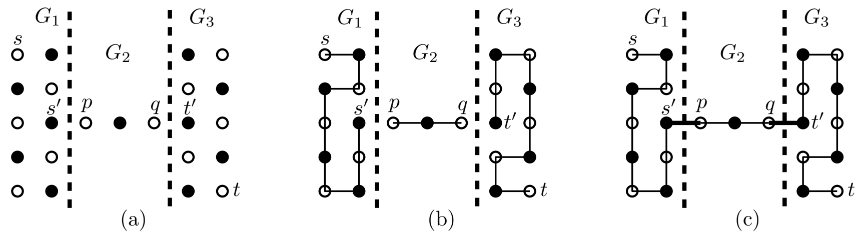

FIGURE 10. Construction of a Hamiltonian path between vertices s and t through subgraphs G_1, G_2, and G_3.

Hamiltonian path. After constructing Hamiltonian paths or cycles in the subgraphs, we combine them to form the desired Hamiltonian path between s and t in G_H. This method ensures that the construction is both feasible and efficient, provided that the partitioning is appropriate and the subgraph properties are carefully handled.

As discussed in the previous section, we consider two cases: Case I ($s_x, t_x \leq x_1 + k_u$) and Case II ($s_x \leq x_1$ and $t_x > x_1 + k_u$). First, we will explain how to construct a Hamiltonian path between two vertices s and t in the case where $y = n - l_u - l_d = 1$.

Lemma 4.1. *Let (G_H, s, t) be acceptable, where $y = 1$. In this case, H-alphabet grid graph G_H contains a Hamiltonian (s,t)-path.*

Proof. The only possible case occurs when $s_x \leq x_1$ and $t_x > x_1 + k_u$. If $s_x, t_x \leq x_1 + k_u$, then (G_H, s, t) falls into one of the forbidden conditions (F1) or (FH1). To construct a Hamiltonian path between s and t, we partition the H-alphabet grid graph G_H into three rectangular subgraphs $G_1 = R(x_1, n)$, $G_2 = R(m - x_1 - x_3, y)$, and $G_3 = R(m - x_1 - k_u, n)$ using two vertical cuts. In this configuration, $s, s' \in G_1$, $t, t' \in G_3$, and the intermediate vertices p and q are in G_2. We set $s' = (x_1, l_u + 1)$, $t' = (x_1 + k_u + 1, l_u + 1)$, $s' \sim p$, and $t' \sim q$, as shown in Figure 10(a). For subgraphs G_1, G_2, and G_3, we find Hamiltonian paths between s and s', between p and q, and between t' and t, respectively, by the algorithm given in [2], as shown in Figure 10(b). These subpaths exist because the subgraphs satisfy the required conditions; otherwise, (G_H, s, t) would violate forbidden condition (F1) or (FH1). After obtaining the Hamiltonian subpaths, we connect s' to p, and q to t', as depicted in Figure 10(c). By combining these connected paths, we construct a Hamiltonian path from s to t in G_H. □

Next, we assume that $y = n - l_u - l_d > 1$ and describe construction of a Hamiltonian path between vertices s and t.

Lemma 4.2. *Let (G_H, s, t) be acceptable, where x_1 is odd and it falls in Case I. In this case, H-alphabet grid graph G_H contains a Hamiltonian (s,t)-path.*

Proof. At the beginning of the proof, we note that k_u is odd and l_u is even. To prove this lemma, we consider two cases.

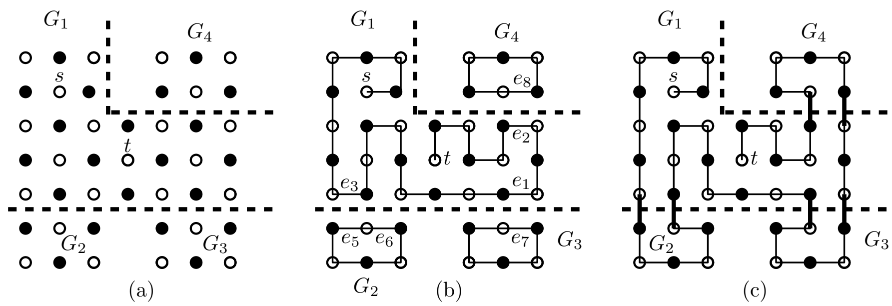

FIGURE 11. Case 1 of Lemma 4.2: (a) partitioning the input graph, (b) constructing Hamiltonian path and cycles, (c) combining the path and cycles to form a Hamiltonian path between s and t in G_H.

Case 1: $(s_y, t_y \leq n - l_d)$, $(s_y, t_y > n - l_d)$, or $(s_y(\text{or } t_y) > n - l_d$ and $l_u < t_y(\text{or } s_y) \leq n - l_d)$. Due to symmetry, it suffices to consider the case $s_y, t_y \leq n - l_d$. In this case, construction of a Hamiltonian path proceeds as follows. By applying one vertical cut and two horizontal cuts, the H-alphabet grid graph G_H is divided into four subgraphs $G_1 = L(m, n - l_d; k_u + x_3, l_u)$, $G_2 = R(x_2, l_d)$, $G_3 = R(x_4, l_d)$, and $G_4 = R(x_3, l_u)$, where $s, t \in G_1$, as illustrated in Figure 11(a). First, we show that (G_1, s, t) is acceptable. Since l_u is even, it follows that $|V_B(G_i)| = |V_W(G_i)|$ for $i = 2, 3, 4$. Furthermore, since $||V_B(G_H)| - |V_W(G_H)|| = 1$, it is easy to verify that $||V_B(G_1)| - |V_W(G_1)|| = 1$. Consequently, s and t in G_1 have the same color as they do in G_H. Therefore, (G_1, s, t) is color-compatible.

Next, we verify that (G_1, s, t) does not satisfy any of the forbidden conditions (F1) or (F3) through (F9). Since $||V_B(G_1)| - |V_W(G_1)|| = 1$, it suffices to show that (G_1, s, t) does not fall into forbidden conditions (F1), (F3), and (F5). Since $x_1 \geq 3$, $y = n - l_u - l_d \geq 3$, and (G_H, s, t) is acceptable, it follows that (G_1, s, t) does not fall under forbidden conditions (F1) or (F3) through (F9). Therefore, we confirm that (G_1, s, t) is acceptable. Up to this point, we have shown that (G_1, s, t) is acceptable. Now, we describe construction of a Hamiltonian path between s and t in G_H. A Hamiltonian path between s and t in G_1 is constructed using the algorithm given in [7], as shown in Figure 11(b). Let $u_1 = (1, n-l_d)$, $u_2 = (2, n-l_d)$, $u_3 = (3, n-l_d)$, $u_4 = (m-1, n-l_d)$, $u_5 = (m, n-l_d)$, $u_6 = (m-1, l_u+1)$, and $u_7 = (m, l_u + 1)$. Since the degrees of vertices u_5 and u_7 in G_1 are both 2, it follows that every Hamiltonian path between s and t must include the edges $e_1 = (u_4, u_5)$ and $e_2 = (u_6, u_7)$. Furthermore, every Hamiltonian path in G_1 must include either $e_3 = (u_1, u_2)$ or $e_4 = (u_2, u_3)$.

By Lemma 2.8, a Hamiltonian cycle exists in subgraphs G_2, G_3, and G_4. Lemma 2.9 also guarantees that the Hamiltonian cycle in G_2 and G_3 (and similarly in G_4) can always be constructed to include all boundary edges of side N (or S). Let $v_1 = (1, n - l_d + 1)$, $v_2 = (2, n - l_d + 1)$, $v_3 = (3, n - l_d + 1)$, $v_4 = (m - 1, n - l_d + 1)$, $v_5 = (m, n - l_d + 1)$, $v_6 = (m - 1, l_u)$, and $v_7 = (m, l_u)$. First, we combine a Hamiltonian path in G_1 with a Hamiltonian cycle in G_2 by using either two parallel edges e_3 and $e_5 = (v_1, v_2)$, or e_4 and $e_6 = (v_2, v_3)$. This results in a Hamiltonian path for $G_1 \cup G_2$, denoted as P_1. Next, we extend path P_1 by combining it with a Hamiltonian cycle in G_3, using two parallel edges e_1

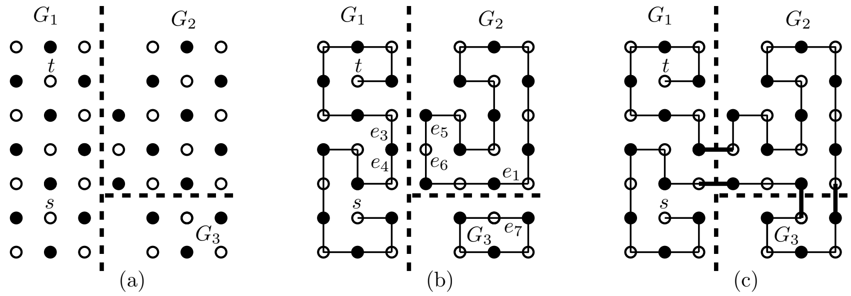

FIGURE 12. Case 2 of Lemma 4.2: (a) partitioning the input graph, (b) constructing Hamiltonian path and cycles, (c) combining the path and cycles to form a Hamiltonian path between s and t in G_H.

and $e_7 = (v_4, v_5)$. This creates a Hamiltonian path for $G_1 \cup G_2 \cup G_3$, denoted as P_2. Finally, we combine path P_2 with a Hamiltonian cycle in G_4, using two parallel edges e_2 and $e_8 = (v_6, v_7)$. This results in a Hamiltonian path between s and t in G_H, as illustrated in Figure 11(c).

Case 2: s_y(or t_y) $\leq l_u$ and t_y(or s_y) $> n - l_d$. In this case, the Hamiltonian path is constructed similarly to Case 1.1, with the difference that there are one vertical cut and one horizontal cut. The subgraphs are defined as $G_1 = R(x_1, n)$, $G_2 = L(m - x_1, n - l_d; k_u, l_u)$, and $G_3 = R(x_4, l_d)$. These subgraphs are illustrated in Figure 12(a). Additionally, in this case, $u_1 = (x_1, l_u + 1)$, $u_2 = (x_1, l_u + 2)$, $u_3 = (x_1, l_u + 3)$, $v_1 = (x_1 + 1, l_u + 1)$, $v_2 = (x_1 + 1, l_u + 2)$, and $v_3 = (x_1 + 1, l_u + 3)$. In the following, it is necessary to show that (G_1, s, t) is acceptable. Since both x_1 and n are odd, it is follows that $|V_W(G_2)| - |V_B(G_2)| = 1$. Furthermore, given that $|V_W(G_H)| - |V_B(G_H)| = 1$, it can be concluded that $|V_B(G_i)| = |V_W(G_i)|$ for $i = 2, 3$. Consequently, (G_1, s, t) is color-compatible. Since $x_1 \geq 3$ and $n \geq 3$, it can be easily verified that (G_1, s, t) does not meet forbidden conditions (F1) and (F2). According to the algorithm [6], it is always possible to construct a Hamiltonian path between s and t in G_1 that includes either e_3 or e_4, as illustrated in Figure 12(b). Moreover, by Lemma 2.10, it is always possible to construct a Hamiltonian cycle in G_2 that includes edges e_5, e_6, and e_1. A Hamiltonian path between s and t is shown in Figure 12(c). $\square$

Lemma 4.3. *Let (G_H, s, t) be acceptable, where x_1 is even, $y = n - l_u - l_d = 3$, and (G_H, s, t) falls under Case I.a. Then H-alphabet grid graph G_H contains a Hamiltonian (s, t)-path.*

Proof. Based on the value of n, we examine the following two cases.

Case 1: $n = 5$. In this case, $l_u = 1$, and either $s_x = x_1$ or $t_x = x_1$. If $s_x, t_x < x_1$, then (G_H, s, t) satisfies forbidden condition (FH4). Without loss of generality, assume $t_x = x_1$. One of the following subcases occurs.

Case 1.1: ($s_y > 3$ and $t_y \leq 3$) or ($s_y < 3$ and $t_y \geq 3$). Based on the symmetry, we consider the case $s_y > 3$ and $t_y \leq 3$. In this case, a Hamiltonian path between vertices s and t in G_H is constructed as follows. First, using one vertical and one horizontal cut grid graph G_H is divided into three subgraphs

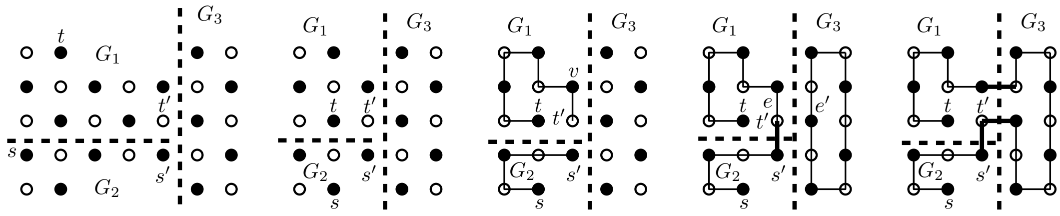

FIGURE 13. Steps for constructing a Hamiltonian (s,t)-path in Lemma 4.3, Case 1.1.

$G_1 = L(x_1+k_u, n_1; k_u, l_u)$, $G_2 = L(x_1+k_u, n-n_1; k_d, l_d)$, and $G_3 = R(m-x_1-k_u, n)$, where $n_1 = l_u+2$, $t \in G_1$, and $s \in G_2$, as illustrated in Figures 13(a) and 13(b). Two auxiliary vertices, $t' \in G_1$ and $s' \in G_2$, are defined as $t' = (x_1+k_u, l_u+2)$ and $s' = (x_1+k_u, l_u+4)$. Since x_1+k_u and l_u+2 are both odd, it follows that t' is a white vertex and s' is a black vertex.

We will now demonstrate that (G_1, t, t') and (G_2, s, s') are acceptable. Since x_3 is even, it is clear that $|V_W(G_3)| = |V_B(G_3)|$. Moreover, since x_1 is even, k_u and l_u are odd, implying $|V_W(G_1)| = |V_B(G_1)|$. Additionally, because $|V_B(G_H)| - |V_W(G_H)| = 1$, it follows that $|V_B(G_2)| - |V_W(G_2)| = 1$. Therefore, (G_1, t, t') and (G_2, s, s') are color-compatible. Since $t_x = x_1$ and $t' = (x_1+k_u, l_u+2)$, it is evident that (G_1, t, t') does not fall under forbidden conditions (F1) and (F3) to (F9). Furthermore, because $s'_x = x_1+k_u$, $s_x \leq x_1$, and $x_1 \neq (x_1, n-l_d)$, it is evident that (G_2, s, s') does not fall under forbidden conditions (F1) and (F3) to (F9). As a result, both (G_1, t, t') and (G_2, s, s') are acceptable, and by Theorem 2.6, there exist Hamiltonian paths between t and t' in G_1 and between s and s' in G_2.

Now, we will describe the process of constructing the final path. Using the algorithm of [7], Hamiltonian paths between t and t' in G_1 and between s and s' in G_2 are constructed, as shown in Figure 13(c). These two paths are connected via edge (t', s'), forming a larger path P_1, as shown in Figure 13(d). According to Lemma 2.8, G_3 contains a Hamiltonian cycle. Lemma 2.9 also guarantees that a Hamiltonian cycle in G_3 can always be constructed in a way such that includes all the boundary edges of side W, as shown in Figure 13(d). Let $v = (x_1+k_u, l_u+1)$, $u = (x_1+k_u+1, l_u+1)$, and $w = (x_1+k_u+1, l_u+2)$. Clearly, edge $e = (t', v)$ exists in any Hamiltonian path between t and t' in G_1. Finally, path P_1 is combined with the Hamiltonian cycle in G_3 through parallel edges e and $e' = (u, w)$. This results in a complete Hamiltonian path between s and t in G_H, as shown in Figure 13(e).

Case 1.2: $(s_y, t_y \leq 3)$, $(s_y = 3$ and $t_y > 3)$, or $(s_y, t_y > 3)$. In this case, $x_1 > 2$. By eliminating symmetry, we only need to consider the cases $s_y, t_y \leq 3$, as the other case $(s_y = 3$ and $t_y > 3)$ or $(s_y, t_y > 3)$ is symmetric to $s_y, t_y \leq 3$. In this case, we can further divide the analysis into the following subcases based on the value of s_x.

Case 1.2.1: $s_x < x_1 - 1$. First, using two vertical cuts and one horizontal cut grid graph G_H is divided into four subgraphs $G_1 = R(m_1, n)$, $G_2 = L(x_1+k_u-m_1, l_u+2; k_u, l_u)$, $G_3 = L(x_1+k_u-m_1, n-l_u-2; k_d, l_d)$, and $G_4 = R(m-x_1-k_u, n)$, where $m_1 = x_1 - 2$, $s \in G_1$ and $t \in G_2$, as illustrated in

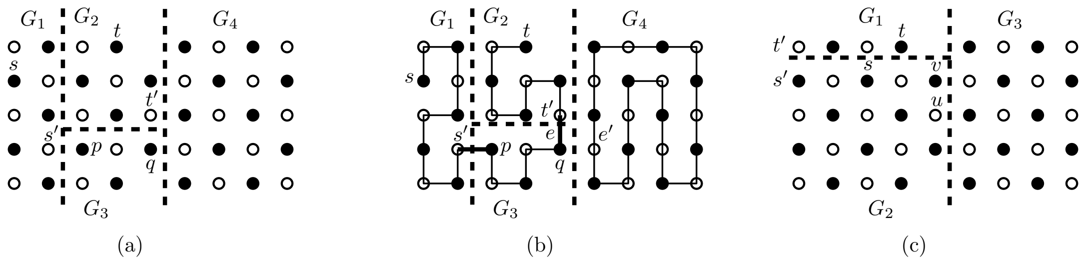

FIGURE 14. Steps for constructing a Hamiltonian (s,t)-path in Lemma 4.3, Case 1.2.

Figure 14(a). Four auxiliary vertices, $t' \in G_2$, $s' \in G_1$, and $p, q \in G_3$, are defined as $t' = (x_1 + k_u, l_u + 2)$, $s' = (m_1, n - l_d)$, $q \sim t'$, and $p \sim s'$. A simple check shows that t' and s' are white and p and q are black.

We will now demonstrate that (G_1, s, s'), (G_2, t, t'), and (G_3, p, q) are acceptable. Since x_2 and $x_1 - 2$ are even, it is clear that $|V_W(G_i)| = |V_B(G_i)|$, where $i = 1, 4$. Moreover, since $x_1 + k_u - m_1$, k_u and l_u are odd, implying $|V_W(G_2)| = |V_B(G_2)|$. Additionally, because $|V_B(G_H)| - |V_W(G_H)| = 1$, it follows that $|V_B(G_3)| - |V_W(G_3)| = 1$. Therefore, (G_1, s, s'), (G_2, t, t'), and (G_3, p, q) are color-compatible. Since $s_y \leq l_u + 2$ and $s'_y = n - l_d$, clearly (G_1, s, s') does not fall under forbidden conditions (F1) and (F2). Since $t_x = x_1$ and $t' = (x_1 + k_u, l_u + 2)$, it is evident that (G_2, t, t') does not fall under forbidden conditions (F1) and (F3) to (F9). Furthermore, because $p_x = x_1 - 1$ and $q_x = x_1 + k_u$, it is evident that (G_3, p, q) does not fall under forbidden conditions (F1) and (F3) to (F9). As a result, (G_1, s, s'), (G_2, t, t'), and (G_3, p, q') are acceptable, and by Theorems 2.4 and 2.6, there exist Hamiltonian paths between s and s' in G_1, between t and t' in G_2, and between p and q in G_3 .

Now, we will describe the process of constructing the final path. Using algorithms in [6, 7], the Hamiltonian paths between s and s' in G_1, between t and t' in G_2, and between p and q in G_3 are constructed, as shown in Figure 14(b). These three paths are connected via the edges (s', p') and (t', q), forming a larger path P_1. According to Lemma 2.8, G_4 contains a Hamiltonian cycle. Lemma 2.9 also guarantees that the Hamiltonian cycle in G_4 can always be constructed to include all boundary edges of face W, as shown in Figure 14(b). Let $u = (x_1 + k_u + 1, l_u + 2)$, and $w = (x_1 + k_u + 1, l_u + 3)$. Finally, path P_1 is combined with the Hamiltonian cycle in G_4 through the parallel edges $e = (q, t')$ and $e' = (u, w)$, as shown in Figure 14(b). This results in a complete Hamiltonian path between s and t in G_H.

Case 1.2.2: $s_x = x_1 - 1$. Here, $s = (x_1 - 1, l_u + 1)$ and $t = (x_1, l_u)$. In this case, the method for constructing the Hamiltonian path follows a similar approach to Case 1.1, with the difference that $G_1 = R(x_1, 1)$, $G_2 = L(x_1 + k_u, n - 1; k_d, l_d)$, $G_3 = R(m - x_1 - k_u, n)$, $t' = (1, 1)$, and $s' = (1, 2)$, as shown in Figure 14(c). Additionally, in this case, $e = (u, v)$, where $u = (x_1 + k_u, l_u + 2)$. A Hamiltonian path between t and t' in G_1 is constructed using the algorithm of [6].

Case 2: $n > 5$. If $s_y, t_y \leq n - l_d$, $s_y, t_y > n - l_d$, or $l_u < s_y\,(\text{or}\, t_y) \leq n - l_d$ and $t_y\,(\text{or}\, s_y) > n - l_d$, a Hamiltonian path between s and t is constructed as in Case 1 of Lemma 4.2. In the special case where $x_1 = 2$, if $s_y, t_y \leq n - l_d$ and $s\,(\text{or}\, t) = (1, n - l_d)$, or if $l_u < s_y\,(\text{or}\, t_y) \leq n - l_d$ and $t_y\,(\text{or}\, s_y) > n - l_d$ and

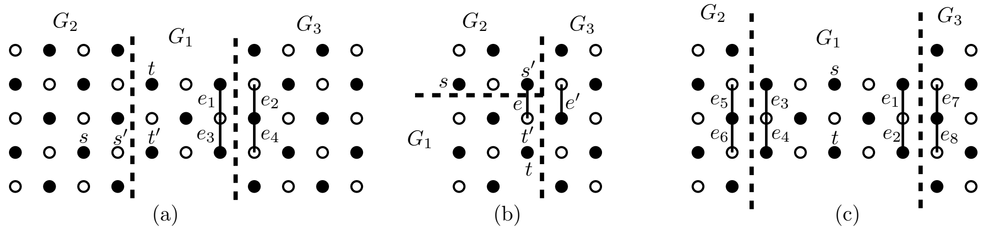

FIGURE 15. Division of G_H into three subgraphs.

$s\,(\text{or}\,t) = (1, l_u + 1)$, then a Hamiltonian path is constructed as in Case 1.1. Finally, if $s_y\,(\text{or}\,t_y) \leq l_u$ and $t_y\,(\text{or}\,s_y) > n - l_d$, a Hamiltonian path is constructed as in Case 1.1. □

Lemma 4.4. *Let (G_H, s, t) be acceptable, where x_1 is even, $y = n - l_u - l_d = 3$, and (G_H, s, t) falls under Cases I.b and I.c. Then, H-alphabet grid graph G_H contains a Hamiltonian (s, t)-path.*

Proof. We consider the following two cases.

Case 1: $s_x \leq x_1$ and $x_1 < t_x \leq x_1 + k_u$. Based on the value of x_1, we further divide this case into two subcases.

Case 1.1: $x_1 > 2$. In this case, the construction of the Hamiltonian path follows a similar approach to Case 1.1 of Lemma 4.3, with the difference that there are two vertical cuts. The subgraphs are defined as $G_1 = R(m - x_1 - x_2, n - l_u - l_d)$, $G_2 = R(x_1, n)$, $G_3 = R(m - x_1 - k_u, n)$, and $t' = (x_1 + 1, l_u + 1)$ if $t \neq (x_1 + 1, l_u + 1)$, otherwise $t' = (x_1 + 1, n - l_d)$, as shown in Figure 15(a). Additionally, in this case, $v_1 = (x_1 + k_u, l_u + 1)$, $v_2 = (x_1 + k_u, l_u + 2)$, $v_3 = (x_1 + k_u, l_u + 3)$, and $u_1, u_2, u_3 \in G_3$, where $u_1 \sim v_1$, $u_2 \sim v_2$, and $u_3 \sim v_3$. A Hamiltonian path between s and s' in G_2, and between t and t' in G_1, is constructed using the algorithm of [6]. In this case, one of two parallel edges $e_1 = (v_1, v_2)$ and $e_2 = (u_1, u_2)$, or $e_3 = (v_2, v_3)$ and $e_4 = (u_2, u_3)$ exists, to combine the path P_1 with the Hamiltonian cycle in G_3.

Case 1.2: $x_1 = 2$. In this case, we consider the scenario where $s_y < n - l_d$. The case where $s_y \geq n - l_d$ is symmetric. If $s_y \neq l_u + 1$ or $s_y = l_u + 1$ and $t \neq (x_1 + 1, n - l_d)$, a Hamiltonian path between s and t is constructed in the same way as in Case 1.1. If $s_y = l_u + 1$ and $t = (x_1 + 1, n - l_d)$, a Hamiltonian path is constructed similarly to Case 1.1 of Lemma 4.3, with the difference that in this case, $G_2 = L(x_1 + k_u, l_u + 1; k_u, l_u)$, $G_1 = L(x_1 + k_u, n - l_u - 1; k_d, l_d)$, and $G_3 = R(m - x_1 - k_u, n)$. Additionally, here $t' = (x_1 + k_u, l_u + 1)$, $s' = (x_1 + k_u, l_u + 2)$, and $e = (s', t')$, as illustrated in Figure 15(b).

Case 2: $x_1 < s_x, t_x \leq x_1 + k_u$. In this case, a Hamiltonian path between s and t is constructed similarly to Case 1.1 of Lemma 4.2, with the difference that there are two vertical cuts here. The resulting subgraphs are $G_1 = R(m - x_1 - x_3, y)$, $G_2 = R(x_1, n)$, and $G_3 = R(m - x_1 - k_u, n)$, as illustrated in Figure 15(c). Additionally, in this case, $u_1 = (x_1 + 1, l_u + 1)$, $u_2 = (x_1 + 1, l_u + 2)$, $u_3 = (x_1 + 1, l_u + 3)$,

$u_4 = (x_1+k_u, l_u+1)$, $u_5 = (x_1+k_u, l_u+1)$, $u_6 = (x_1+k_u, l_u+3)$, $v_1, v_2, v_3 \in G_2$, $v_4, v_5, v_6 \in G_3$, $v_1 \sim u_1$, $v_2 \sim u_2$, $v_3 \sim u_3$, $v_4 \sim u_4$, $v_5 \sim u_5$, and $v_6 \sim u_6$.

In the following, it is necessary to show that (G_1, s, t) is acceptable. Since x_1 and x_3 are even, it follows that $|V_W(G_i)| = |V_B(G_i)|$ for $i = 2, 3$. Furthermore, given that $||V_B(G_H)| - |V_W(G_H)|| = 1$, we deduce that $||V_B(G_1)| - |V_W(G_1)|| = 1$. Thus, (G_1, s, t) is color-compatible. Since $y = 3$ and $k_u \geq 1$, the subgraph (G_2, s, t) does not meet forbidden conditions (F1) and (F2). A Hamiltonian path between s and t in G_1 is constructed using the algorithm of [6]. Since $n > 1$, $x_1 > 1$, and $x_2 > 1$, Lemma 2.8 ensures that the rectangular subgraphs G_2 and G_3 have Hamiltonian cycles. Moreover, according to Lemma 2.9, Hamiltonian cycles in G_2 and G_3 can always be constructed to include all boundary edges of faces E and W, respectively.

Additionally, in this case, the edges used to combine the Hamiltonian path between s and t in G_1 with the Hamiltonian cycles in G_2 and G_3, are more than one, as explained below. First, we merge a Hamiltonian path in (G_1, s, t) with a Hamiltonian cycle in G_2 via one of the two parallel edges $e_3 = (u_1, u_2)$ and $e_5 = (v_1, v_2)$, or $e_4 = (u_2, u_3)$ and $e_6 = (v_2, v_3)$, resulting in a larger Hamiltonian path, denoted as P_1. Next, path P_1 is merged with a Hamiltonian cycle in G_3 through one of two parallel edges $e_1 = (u_4, u_5)$ and $e_7 = (v_4, v_5)$, or $e_2 = (u_5, u_6)$ and $e_8 = (v_5, v_6)$, resulting in a final Hamiltonian path in (G_H, s, t), as shown in Figure 15(c). □

Lemma 4.5. *Let (G_H, s, t) be acceptable, where x_1 is even, $n > 5$, and it falls under Case I. In this case, H-alphabet grid graph G_H contains a Hamiltonian (s, t)-path.*

Proof. In this case, $y = n - l_u - l_d > 3$. The following cases are considered for the proof.

Case 1: $s_y, t_y \leq n - l_d - 2$ or $s_y, t_y > n - l_d - 2$. We only consider the case $s_y, t_y \leq n - l_d - 2$. The case $s_y, t_y > n - l_d - 2$ is symmetric. In this case, the Hamiltonian path is constructed similarly to Case 1 of Lemma 4.2, with the difference that there are two vertical cuts. The subgraphs are defined as $G_1 = L(x_1+k_u, n_1; k_u, l_u)$, $G_2 = R(m-x_1-k_u, n)$, and $G_3 = L(x_1+k_u, n-n_1; k_d, l_d)$, where $n_1 = n-l_d-2$. These subgraphs are illustrated in Figure 16(a). Additionally, in this case, $u_1 = (x_1 + k_u, l_u + 1)$, $u_2 = (x_1 + k_u, l_u + 2)$, $u_1 = (x_1 + k_u, l_u + 3)$, $u_4 = (x_1 + k_u + 1, n - l_d)$, $u_5 = (x_1 + k_u + 1, n - l_d - 1)$, $v_1, v_2, v_3 \in G_2$, $v_4, v_5 \in G_3$, $v_1 \sim u_1$, $v_2 \sim u_2$, $v_3 \sim u_3$, $v_4 \sim u_4$, and $v_5 \sim u_5$. In the following, it is necessary to show that (G_1, s, t) is acceptable. Let l_u is odd (resp. even), then n_1 is even (resp. odd). Since $x_1 + k_u$ and k_u are odd, it follows that $||V_B(G_1)| - |V_W(G_1)|| = 1$. Additionally, given that $||V_B(G_H)| - |V_W(G_H)|| = 1$, it is straightforward to verify that $|V_B(G_i)| = |V_W(G_i)|$ for $i = 2, 3$. Since $||V_B(G_H)| - |V_W(G_H)|| = 1$ and $||V_B(G_1)| - |V_W(G_1)|| = 1$, the color parity holds for subgraph G_1. Therefore, (G_1, s, t) is color-compatible. Since $n - l_u - l_d - 2 \geq 3$ and (G_H, s, t) is acceptable, it follows that (G_1, s, t) does not fall into forbidden conditions (F1), and (F3) to (F9). Therefore, (G_1, s, t) is confirmed to be acceptable.

Case 2: s_y(or t_y) $\leq n - l_d - 2$ and t_y(or s_y) $> n - l_d - 2$. Without loss of generality, assume that $t_y \leq n - l_d - 2$ and $s_y > n - l_d - 2$. For t_y, we consider the following two cases.

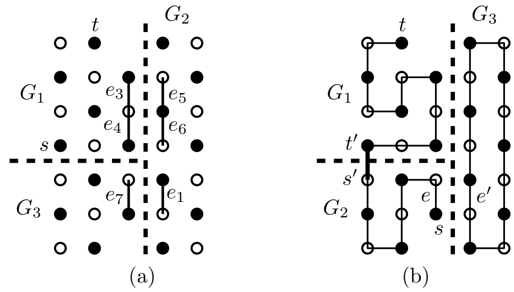

FIGURE 16. The proof of Lemma 4.5, division of G_H into three subgraphs.

Case 2.1: $l_u + 2 < t_y \leq n - l_d - 2$. This case is symmetric to Case 1.

Case 2.2: $t_y \leq l_u + 2$. A Hamiltonian path between s and t is constructed similarly to Case 1.1 of Lemma 4.3, with the key difference that $n_1 = n - l_d - 2$, $s' = (1, n_1 + 1)$, and $t' = (1, n_1)$. Additionally, in this case, $v = (x_1 + k_u, n - l_d - 1)$, $z = (x_1 + k_u, n - l_d)$, $u = (x_1 + k_u + 1, n - l_d)$, and $w = (x_1 + k_u + 1, n - l_d - 1)$, as illustrated in Figure 16(b). Note that since $s'_x = t'_x = 1$ and $s_x, t_x \geq 1$, it is straightforward to verify that (G_2, s, s') and (G_1, t, t') do not fall under forbidden conditions (F1) and (F3) to (F9). Since the degree of vertex v in G_2 is 2, edge $e = (z, v)$ must be included in every Hamiltonian path between s and s' in G_2. This ensures that $e = (z, v)$ is part of path P_1. □

Lemma 4.6. *Let (G_H, s, t) be acceptable and it falls under Case IV. In this case, H-alphabet grid graph G_H contains a Hamiltonian (s,t)-path.*

Proof. The following cases are considered based on the value of x_1.

Case 1: x_1 is even. Two possible cases are considered.

Case 1.1: $x_1 > 2$. A Hamiltonian (s,t)-path is constructed as in Lemma 4.1, with the difference that in this case, $s' = (x_1, l_u + 1)$ and $t' = (x_1 + k_u + 1, n - l_d)$.

Case 1.2: $x_1 = 2$. If $s \neq (1, l_u + 1)$ and $t \neq (m, n - l_d)$, a Hamiltonian (s,t)-path is constructed as described in Case 1.1. Now, let $s = (1, l_u + 1)$ or $t = (m, n - l_d)$. Due to symmetry, consider the case $s = (1, l_u + 1)$. If $t \neq (m, l_u + 1)$, a Hamiltonian (s,t)-path is constructed similarly to Lemma 4.1, with the difference that in this case $s' = (x_1, n - l_d)$ and $t' = (m - 1, l_u + 1)$. If $t = (m, l_u + 1)$, a Hamiltonian (s,t)-path is constructed as in Case 1.2.1 of Lemma 4.3, with the following modifications: $G_1 = R(x_1, n)$, $G_3 = R(m - x_1 - x_3, n - l_u - l_d)$, $G_2 = R(m - x_1 - k_u, t_y)$, and $G_4 = R(m - x_1 - k_u, n - t_y)$. Here, $s' = (x_1, n - l_d)$, $t' = (m - 1, l_u + 1)$, $e = (u_1, u_2)$, and $e' = (u_3, u_4)$, where $u_1 = (x_1 + k_u, n - l_d)$, $\quad u_2 = (x_1 + k_u, n - l_d - 1)$, $\quad u_3, u_4 \in G_4$, $\quad u_3 \sim u_1$, $\quad u_4 \sim u_3$.

Case 2: x is odd. If $s \neq (x_1, l_u + 1)$ and $t \neq (x_1 + k_u + 1, n - l_d)$, a Hamiltonian (s,t)-path is constructed as described in Case 1.1. Now, let $s = (x_1, l_u + 1)$ or $t = (x_1 + k_u + 1, n - l_d)$. Due to symmetry, consider the case $s = (x_1, l_u + 1)$. If $t \neq (x_1 + k_u + 1, l_u + 1)$, a Hamiltonian (s,t)-path is constructed similarly to Case 1.1, with the difference that in this case $s' = (x_1, n - l_d)$ and $t' = (x_1 + k_u + 1, l_u + 1)$. If $t = (x_1 + k_u + 1, l_u + 1)$, a Hamiltonian (s,t)-path is constructed as in Case 1 of Lemma 4.2.

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□

Theorem 4.7. *Assuming that the forbidden conditions do not hold, the cases presented in Lemmas 4.1–4.6 cover all possible placements of the given vertices s and t in a given odd-sized H-alphabet grid graph.*

Proof. Based on the value of y, we consider the following two cases.

Case 1: $y = 1$. A Hamiltonian path between s and t in G_H is constructed based on Lemma 4.1.

Case 2: $y > 1$. Based on the x-coordinates of the vertices s and t, we consider the following cases.

Case 2.1: $s_x, t_x \leq x_1$ (Case I.a).

Case 2.1.1: x_1 is odd. A Hamiltonian path between s and t in G_H is constructed based on Lemma 4.2.

Case 2.1.2: x_1 is even. If $y = 3$, then a Hamiltonian path between s and t in G_H is constructed based on Lemma 4.3. If $y > 3$, then a Hamiltonian path between s and t in G_H is constructed based on Lemma 4.5.

Case 2.2: $s_x \leq x_1$ and $x_1 < t_x \leq x_1 + k_u$ (Case I.b). We distinguish the following two subcases based on the value of x_1.

Case 2.2.1: x_1 is odd. A Hamiltonian path between s and t in G_H is constructed based on Lemma 4.2.

Case 2.2.2: x_1 is even. If $y = 3$, then a Hamiltonian path between s and t in G_H is constructed based on Lemma 4.4. If $y > 3$, then a Hamiltonian path between s and t in G_H is constructed based on Lemma 4.5.

Case 2.3: $x_1 < s_x, t_x \leq x_1 + k_u$ (Case I.c).

Case 2.3.1: x_1 is odd. A Hamiltonian path between s and t in G_H is constructed based on Lemma 4.2.

Case 2.3.2: x_1 is even. If $y = 3$, then a Hamiltonian path between s and t in G_H is constructed based on Lemma 4.4. If $y > 3$, then a Hamiltonian path between s and t in G_H is constructed based on Lemma 4.5.

Case 2.4: $s_x, t_x > x_1 + k_u$ (Case II). This case is isomorphic to Case 2.1.

Case 2.5: $x_1 < s_x \leq x_1 + k_u$ and $t_x > x_1 + k_u$ (Case III). This case is isomorphic to Case 2.2.

Case 2.6: $s_x \leq x_1$ and $t_x > x_1 + k_u$ (Case IV). A Hamiltonian path between s and t in G_H is constructed based on Lemma 4.5. □

Theorem 4.8. *Given an odd-sized H-alphabet grid graph G_H and two given vertices s and t, a Hamiltonian (s, t)-path can be constructed for it in linear time.*

Proof. According to Lemmas 4.1–4.5, constructing a Hamiltonian (s, t)-path for G_H involves partitioning G_H into several subgraphs. This partitioning is performed in $O(1)$ time. Next, the Hamiltonian cycles and paths for the subgraphs are constructed. The basic cases include rectangular and L-shaped grid graphs, for which the Hamiltonian cycles and paths are computed in linear time, by Theorem 2.7 and Lemma 2.8. The merging of Hamiltonian cycles and paths of the subgraphs is also performed in $O(1)$ time. The total cost of partitioning and merging operations is linear with respect to the total order of the graph. Therefore, the algorithm runs in linear time. □

5. Conclusion

In this paper, we addressed the Hamiltonian (s,t)-path problem in odd-sized H-alphabet grid graphs. We first derived the conditions under which a Hamiltonian (s,t)-path does not exist in odd-sized H-alphabet grid graphs. Additionally, for the cases where a Hamiltonian (s,t)-path there exists, we proposed a linear-time algorithm to find such a path. The results obtained in this work contribute significantly to the understanding of the Hamiltonian path problem in grid graphs.

Declarations

- Conflict of interest/Competing interests
  The authors declare that they have no competing of interests regarding the publication of this paper.
- Authors' contributions
  Marzieh Ghanbarian-Alavijeh and F. Keshavarz-Kohjerdi developed the algorithm and wrote the main manuscript text. The figures are prepared by Marzieh Ghanbarian-Alavijeh and F. Keshavarz- Kohjerdi. All authors reviewed the paper.

References

[1] T .W. Chang, O. Navrtil, S. L. Peng, The end-to-end longest path problem on a mesh with a missing vertex, *Front. Artif. Intell. Appl.*, **274** (2015) 59–66.

[2] S. D. Chen, H. Shen and R. Topor, An efficient algorithm for constructing Hamiltonian paths in meshe, *Parallel Comput.*, **28** no. 9 (2002) 1293–1305.

[3] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, CA, 1979.

[4] K. Hamada, A picturesque maze generation algorithm with any given endpoints, *Journal of Information Processing*, **21** no. 3 (2013) 393–397.

[5] C. Icking, T. Kamphans, R. Klein and E. Langetepe, Exploring simple grid polygons, Computing and combinatorics, COCOON 2005, Lecture Notes in Comput. Sci., **3595**, Springer, Berlin, 2005 524–533.

[6] A. Itai, C. H. Papadimitriou and J. L. Szwarcfiter, Hamiltonian paths in grid graphs, *SIAM J. Comput.*, **11** no. 4 (1982) 676–686.

[7] F. Keshavarz-Kohjerdi and A. Bagheri, Hamiltonian paths in L-shaped grid graphs, *Theoret. Comput. Sci.*, 621 (2016) 37–56.

[8] F. Keshavarz-Kohjerdi and A. Bagheri, Finding Hamiltonian cycles of truncated rectangular grid graphs in linear time, *Appl. Math. Comput.*, **436** (2023) 14 pp.

[9] F. Keshavarz-Kohjerdi, and A. Bagheri, Hamiltonian (s,t)-paths in solid supergrid graphs, *Comput. Appl. Math.*, **43** no. 3 (2024) 24 pp.

[10] M. Kreh, A note on Hamiltonian cycles in planar graphs, *Discrete Math. Algorithms Appl.*, **15** no. 1 (2023) 5 pp.

[11] R. I. Nishat, V. Srinivasan and S. Whitesides, 1-complex s,t Hamiltonian paths: structure and reconfiguration in rectangular grids, J. Graph Algorithms Appl., **27** no. 4 (2023) 281–327.

[12] R. I. Nishat, V. Srinivasan and S. Whitesides, Reconfiguring simple s,t Hamiltonian paths in rectangular grid graphs, *Combinatorial Algorithms. IWOCA 2021*, Lecture Notes in Comput. Sci., Springer, Cham, (2021) 501–515.

[13] R. I. Nishat, V. Srinivasan and S. Whitesides, The hamiltonian path graph is connected for simple s,t paths in rectangular grid graphs, *J. Comb. Optim.*, **48** no. 31 (2024) 48 pp.

[14] R. I. Nishat and S. Whitesides, Reconfiguring Hamiltonian cycles in L-shaped grid graphs, Graph-theoretic concepts in computer science, WG 2019., Lecture Notes in Comput. Sci., Springer, Cham, (2019) 325–337.

[15] R. I. Nishat and S. Whitesides, Reconfiguration of Hamiltonian cycles in rectangular grid graphs, *Internat. J. Found. Comput. Sci.*, **34** no. 07 (2023) 773–793.

[16] M. S. Rahman and M. Kaykobad, On Hamiltonian cycles and Hamiltonian paths, *Inform. Process. Lett.*, **94** no. 1 (2005) 37–41.

[17] A. N. M. Salman, E. T. Baskoro and H. J. Broersma, A note concerning Hamiltonian cycles in some classes of grid graphs, *J. Math. Fund. Sci.*, **35** no. 1 (2003) 65–70.

[18] B. Seluk and A. N. A. Tankl, Hamiltonian path, routing, broadcasting algorithms for connected square network graphs, *Int. J. Eng. Sci. Technol.*, **44** (2023).

[19] C. Umans and W. Lenhart, Hamiltonian cycles in solid grid graphs, *Proceedings of 38th Annual Symposium on Foundations of Computer Science, FOCS?*, Miami Beach, FL, USA, (1997) 496–505.

[20] Y. Xie, On Hamiltonian properties of grid graph, *International Conference on Intelligent Computing, Automation and Applications (ICAA)*, (2021) 5–9.

[21] J. M. Xu and M. Ma, Survey on path and cycle embedding in some networks, *Front. Math. China*, **4** no. 2 (2009) 217–252.

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