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TOTAL AND PAIRED DOMINATION NUMBERS OF SOME WHEEL-RELATED GRAPHS

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Abstract. Let G be a graph without isolated vertices. A total dominating set of G is a set $D \subseteq V(G)$ such that every vertex of G is adjacent to some vertex in D. A paired dominating set of G is a total dominating set whose induced subgraph has a perfect matching. The total (paired) domination number of G is the minimum cardinality of a total (paired) dominating set of G. In this paper, we determine the total and the paired domination numbers of some wheel-related graphs. We also give upper bounds on the total and the paired domination numbers of closed helm graphs and web graphs. Moreover, we determine the paired domination numbers of Jahangir graphs and correct some results on the total domination numbers presented by Mtarneh et al. (Malays. J. Math. Sci., 13(S) (2019) 113–121).

1. Introduction

Let G be a graph without isolated vertices with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the *neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$. If v is adjacent to a vertex of degree one, then it is called a *support vertex*. For a set $D \subseteq V(G)$, the *neighborhood* of D is the set $N(D) = \bigcup_{v \in D} N(v)$. The *induced subgraph* $G[D]$ is the graph whose vertex set is D and whose edge set consists of all edges in $E(G)$ having both endpoints in D.

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A *total dominating set* of G, defined by Cockaye et al. [3], is a set $D \subseteq V(G)$ such that $N(D) = V(G)$. A *paired dominating set* of G, introduced by Haynes and Slater [8], is a set $D \subseteq V(G)$ such that $N(D) = V(G)$ and $G[D]$ contains a perfect matching. If D is a paired dominating set with a perfect matching M, then two vertices u and v are said to be *paired* if the edge $uv \in M$. The minimum cardinality of a total dominating set (respectively, paired dominating set) of G is called the *total domination number* $\gamma_t(G)$ (respectively, *paired domination number* $\gamma_{pr}(G)$). A $\gamma_t(G)$-*set* is a total dominating set with cardinality $\gamma_t(G)$; a $\gamma_{pr}(G)$-*set* is defined analogously. Note that $\gamma_{pr}(G)$ is an even integer, and $\gamma_{pr}(G) \geq \gamma_t(G)$ since a paired dominating set of G is also a total dominating set of G.

Let D be a total (paired) dominating set of G. Then we say that a vertex $v \in D$ *dominates* every vertex in $N(v)$. We note that a vertex of degree one must be dominated by its support vertex, so we can easily get the following lemma.

Lemma 1.1. *Every support vertex of a graph G belongs to every $\gamma_t(G)$-set and $\gamma_{pr}(G)$-set.*

A *path* and a *cycle* with p vertices are denoted by P_p and C_p, respectively. Henning [9] determined the total domination numbers of paths and cycles in Lemma 1.2. In [8], the authors calculated the paired domination numbers of paths and cycles in Lemma 1.3. There are also many research articles on total domination numbers or paired domination numbers of some graph operations [1, 2, 12, 13, 21] and some families of graphs, for example, lollipop graphs [5], windmill graphs [6], 3-regular Knödel graphs [18], Jahangir graphs [16, 19], hexagonal grids [15, 17], Cartesian products of two paths [7, 14], Cartesian products of two cycles [11], and Cartesian products of paths and cycles [4, 10].

Lemma 1.2. [9] *For any integer $p \geq 3$, $\gamma_t(P_p) = \gamma_t(C_p) = \lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor$.*

Lemma 1.3. [8] *For any integer $p \geq 3$, $\gamma_{pr}(P_p) = \gamma_{pr}(C_p) = 2\lceil \frac{p}{4} \rceil$.*

We next give the definitions of wheel graphs and its related graphs. Let $q \geq 3$ be an integer. Then

(1) the *wheel graph* W_q is is obtained by connecting the vertex c to all vertices of C_q, where $V(C_q) = \{u_1, u_2, \ldots, u_q\}$,

(2) the *helm graph* H_q is obtained from the wheel graph W_q by adding the vertices $v_1, v_2, \ldots, v_q$ and the edge $u_i v_i$ for all $i \in \{1, 2, \ldots, q\}$,

(3) the *flower graph* Fl_q is obtained from the helm graph H_q by adding the edge cv_i for all $i \in \{1, 2, \ldots, q\}$, and

(4) the *sunflower graph* Sf_q is obtained from the helm graph H_q by adding the edges $v_i u_{i+1}$ for all $i \in \{1, 2, \ldots, q-1\}$ and $v_q u_1$.

The wheel graph W_q and the helm graph H_q are shown in Figure 1, while the flower graph Fl_6 and the sunflower graph Sf_6 are shown in Figure 2.

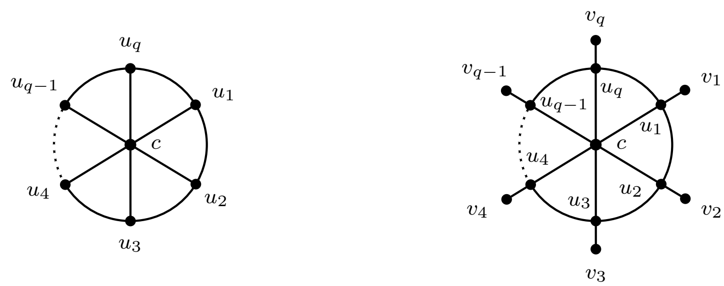

FIGURE 1. The wheel graph W_q (left) and the helm graph H_q (right)

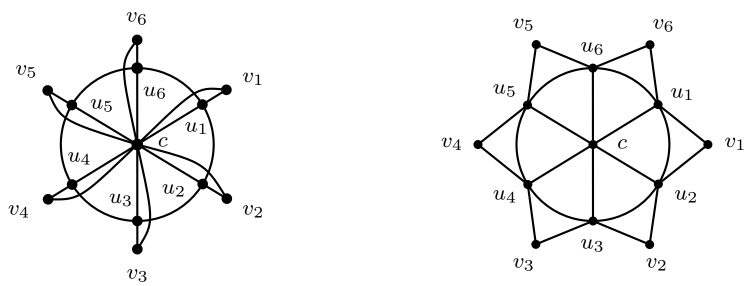

FIGURE 2. The flower graph Fl_6 (left) and the sunflower graph Sf_6 (right)

Let $p \geq 1$ and $q \geq 3$ be integers. Let C_{pq} be the cycle with $V(C_{pq}) = \{u_{i,j} : 1 \leq i \leq p, 1 \leq j \leq q\}$ and $E(C_{pq}) = \{u_{i,j}u_{i+1,j} : 1 \leq i \leq p-1, 1 \leq j \leq q\} \cup \{u_{p,j}u_{1,j+1} : 1 \leq j \leq q-1\} \cup \{u_{p,q}u_{1,1}\}$. The *Jahangir graph* $J_{p,q}$ is obtained from the cycle C_{pq} by adding the vertex c and the edge $cu_{1,j}$ for all $j \in \{1, 2, \ldots, q\}$. We note that $J_{1,q}$ is the wheel graph W_q and $J_{2,q}$ is known as the *gear graph*. Figure 3 illustrates the Jahangir graphs $J_{2,q}$ and $J_{3,q}$.

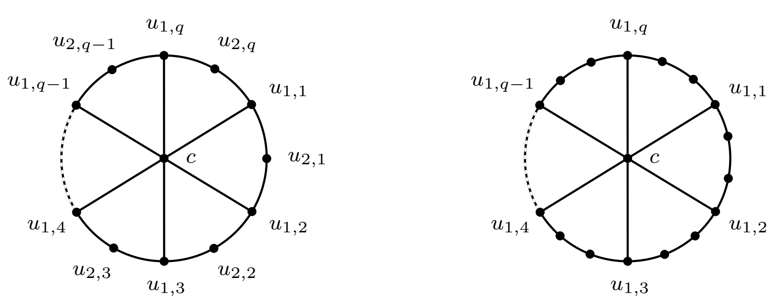

FIGURE 3. The Jahangir graphs $J_{2,q}$ (left) and $J_{3,q}$ (right)

The *Cartesian product* of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ whose vertices (u, v) and (u', v') are adjacent if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

Let $P_p : 1, 2, \ldots, p$ be the path with p vertices and $C_q : 1, 2, \ldots, q$ be the cycle with q vertices. We use $u_{i,j}$ to denote the vertex of $P_p \square C_q$ corresponding to the vertex (i, j). For any integers $p \geq 1$ and $q \geq 3$,

(1) the *closed helm graph* $CH_{p,q}$ is obtained from $P_p \square C_q$ by adding the vertex c and the edge $cu_{1,j}$ for all $j \in \{1, 2, \ldots, q\}$, and

(2) the *web graph* $W_{p,q}$ is obtained from $CH_{p,q}$ by adding the vertices $v_1, v_2, \ldots, v_q$ and the edge $u_{p,j}v_j$ for all $j \in \{1, 2, \ldots, q\}$.

We observe that $CH_{1,q} \cong W_q$ and $W_{1,q} \cong H_q$. The closed helm graph $CH_{p,q}$ and the web graph $W_{p,q}$ are illustrated in Figure 4.

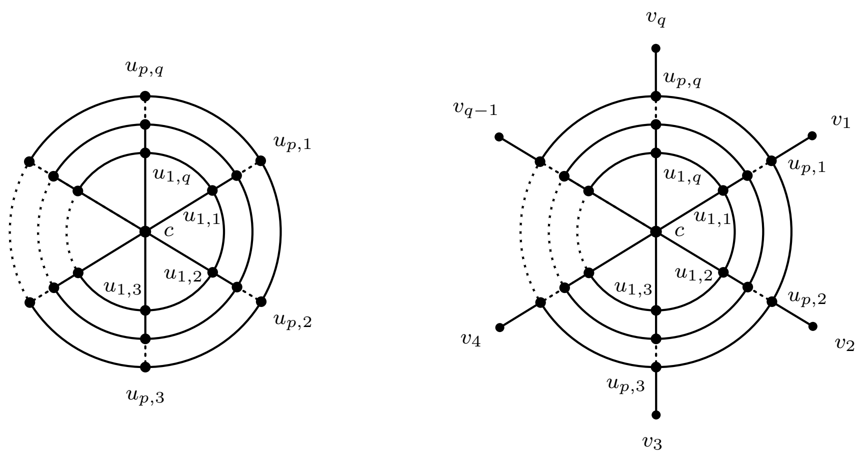

FIGURE 4. The closed helm graph $CH_{p,q}$ (left) and the web graph $W_{p,q}$ (right)

This paper is organized as follows. Section 2 presents the total and the paired domination numbers of wheel graphs, flower graphs, helm graphs, and sunflower graphs. In Section 3, we first revise some results on the total domination numbers of Jahangir graphs given in [19], and we then compute the paired domination numbers. The total and the paired domination numbers of some closed helm graphs and web graphs are provided in Section 4 where we also give some upper bounds for the other cases. Finally, we provide some open problems in Section 5.

2. Wheel, Flower, Helm, and Sunflower Graphs

Note that $\{c, u_1\}$ is a minimum total (paired) dominating set of wheel graphs and flower graphs, so we obtain the following obvious result.

Theorem 2.1. *For any integer $q \geq 3$, $\gamma_t(W_q) = \gamma_{pr}(W_q) = \gamma_t(Fl_q) = \gamma_{pr}(Fl_q) = 2$.*

Theorem 2.2. *For any integer $q \geq 3$,*

$$\gamma_t(H_q) = q \text{ and } \gamma_{pr}(H_q) = \begin{cases} q & \text{if } q \text{ is even;} \\ q+1 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. By Lemma 1.1, the vertices $u_1, u_2, \ldots, u_q$ are in every $\gamma_t(H_q)$-set. Hence, $\gamma_t(H_q) \geq q$. Since $D = \{u_1, u_2, \ldots, u_q\}$ is a total dominating set of H_q, $\gamma_t(H_q) \leq |D| = q$. It follows that $\gamma_t(H_q) = q$. Moreover, D is a paired dominating set of H_q if q is even, and $D \cup \{c\}$ is a paired dominating set of H_q

if q is odd. Thus, $\gamma_{pr}(H_q) \le q$ if q is even, and $\gamma_{pr}(H_q) \le q+1$ if q is odd. Since $\gamma_{pr}(H_q) \ge \gamma_t(H_q) = q$ and $\gamma_{pr}(H_q)$ is even, we get $\gamma_{pr}(H_q) = q$ if q is even, and $\gamma_{pr}(H_q) = q+1$ if q is odd. $\square$

Theorem 2.3. *For any integer $q \ge 3$,*

$$\gamma_t(Sf_q) = \begin{cases} 2 & \text{if } q = 3; \\ \lceil \frac{q}{2} \rceil + 1 & \text{if } q \ge 4; \end{cases} \text{ and } \gamma_{pr}(Sf_q) = 2\lceil \frac{q}{3} \rceil.$$

Proof. Clearly, $\gamma_t(Sf_3) = 2$. Let $q \ge 4$ and D be a $\gamma_t(Sf_q)$-set. To dominate $v_1, v_2, \ldots, v_q$, D contains at least $\lceil \frac{q}{2} \rceil$ vertices from $\{u_1, u_2, \ldots, u_q\}$. If $|D| = \lceil \frac{q}{2} \rceil$, then without loss of generality, $D = \{u_i : i \equiv 1 \pmod 2\}$. We can see that there is some vertex in D that is not dominated. This contradicts the fact that D is a total dominating set, so $|D| \ge \lceil \frac{q}{2} \rceil + 1$. Since $\{c\} \cup \{u_i : i \equiv 1 \pmod 2\}$ is a total dominating set with cardinality $\lceil \frac{q}{2} \rceil + 1$, we get that $\gamma_t(Sf_q) = \lceil \frac{q}{2} \rceil + 1$.

We note that any two adjacent vertices can dominate at most three vertices in $\{v_1, v_2, \ldots, v_q\}$, so $\gamma_{pr}(Sf_q) \ge 2\lceil \frac{q}{3} \rceil$. If $q \equiv 0, 2 \pmod 3$, let $D = \{u_i, u_{i+1} : i \equiv 1 \pmod 3\}$; otherwise, let $D = \{u_i, u_{i+1} : i \equiv 1 \pmod 3, i \ne q\} \cup \{u_{q-1}, u_q\}$. Then D is a paired dominating set with cardinality $2\lceil \frac{q}{3} \rceil$, so $\gamma_{pr}(Sf_q) = 2\lceil \frac{q}{3} \rceil$. $\square$

3. **Jahangir Graphs**

This section discusses the total and the paired domination numbers of Jahangir graphs. The total domination number of Jahangir graph $J_{p,q}$ for $q \ge 3$ is determined by Mojdeh and Ghameshlou [16] when $p = 2$ and by Mtarneh *et al.* [19] when $p \ge 3$.

Lemma 3.1. *For any integer $q \ge 3$,*

(1) [16] $\gamma_t(J_{2,q}) = \lceil \frac{q}{2} \rceil + 1$ *and*
(2) [19] $\gamma_t(J_{3,q}) = q + 1$ *and* $\gamma_t(J_{p,q}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ *for any integer $p \ge 4$.*

Consider the Jahangir graph $J_{7,3}$. Lemma 3.1(2) gives that $\gamma_t(J_{7,3}) = 11$, but $\{c, u_{1,1}, u_{4,1}, u_{5,1}, u_{1,2}, u_{4,2}, u_{5,2}, u_{1,3}, u_{4,3}, u_{5,3}\}$ is a total dominating set of $J_{7,3}$ with cardinality 10. It seems that, for $p \equiv 1, 2, 3 \pmod 4$, the value $\gamma_t(J_{p,q}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ does not hold for some value q. We correct this mistake in the next theorem.

Theorem 3.2. *Let $p \ge 4$ and $q \ge 3$ be integers. Then*

$$\gamma_t(J_{p,q}) = \begin{cases} \frac{pq}{2} & \text{if } p \equiv 0 \pmod{4}; \\ \frac{(p-1)q}{2} + 2 & \text{if } p \equiv 1 \pmod{4}; \\ \lceil \frac{(p-1)q}{2} \rceil + 1 & \text{if } p \equiv 2, 3 \pmod{4}. \end{cases}$$

Proof. If $p \equiv 0 \pmod 4$, then $\gamma_t(J_{p,q}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ seems to be correct and it can be simplified as $\gamma_t(J_{p,q}) = \frac{pq}{2}$.

Let $p = 4k+1$ for some $k \geq 1$ and $D = \{c, u_{1,1}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, 1 \leq j \leq q\}$. Then D is a total dominating set of $J_{p,q}$ with $|D| = \frac{(p-1)q}{2} + 2$. Moreover, $|D| \leq |S|$ for every total dominating set S with $c \in S$ and $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap S| = 1$. We claim that among all total dominating sets containing the vertex c, D is minimum. Suppose that D' is a total dominating set containing c with $|D'| < |D|$. Then $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'| \geq 2$. Without loss of generality, we may assume that $u_{1,1}, u_{1,j} \in D'$ for some $j \in \{2, 3, \ldots, q\}$. Note that D' contains at least $2k$ vertices from $\{u_{2,j-1}, u_{3,j-1}, \ldots, u_{p,j-1}\}$, and at least $2k$ vertices from $\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}$. Thus, $D'' = (D' \setminus \{u_{2,j-1}, \ldots, u_{p,j-1}, u_{1,j}, u_{2,j}, \ldots, u_{p,j}\}) \cup \{u_{i,j-1}, u_{i+1,j-1}, u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4\}$ is a total dominating set of $J_{p,q}$ with $|D''| < |D'|$ and $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| < |\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'|$. We can repeat this process until we obtain that $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| = 1$. We have a contradiction since $|D| \leq |D''| < |D'| < |D|$, so the claim holds. Next, we show that D is a $\gamma_t(J_{p,q})$-set. Assume that $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$. Then $c \notin \widehat{D}$ and $\widehat{D}$ is also a total dominating set of C_{pq}, so $|\widehat{D}| \geq \gamma_t(C_{pq}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ by Lemma 1.2. Then

$$|\widehat{D}| \geq \begin{cases} 8kl + 2l \geq 8kl + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l, l \geq 1; \\ 8kl + 2k + 2l + 1 > 8kl + 2k + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l+1, l \geq 1; \\ 8kl + 4k + 2l + 2 > 8kl + 4k + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l+2, l \geq 1; \\ 8kl + 6k + 2l + 2 \geq 8kl + 6k + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l+3, l \geq 0; \end{cases}$$

which contradicts with $|\widehat{D}| < |D|$. Therefore, $\gamma_t(J_{p,q}) = \frac{(p-1)q}{2} + 2$.

Let $p = 4k+2$ for some $k \geq 1$ and $D = \{c\} \cup \{u_{1,j} : j \equiv 1 \pmod 2\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4, j \equiv 1 \pmod 2\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, j \equiv 0 \pmod 2\}$. Then D is a total dominating set of $J_{p,q}$ with $|D| = \lceil \frac{(p-1)q}{2} \rceil + 1$. We show that among all total dominating sets containing the vertex c, D is minimum. Suppose that D' is total dominating set with $c \in D'$ and $|D'| < |D|$. If $u_{1,j} \in D'$ for all $j \in \{1, 2, \ldots, q\}$, then $|\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\} \cap D'| \geq 2k$, so $|D'| \geq 1 + q + 2kq = 1 + (1+2k)q = 1 + \lceil \frac{(4k+2)q}{2} \rceil \geq 1 + \lceil \frac{(4k+1)q}{2} \rceil = 1 + \lceil \frac{(p-1)q}{2} \rceil = |D|$, a contradiction. Thus, $u_{1,j} \notin D'$ for some $j \in \{1, 2, \ldots, q\}$. If $u_{1,j+1} \notin D'$, then $|\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\} \cap D'| \geq 2k+1$. Hence, $D'' = (D' \setminus \{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}) \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4\} \cup \{u_{1,j+1}\}$ is a total dominating set of $J_{p,q}$ with $|D''| \leq |D'|$, so we can assume that if $u_{1,j} \notin D'$, then $u_{1,j+1} \in D'$. This implies that $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'| \geq \lceil \frac{q}{2} \rceil$ and $|\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\} \cap D'| \geq 2k$ for each $j \in \{1, 2, \ldots, q\}$. Therefore, $|D'| \geq 1 + \lceil \frac{q}{2} \rceil + 2kq = 1 + \lceil \frac{q}{2} + \frac{(p-2)q}{2} \rceil = 1 + \lceil \frac{(p-1)q}{2} \rceil = |D|$, a contradiction. The claim follows. Next, we prove that D is a $\gamma_t(J_{p,q})$-set. If $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$, then $c \notin \widehat{D}$. Note that $|\widehat{D}| \geq \gamma_t(C_{pq}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$. Then

$$|\widehat{D}| \geq \begin{cases} 4kl + 2l > 4kl + l + 1 = \lceil \frac{(p-1)q}{2} \rceil + 1 = |D| & \text{if } q = 2l, l \geq 2; \\ 4kl + 2k + 2l + 2 > 4kl + 2k + l + 2 = \lceil \frac{(p-1)q}{2} \rceil + 1 = |D| & \text{if } q = 2l+1, l \geq 1; \end{cases}$$

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which is a contradiction. It follows that $\gamma_t(J_{p,q}) = \lceil \frac{(p-1)q}{2} \rceil + 1$.

Let $p = 4k+3$ for some $k \geq 1$ and $D = \{c\} \cup \{u_{1,j} : 1 \leq j \leq q\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4, 1 \leq j \leq q\}$. Then D is a total dominating set of $J_{p,q}$ with $|D| = \lceil \frac{(p-1)q}{2} \rceil + 1$. Note that among all total dominating sets containing c and $u_{1,1}, u_{1,2}, \ldots, u_{1,q}$, D is minimum. If D' is a total dominating set containing c and $|D'| < |D|$, then $u_{1,j} \notin D'$ for some $j \in \{1, 2, \ldots, q\}$. We can check that D' contains at least $2k+1$ vertices from $\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}$ if $u_{1,j+1} \in D'$, and at least $2k+2$ vertices from $\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}$ if $u_{1,j+1} \notin D'$. If $u_{1,j+1} \in D'$, let $D'' = (D' \setminus \{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}) \cup \{u_{1,j}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4\}$; otherwise, let $D'' = (D' \setminus \{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}) \cup \{u_{1,j}, u_{1,j+1}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4\}$. Now, we conclude that D'' is a total dominating set with $|D''| \leq |D'|$ and $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| > |\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'|$. We repeat this process until we obtain that $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| = q$. We see that $|D| \leq |D''| \leq |D'| < |D|$, a contradiction. This follows that among all total dominating sets containing c, D is minimum. We next prove that D is a $\gamma_t(J_{p,q})$-set. If $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$, then $c \notin \widehat{D}$. We have a contradiction because $|\widehat{D}| \geq \gamma_t(C_{pq}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor \geq \lfloor \frac{pq+2}{4} + \frac{pq+3}{4} \rfloor - 1 \geq \lfloor \frac{pq}{2} \rfloor \geq \lfloor \frac{pq}{2} - \frac{q}{2} + 1 \rfloor = \lfloor \frac{(p-1)q}{2} \rfloor + 1 = \lceil \frac{(p-1)q}{2} \rceil + 1 = |D|$. This completes the proof. □

It is straighforward to observe that, for all $p \geq 4$, the results in Theorem 3.2 give the smaller values than the results in Lemma 3.1(2) when $p \equiv 1 \pmod 4$ and $q \geq 5$, $p \equiv 2 \pmod 4$ and $q \geq 3$, and $p \equiv 3 \pmod 4$ and $q \geq 3$.

Next, we compute the paired domination number of $J_{p,q}$ for all $p \geq 2$ and $q \geq 3$.

Theorem 3.3. *Let $p \geq 2$ and $q \geq 3$ be integers. Then*

$$\gamma_{pr}(J_{p,q}) = \begin{cases} 2\lceil \frac{pq}{4} \rceil & \text{if } p \equiv 0, 2 \pmod 4; \\ \frac{(p-1)q}{2} + 2 & \text{if } p \equiv 1 \pmod 4; \\ 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. We first determine $\gamma_{pr}(J_{p,q})$ for $p \equiv 0, 1 \pmod 4$. By the fact that $\gamma_{pr}(J_{p,q}) \geq \gamma_t(J_{p,q})$ and Theorem 3.2, we immediately obtain the lower bound of $\gamma_{pr}(J_{p,q})$ for $p \equiv 0, 1 \pmod 4$. For the upper bounds, we observe that $\{u_{i,j}, u_{i+1,j} : i \equiv 1 \pmod 4, 1 \leq j \leq q\}$ is a paired dominating set of $J_{p,q}$ with cardinality $2\lceil \frac{pq}{4} \rceil$ if $p \equiv 0 \pmod 4$, and $\{c, u_{1,1}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, 1 \leq j \leq q\}$ is a paired dominating set of $J_{p,q}$ with cardinality $\frac{(p-1)q}{2} + 2$ if $p \equiv 1 \pmod 4$.

Let $p = 4k+2$ for some $k \geq 0$. Then $\{u_{1,j}, u_{2,j}, u_{5,j}, u_{6,j} : j \equiv 1 \pmod 2\} \cup \{u_{3,j}, u_{4,j} : j \equiv 0 \pmod 2\}$ is a paired dominating set of $J_{p,q}$ with cardinality $2\lceil \frac{pq}{4} \rceil$, so $\gamma_{pr}(J_{p,q}) \leq 2\lceil \frac{pq}{4} \rceil$. Next, we show that $\gamma_{pr}(J_{p,q}) \geq 2\lceil \frac{pq}{4} \rceil$. Let D be a $\gamma_{pr}(J_{p,q})$-set. If $c \notin D$, then D is also a paired dominating set of C_{pq}, and thus $|D| \geq \gamma_{pr}(C_{pq}) = 2\lceil \frac{pq}{4} \rceil$. Next, without loss of generality, we assume that the pair $\{c, u_{1,1}\} \subseteq D$. Then $|\{u_{3,1}, u_{4,1}, \ldots, u_{p,1}\} \cap D| \geq 2k$. To dominate $u_{2,2j}, \ldots, u_{p,2j}, u_{1,2j+1}, u_{2,2j+1}, \ldots, u_{p,2j+1}$ for each $1 \leq j \leq \lfloor \frac{q-2}{2} \rfloor$, D contains at least $4k+2$ vertices from them. If $q = 2l$ for some $l \geq 2$, then

$|\{u_{2,q}, u_{3,q}, \ldots, u_{p-1,q}\} \cap D| \geq 2k$, and hence $|D| \geq 2 + (2k) + \lfloor \frac{q-2}{2} \rfloor (4k+2) + (2k) = 4kl + 2l = 2\lceil \frac{pq}{4} \rceil$. If $q = 2l+1$ for some $l \geq 1$, then $|\{u_{2,q-1}, \ldots, u_{p,q-1}, u_{1,q}, u_{2,q}, \ldots, u_{p-1,q}\} \cap D| \geq 4k+2$, so $|D| \geq 2 + (2k) + \lfloor \frac{q-2}{2} \rfloor (4k+2) + (4k+2) = 4kl + 2l + 2k + 2 = 2\lceil \frac{pq}{4} \rceil$.

Let $p = 4k+3$ for some $k \geq 0$. Define the set $E = \{c\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, i < p, j \equiv 0 \pmod 3\} \cup \{u_{p,j} : j \equiv 0 \pmod 3\} \cup \{u_{1,j} : j \equiv 1 \pmod 3\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4, j \equiv 1 \pmod 3\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 1 \pmod 4, j \equiv 2 \pmod 3\}$. If $q \equiv 1, 2 \pmod 3$ (respectively, $q \equiv 0 \pmod 3$), then $D = E$ (respectively, $D = E \cup \{u_{p-1,q}\}$) is a paired dominating set of $J_{p,q}$ with $|D| = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. We claim that among all paired dominating sets containing c, D is minimum. Let D' be any paired dominating set of $J_{p,q}$ containing c. Without loss of generality, let c and $u_{1,1}$ be paired in D'. Then $|\{u_{3,1}, \ldots, u_{p,1}, u_{1,2}, u_{2,2}, \ldots, u_{p,2}\} \cap D'| \geq 4k+2$. To dominate $u_{2,3j}, \ldots, u_{p,3j}, u_{1,3j+1}, u_{2,3j+1}, \ldots, u_{p,3j+1}, u_{1,3j+2}, u_{2,3j+2}, \ldots, u_{p,3j+2}$ for each $1 \leq j \leq \lfloor \frac{q-3}{3} \rfloor$, D' contains at least $6k+4$ vertices from them. If $q = 3l$ for some $l \geq 1$, then $|\{u_{2,q}, u_{3,q}, \ldots, u_{p-1,q}\} \cap D'| \geq 2k+2$, and hence $|D'| \geq 2 + (4k+2) + \lfloor \frac{q-3}{3} \rfloor (6k+4) + (2k+2) = 6kl + 4l + 2 = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. If $q = 3l+1$ for some $l \geq 1$, then $|\{u_{2,q-1}, \ldots, u_{p,q-1}, u_{2,q}, \ldots, u_{p-1,q}\} \cap D'| \geq 4k+2$, so $|D'| \geq 2 + (4k+2) + \lfloor \frac{q-3}{3} \rfloor (6k+4) + (4k+2) = 6kl + 2k + 4l + 2 = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. If $q = 3l+2$ for some $l \geq 1$, then $|\{u_{2,q-2}, \ldots, u_{p,q-2}, u_{2,q-1}, \ldots, u_{p,q-1}, u_{2,q}, \ldots, u_{p-1,q}\} \cap D'| \geq 6k+4$, so $|D'| \geq 2 + (4k+2) + \lfloor \frac{q-3}{3} \rfloor (6k+4) + (6k+4) = 6kl + 4k + 4l + 4 = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. Now, the claim holds. We next prove that D is a $\gamma_{pr}(J_{p,q})$-set. If D'' is a paired dominating set with $|D''| < |D|$, then $c \notin D''$. Note that D'' is also a paired dominating set of C_{pq}, so $|D''| \geq \gamma_{pr}(C_{pq}) = 2\lceil \frac{pq}{4} \rceil \geq 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil = |D|$, a contradiction. We conclude that $\gamma_{pr}(J_{p,q}) = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. □

4. Closed Helm Graphs and Web Graphs

In this section, we determine the total and the paired domination numbers of a closed helm graph $CH_{p,q}$ and a web graph $W_{p,q}$ for some p and q. We also provide their upper bounds for other values of p and q.

Throughout this section, for each $i \in \{1, 2, \ldots, p\}$, let $C^i = \{u_{i,j} : 1 \leq j \leq q\}$ be the set of vertices in i th cycle of $CH_{p,q}$ and $W_{p,q}$.

4.1. Total and Paired Domination Numbers of Some Closed Helm Graphs.

Before we present the total and the paired domination numbers of some closed helm graphs, we recall some useful results.

The total and the paired domination numbers of $P_p \square C_q$ were determined by Hu *et al.* [10] for $p \geq 2$ and $q \in \{3, 4\}$ and by Eakawinrujee [4] for $p \in \{2, 3, 4\}$ and $q \geq 5$.

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Lemma 4.1. [10] *For any integer $p \geq 2$,*

$$\gamma_t(P_p\Box C_3) = \begin{cases} \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 0, 1 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 3, 4 \pmod 5; \end{cases}$$

and

$$\gamma_{pr}(P_p\Box C_3) = \begin{cases} \lceil \frac{4p}{5} \rceil + 2 & \text{if } p \equiv 0 \pmod 5; \\ \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 1, 3 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 4 \pmod 5. \end{cases}$$

Lemma 4.2. [10] *For any integer $p \geq 2$,*

$$\gamma_t(P_p\Box C_4) = \gamma_{pr}(P_p\Box C_4) = \begin{cases} p+1 & \text{if } p \text{ is odd}; \\ p+2 & \text{if } p \text{ is even}. \end{cases}$$

Lemma 4.3. [4] *For any integer $q \geq 5$, $\gamma_t(P_2\Box C_q) = \gamma_{pr}(P_2\Box C_q) = 2\lceil \frac{q}{3} \rceil$.*

Lemma 4.4. [4] *For any integer $q \geq 5$,*

$$\gamma_t(P_3\Box C_q) = q$$

and

$$\gamma_{pr}(P_3\Box C_q) = \begin{cases} q & \text{if } q \text{ is even}; \\ q+1 & \text{if } q \text{ is odd}. \end{cases}$$

Lemma 4.5. [4] *For any integer $q \geq 5$,*

$$\gamma_t(P_4\Box C_q) = \gamma_{pr}(P_4\Box C_q) = \begin{cases} \lfloor \frac{6q+8}{5} \rfloor - 1 & \text{if } q \equiv 0 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor & \text{if } q \equiv 1, 2, 4 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor + 1 & \text{if } q \equiv 3 \pmod 5. \end{cases}$$

Next, we compute the total and the paired domination numbers of $CH_{p,q}$, where the values of p and q are divided as follows: $p \geq 2$ and $q = 3$; $p \geq 2$ and $q = 4$; $p = 2$ and $q \geq 5$; $p = 3$ and $q \geq 5$; $p = 4$ and $q \geq 5$.

Theorem 4.6. *For any integer $p \geq 2$,*

$$\gamma_t(CH_{p,3}) = \begin{cases} \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 0, 1 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 3, 4 \pmod 5; \end{cases}$$

and

$$\gamma_{pr}(CH_{p,3}) = \begin{cases} \lceil \frac{4p}{5} \rceil + 2 & \text{if } p \equiv 0 \pmod 5; \\ \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 1, 3 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 4 \pmod 5. \end{cases}$$

Proof. Let D be a $\gamma_t(CH_{p,3})$-set. If $c \in D$, then, without loss of generality, $u_{1,1} \in D$ to dominate c. It follows that $u_{2,1} \notin D$; otherwise, $D \setminus \{c\}$ is a total dominating set of $CH_{p,3}$ with cardinality less than D. Hence, $D' = (D \setminus \{c\}) \cup \{u_{2,1}\}$ is a total dominating set of $CH_{p,3}$ with $|D'| = |D|$, so we can assume that $c \notin D$. Then D is also a total dominating set of $P_p \square C_3$, and thus $|D| \geq \gamma_t(P_p \square C_3)$. By Lemma 4.1, we get

$$\gamma_t(CH_{p,3}) = |D| \geq \gamma_t(P_p \square C_3) = \begin{cases} \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 0, 1 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 3, 4 \pmod 5). \end{cases}$$

Since $\gamma_{pr}(CH_{p,3}) \geq \gamma_t(CH_{p,3})$ and $\gamma_{pr}(CH_{p,3})$ is even, $\gamma_{pr}(CH_{p,3}) \geq \lceil \frac{4p}{5} \rceil$ if $p \equiv 2, 4 \pmod 5$, $\gamma_{pr}(CH_{p,3}) \geq \lceil \frac{4p}{5} \rceil + 1$ if $p \equiv 1, 3 \pmod 5$, and $\gamma_{pr}(CH_{p,3}) \geq \lceil \frac{4p}{5} \rceil + 2$ if $p \equiv 0 \pmod 5$.

To obtain the upper bounds of $\gamma_t(CH_{p,3})$ and $\gamma_{pr}(CH_{p,3})$, we let

$$D = \{u_{i,2} : i \equiv 1, 2 \pmod 5)\} \cup \{u_{i,1}, u_{i,3} : i \equiv 4 \pmod 5)\}.$$

If $p \equiv 2, 4 \pmod 5$, then D is a total dominating set of $CH_{p,3}$, so $\gamma_t(CH_{p,3}) \leq |D| = \lceil \frac{4p}{5} \rceil$. If $p \equiv 3 \pmod 5$, then $D \cup \{u_{p,2}\}$ is a total dominating set of $CH_{p,3}$, so $\gamma_t(CH_{p,3}) \leq |D| + 1 = \lceil \frac{4p}{5} \rceil$. If $p \equiv 0, 1 \pmod 5$, then $D \cup \{u_{p-1,2}\}$ is a total dominating set of $CH_{p,3}$, so $\gamma_t(CH_{p,3}) \leq |D| + 1 = \lceil \frac{4p}{5} \rceil + 1$. Now, the theorem holds for $\gamma_t(CH_{p,3})$.

If $p \equiv 2, 4 \pmod 5$, then D is a paired dominating set of $CH_{p,3}$, so $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil$. If $p \equiv 1 \pmod 5$, then $D \cup \{u_{p,3}\}$ is a paired dominating set of $CH_{p,3}$, so $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil + 1$. If $p \equiv 0, 3 \pmod 5$, then $D \cup \{u_{p,1}, u_{p,3}\}$ is a paired dominating set of $CH_{p,3}$, so $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil + 2$ if $p \equiv 0 \pmod 5$, and $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil + 1$ if $p \equiv 3 \pmod 5$. The theorem follows. $\square$

Theorem 4.7. *For any integer $p \geq 2$,*

$$\gamma_t(CH_{p,4}) = \gamma_{pr}(CH_{p,4}) = \begin{cases} p + 1 & \text{if } p \text{ is odd;} \\ p + 2 & \text{if } p \text{ is even.} \end{cases}$$

Proof. Let D be a $\gamma_t(CH_{p,4})$-set. If $c \in D$, then, without loss of generality, $u_{1,1} \in D$, and thus $u_{1,2} \notin D$. Therefore, $D' = (D \setminus \{c\}) \cup \{u_{1,2}\}$ is a total dominating set of $CH_{p,4}$ with $|D'| = |D|$, so we can assume that $c \notin D$. Then D is also a total dominating set of $P_p \square C_4$. By Lemma 4.2, we have

$$\gamma_t(CH_{p,4}) = |D| \geq \gamma_t(P_p \square C_4) = \begin{cases} p + 1 & \text{if } p \text{ is odd;} \\ p + 2 & \text{if } p \text{ is even.} \end{cases}$$

Note that $\gamma_{pr}(CH_{p,4}) \geq \gamma_t(CH_{p,4})$. To complete this theorem, we only determine the upper bound of $\gamma_{pr}(CH_{p,4})$. Let $D = \{u_{i,1}, u_{i,2} : i \equiv 1 \pmod 4)\} \cup \{u_{i,3}, u_{i,4} : i \equiv 3 \pmod 4)\}$. Then D is a paired dominating set of $CH_{p,4}$ with cardinality $p+1$ if p is odd, and $D \cup \{u_{p,1}, u_{p,2}\}$ is a paired dominating set of $CH_{p,4}$ with cardinality $p + 2$ if p is even. $\square$

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Theorem 4.8. *For any integer $q \geq 5$,*

$$\gamma_t(CH_{2,q}) = \begin{cases} 4 & \text{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil & \text{if } q \equiv 0, 1 \pmod{4}; \\ 2\lceil \frac{q+3}{4} \rceil - 1 & \text{if } q \equiv 2, 3 \pmod{4} \text{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(CH_{2,q}) = \begin{cases} 4 & \text{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil & \text{otherwise.} \end{cases}$$

Proof. Obviously, $\{u_{1,1}, u_{2,1}, u_{1,4}, u_{2,4}\}$ is a $\gamma_t(CH_{2,6})$-set and a $\gamma_{pr}(CH_{2,6})$-set, so let $q \neq 6$. If $q \equiv 0, 1 \pmod 4$, let $D = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 3 \pmod 4\}$; otherwise, let $D = \{c, u_{1,1}, u_{2,1}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 0 \pmod 4\}$. Then D is a total dominating set of $CH_{2,q}$ with $|D| = 2\lceil \frac{q+3}{4} \rceil$ if $q \equiv 0, 1 \pmod 4$, and $|D| = 2\lceil \frac{q+3}{4} \rceil - 1$ if $q \equiv 2, 3 \pmod 4$. We first show that among all total dominating sets containing c, D is the minimum. Suppose on the contrary that there is a total dominating set D' containing c such that $|D'| < |D|$. By the construction of D, we have $|D| \leq |S|$ for every total dominating set S with $c \in S$ and $|S \cap C^1| = 1$. Without loss of generality, we assume that D' contains at least two vertices $u_{1,1}$ and $u_{1,l}$ of C^1 for some $l \neq 1$. Hence, $D'' = (D' \setminus \{u_{1,l}\}) \cup \{u_{2,l+1}\}$ is a total dominating set with $|D''| \leq |D'|$ and $|D'' \cap C^1| < |D' \cap C^1|$. We can repeat this process until we obtain a total dominating set D'' with $|D'' \cap C^1| = 1$. Therefore, $|D| \leq |D''| \leq |D'| < |D|$, a contradiction.

Next, we claim that D is a $\gamma_t(CH_{2,q})$-set. Suppose that $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$. Then $c \notin \widehat{D}$, and thus $\widehat{D}$ is a total dominating set of $P_2 \square C_q$. By Lemma 4.3, $|\widehat{D}| \geq \gamma_t(P_2 \square C_q) = 2\lceil \frac{q}{3} \rceil$. Then

$$|\widehat{D}| \geq \begin{cases} 2\lceil \frac{4k}{3} \rceil = 2(k + \lceil \frac{k}{3} \rceil) \geq 2(k+1) = 2\lceil \frac{q+3}{4} \rceil = |D| & \text{if } q = 4k, k \geq 2; \\ 2\lceil \frac{4k+1}{3} \rceil = 2(k + \lceil \frac{k+1}{3} \rceil) \geq 2(k+1) = 2\lceil \frac{q+3}{4} \rceil = |D| & \text{if } q = 4k+1, k \geq 1; \\ 2\lceil \frac{4k+2}{3} \rceil = 2(k + \lceil \frac{k+2}{3} \rceil) > 2(k+2) - 1 = 2\lceil \frac{q+3}{4} \rceil - 1 = |D| & \text{if } q = 4k+2, k \geq 2; \\ 2\lceil \frac{4k+3}{3} \rceil = 2(k + \lceil \frac{k+3}{3} \rceil) > 2(k+2) - 1 = 2\lceil \frac{q+3}{4} \rceil - 1 = |D| & \text{if } q = 4k+3, k \geq 1; \end{cases}$$

contradicting the assumption $|\widehat{D}| < |D|$, so our claim holds.

Since $\gamma_{pr}(CH_{2,q}) \geq \gamma_t(CH_{2,q})$ and $\gamma_{pr}(CH_{2,q})$ is even, we get $\gamma_{pr}(CH_{2,q}) \geq 2\lceil \frac{q+3}{4} \rceil$. If $q \equiv 0, 1 \pmod 4$, let $D = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 3 \pmod 4\}$; otherwise, let $D = \{c, u_{1,1}, u_{2,1}, u_{2,2}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 0 \pmod 4\}$. Then D is a paired dominating set of $CH_{2,q}$ with cardinality $2\lceil \frac{q+3}{4} \rceil$, and thus $\gamma_{pr}(CH_{2,q}) = 2\lceil \frac{q+3}{4} \rceil$. □

Theorem 4.9. *For any integer $q \geq 5$,*

$$\gamma_t(CH_{3,q}) = \begin{cases} 2\lceil \frac{q+2}{3} \rceil & \text{if } q \equiv 0,2,3 \pmod{6}; \\ 2\lceil \frac{q+3}{3} \rceil - 1 & \text{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(CH_{3,q}) = 2\lceil \frac{q+3}{3} \rceil.$$

Proof. Let $E_1 = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{3,j} : j \equiv 0 \pmod{3}\}$, $E_2 = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{3,j} : j \equiv 2 \pmod{3}\}$, and $E_3 = \{c, u_{1,1}, u_{2,1}\} \cup \{u_{3,j}, u_{3,j+1} : j \equiv 3 \pmod{6}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 0 \pmod{6}\}$. If $q \equiv 0, 3 \pmod{6}$ (respectively, $q \equiv 2 \pmod{6}$ and $q \equiv 1, 4, 5 \pmod{6}$), then $D = E_1$ (respectively, $D = E_2$ and $D = E_3$) is a total dominating set of $CH_{3,q}$ with cardinality $2\lceil \frac{q+2}{3} \rceil$ (respectively, $2\lceil \frac{q+2}{3} \rceil$ and $|D| = 2\lceil \frac{q+3}{3} \rceil - 1$). We show that among all total dominating sets containing the vertex c, D is the minimum. Assume that D' is a total dominating set with $c \in D'$ and $|D'| < |D|$. Let $S = \{v : v \notin N(c)\}$. Then the induced subgraph $CH_{3,q}[S]$ contains $P_2 \Box C_q$ and the vertex c. Thus, D' contains at least $2\lceil \frac{q}{3} \rceil$ vertices to dominate $P_2 \Box C_q$ by Lemma 4.3 and one vertex of C^1 to dominate c. Hence, $|D'| \geq 2\lceil \frac{q}{3} \rceil + 2 = 2\lceil \frac{q+3}{3} \rceil \geq |D|$, a contradiction.

We next show that D is a $\gamma_t(CH_{3,q})$-set. Suppose that $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$. Then $c \notin \widehat{D}$, and hence $\widehat{D}$ is also a total dominating set of $P_3 \Box C_q$. By Lemma 4.4, $|\widehat{D}| \geq \gamma_t(P_3 \Box C_q) = q$. Then

$$|\widehat{D}| \geq \begin{cases} 6k \geq 4k+2 = 2(2k+1) = 2\lceil \frac{q+2}{3} \rceil = |D| & \text{if } q = 6k, k \geq 1; \\ 6k+1 \geq 4k+3 = 2(2k+2) - 1 = 2\lceil \frac{q+3}{3} \rceil - 1 = |D| & \text{if } q = 6k+1, k \geq 1; \\ 6k+2 \geq 4k+4 = 2(2k+2) = 2\lceil \frac{q+2}{3} \rceil = |D| & \text{if } q = 6k+2, k \geq 1; \\ 6k+3 > 4k+4 = 2(2k+2) = 2\lceil \frac{q+2}{3} \rceil = |D| & \text{if } q = 6k+3, k \geq 1; \\ 6k+4 > 4k+5 = 2(2k+3) - 1 = 2\lceil \frac{q+3}{3} \rceil - 1 = |D| & \text{if } q = 6k+4, k \geq 1; \\ 6k+5 \geq 4k+5 = 2(2k+3) - 1 = 2\lceil \frac{q+3}{3} \rceil - 1 = |D| & \text{if } q = 6k+5, k \geq 0; \end{cases}$$

which contradicts with our assumption $|\widehat{D}| < |D|$.

Note that $\gamma_{pr}(CH_{3,q}) \geq \gamma_t(CH_{3,q})$. If $q \equiv 0, 2, 3 \pmod{6}$, then $\gamma_{pr}(CH_{3,q}) \geq 2\lceil \frac{q+2}{3} \rceil = 2\lceil \frac{q+3}{3} \rceil$. Since $\gamma_{pr}(CH_{3,q})$ is even, $\gamma_{pr}(CH_{3,q}) \geq 2\lceil \frac{q+3}{3} \rceil$ for $q \equiv 1, 4, 5 \pmod{6}$. Consider the sets E_1, E_2, and E_3 defined above. We observe that E_1 (respectively, E_2) is a paired dominating set with cardinality $2\lceil \frac{q+2}{3} \rceil = 2\lceil \frac{q+3}{3} \rceil$ if $q \equiv 0, 3 \pmod{6}$ (respectively, $q \equiv 2 \pmod{6}$). If $q \equiv 1, 4, 5 \pmod{6}$, then $E_3 \cup \{u_{3,1}\}$ is a paired dominating set with cardinality $2\lceil \frac{q+3}{3} \rceil$. This completes the proof. □

Theorem 4.10. *For any integer $q \geq 5$,*

$$\gamma_t(CH_{4,q}) = \begin{cases} 6 & \text{if } q = 5; \\ q+2 & \text{if } q \geq 6; \end{cases}$$

and

$$\gamma_{pr}(CH_{4,q}) = \begin{cases} 6 & \text{if } q = 5; \\ q+2 & \text{if } q \text{ is even}; \\ q+3 & \text{if } q \text{ is odd and } q \neq 5. \end{cases}$$

Proof. If $q = 5$, then it is clear that $\{u_{2,1}, u_{3,1}, u_{1,3}, u_{1,4}, u_{4,3}, u_{4,4}\}$ is both a $\gamma_t(CH_{4,q})$-set and a $\gamma_{pr}(CH_{4,q})$-set, so we let $q \geq 6$. Let $D = \{c, u_{1,q}\} \cup \{u_{3,j} : 1 \leq j \leq q\}$. Then D is a total dominating set of $CH_{4,q}$ with $|D| = q+2$. Let D' be a total dominating set containing c. Then the induced subgraph $CH_{4,q}[\{v : v \notin N(c)\}]$ consists of $P_3 \Box C_q$ and the vertex c. By Theorem 4.4, D' contains at least q vertices to dominate $P_3 \Box C_q$. Also, it contains one vertex of C^1 to dominate c. Hence, $|D'| \geq q+2$. We conclude that among all total dominating sets of $CH_{4,q}$ containing c, D is minimum.

Next, we show that D is a $\gamma_t(CH_{4,q})$-set. If $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$, then $c \notin \widehat{D}$, so $\widehat{D}$ is also a total dominating set of $P_4 \Box C_q$. Lemma 4.5 gives that

$$|\widehat{D}| \geq \gamma_t(P_4 \Box C_q) = \begin{cases} \lfloor \frac{6q+8}{5} \rfloor - 1 & \text{if } q \equiv 0 \pmod{5}; \\ \lfloor \frac{6q+8}{5} \rfloor & \text{if } q \equiv 1,2,4 \pmod{5}; \\ \lfloor \frac{6q+8}{5} \rfloor + 1 & \text{if } q \equiv 3 \pmod{5}. \end{cases}$$

It is easy to check that $|\widehat{D}| \geq |D|$ for all $q \geq 6$, a contradiction. Therefore, $\gamma_t(CH_{4,q}) = q+2$.

Since $\gamma_{pr}(CH_{4,q}) \geq \gamma_t(CH_{4,q})$ and $\gamma_{pr}(CH_{4,q})$ is even, $\gamma_{pr}(CH_{4,q}) \geq q+2$ if q is even, and $\gamma_{pr}(CH_{4,q}) \geq q+3$ if q is odd. Note that D (respectively, $D \cup \{u_{2,1}\}$) defined above is a paired dominating set if q is even (respectively, odd). The theorem follows. □

4.2. **Upper Bounds of $\gamma_t(CH_{p,q})$ and $\gamma_{pr}(CH_{p,q})$.** Recently, we determined the values of $\gamma_t(CH_{p,q})$ and $\gamma_{pr}(CH_{p,q})$ for some values of p and q. We next present their upper bounds for $p, q \geq 5$. Before we can achieve these bounds, we need the following results.

Lemma 4.11. [10] *For any integer $p \geq 4$,*

$$\gamma_t(P_p \Box C_5) \leq \begin{cases} \lceil \frac{9p}{7} \rceil & \text{if } p \equiv 4 \pmod{7}; \\ \lceil \frac{9p}{7} \rceil + 1 & \text{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(P_p \Box C_5) \leq \begin{cases} \lceil \frac{4p}{3} \rceil & \text{if } p \equiv 1 \pmod{3}; \\ \lceil \frac{4p}{3} \rceil + 1 & \text{if } p \equiv 2 \pmod{3}; \\ \lceil \frac{4p}{3} \rceil + 2 & \text{if } p \equiv 0 \pmod{3}. \end{cases}$$

Lemma 4.12. [4] *For any integer $q \geq 6$,*

$$\gamma_t(P_5 \Box C_q) \leq \begin{cases} 10 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor & \text{if } q \equiv 0 \pmod 4; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \equiv 1,2,3 \pmod 4 \text{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(P_5 \Box C_q) \leq \begin{cases} 10 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor & \text{if } q \equiv 0 \pmod 4; \\ \lfloor \frac{3q}{2} \rfloor + 3 & \text{if } q \equiv 1,2 \pmod 4 \text{ and } q \neq 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

Lemma 4.13. [4] *For any integer $q \geq 6$,*

$$\gamma_t(P_6 \Box C_q) \leq \gamma_{pr}(P_6 \Box C_q) \leq \begin{cases} \lfloor \frac{12q}{7} \rfloor & \text{if } q \equiv 0 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 3 & \text{if } q \equiv 1,2 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 1 & \text{if } q \equiv 3 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 2 & \text{if } q \equiv 4,6 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 4 & \text{if } q \equiv 5 \pmod 7). \end{cases}$$

The following lemma shows the upper bounds of $\gamma_t(P_p \Box C_q)$ and $\gamma_{pr}(P_p \Box C_q)$ for arbitrary $p \geq 7$ and $q \geq 6$.

Lemma 4.14. [4] *For any integers $p \geq 7$ and $q \geq 6$,*

$$\gamma_t(P_p \Box C_q) \leq \gamma_{pr}(P_p \Box C_q) \leq 2\lceil \frac{p+1}{2} \rceil \lceil \frac{q}{4} \rceil.$$

For some special values of p and q, Lemmas 4.15 and 4.16 provide better bounds than ones in Lemma 4.14.

Lemma 4.15. [4] *If $7 \leq p \equiv 1 \pmod 2$ and $9 \leq q \equiv 1 \pmod 4$, then $\gamma_t(P_p \Box C_q) \leq \frac{(p+1)(q+1)}{4}$.*

Lemma 4.16. [4] *If $8 \leq p \equiv 0 \pmod 2$ and $q \geq 6$, then*

$$\gamma_t(P_p \Box C_q) \leq \begin{cases} \frac{(p+2)(q+1)}{4} - 2 & \text{if } q \equiv 1,3 \pmod 4; \\ \frac{(p+2)(q+2)}{4} - 4 & \text{if } q \equiv 2 \pmod 4; \end{cases}$$

and

$$\gamma_{pr}(P_p \Box C_q) \leq \begin{cases} \frac{(p+2)(q+3)}{4} - 4 & \text{if } q \equiv 1 \pmod 4; \\ \frac{(p+2)(q+2)}{4} - 4 & \text{if } q \equiv 2 \pmod 4; \\ \frac{(p+2)(q+1)}{4} - 2 & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

Let H be the graph obtained from $CH_{p,q}$ by deleting the vertices c and $u_{1,j}$ for all $j \in \{1, 2, \ldots, q\}$. Note that the union of $\{c, u_{1,1}\}$ and a $\gamma_t(H)$-set (respectively, a $\gamma_{pr}(H)$-set) is a total dominating set (respectively, a paired dominating set) of $CH_{p,q}$. Then we have the following lemma where we replace H by $P_{p-1}\square C_q$.

Lemma 4.17. *If $p \geq 2$ and $q \geq 3$ are integers, then $\gamma_t(CH_{p,q}) \leq \gamma_t(P_{p-1}\square C_q) + 2$ and $\gamma_{pr}(CH_{p,q}) \leq \gamma_{pr}(P_{p-1}\square C_q) + 2$.*

By using Lemma 4.17 together with Lemmas 4.11, 4.5, 4.12, 4.13, 4.14, 4.15, 4.16, we can easily get Theorems 4.18, 4.19, 4.20, 4.21, 4.22, 4.23, 4.24, respectively.

Theorem 4.18. *For any integer $p \geq 5$,*

$$\gamma_t(CH_{p,5}) \leq \begin{cases} \lceil \frac{9(p-1)}{7} \rceil + 2 & \text{if } p \equiv 5 \pmod{7}; \\ \lceil \frac{9(p-1)}{7} \rceil + 3 & \text{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(CH_{p,5}) \leq \begin{cases} \lceil \frac{4(p-1)}{3} \rceil + 3 & \text{if } p \equiv 0 \pmod{3}; \\ \lceil \frac{4(p-1)}{3} \rceil + 4 & \text{if } p \equiv 1 \pmod{3}; \\ \lceil \frac{4(p-1)}{3} \rceil + 2 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Theorem 4.19. *For any integer $q \geq 6$,*

$$\gamma_t(CH_{5,q}) \leq \gamma_{pr}(CH_{5,q}) \leq \begin{cases} \lfloor \frac{6q+8}{5} \rfloor + 1 & \text{if } q \equiv 0 \pmod{5}; \\ \lfloor \frac{6q+8}{5} \rfloor + 2 & \text{if } q \equiv 1,2,4 \pmod{5}; \\ \lfloor \frac{6q+8}{5} \rfloor + 3 & \text{if } q \equiv 3 \pmod{5}. \end{cases}$$

Theorem 4.20. *For any integer $q \geq 6$,*

$$\gamma_t(CH_{6,q}) \leq \begin{cases} 12 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \equiv 0 \pmod{4}; \\ \lfloor \frac{3q}{2} \rfloor + 4 & \text{if } q \equiv 1,2,3 \pmod{4} \text{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(CH_{6,q}) \leq \begin{cases} 12 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \equiv 0 \pmod{4}; \\ \lfloor \frac{3q}{2} \rfloor + 5 & \text{if } q \equiv 1,2 \pmod{4} \text{ and } q \neq 6; \\ \lfloor \frac{3q}{2} \rfloor + 4 & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Theorem 4.21. *For any integer $q \geq 6$,*

$$\gamma_t(CH_{7,q}) \leq \gamma_{pr}(CH_{7,q}) \leq \begin{cases} \lfloor \frac{12q}{7} \rfloor + 2 & \text{if } q \equiv 0 \pmod{7}; \\ \lfloor \frac{12q}{7} \rfloor + 5 & \text{if } q \equiv 1, 2 \pmod{7}; \\ \lfloor \frac{12q}{7} \rfloor + 3 & \text{if } q \equiv 3 \pmod{7}; \\ \lfloor \frac{12q}{7} \rfloor + 4 & \text{if } q \equiv 4, 6 \pmod{7}; \\ \lfloor \frac{12q}{7} \rfloor + 6 & \text{if } q \equiv 5 \pmod{7}. \end{cases}$$

Theorem 4.22. *For any integers $p \geq 8$ and $q \geq 6$, $\gamma_t(CH_{p,q}) \leq \gamma_{pr}(CH_{p,q}) \leq 2\lceil \frac{p}{2} \rceil \lceil \frac{q}{4} \rceil + 2$.*

The next two theorems give better bounds than ones in Theorem 4.22 for some special values of p and q.

Theorem 4.23. *If $8 \leq p \equiv 0 \pmod{2}$ and $9 \leq q \equiv 1 \pmod{4}$, then $\gamma_t(CH_{p,q}) \leq \frac{p(q+1)}{4} + 2$.*

Theorem 4.24. *If $9 \leq p \equiv 1 \pmod{2}$ and $q \geq 6$, then*

$$\gamma_t(CH_{p,q}) \leq \begin{cases} \frac{(p+1)(q+1)}{4} & \text{if } q \equiv 1, 3 \pmod{4}; \\ \frac{(p+1)(q+2)}{4} - 2 & \text{if } q \equiv 2 \pmod{4}; \end{cases}$$

and

$$\gamma_{pr}(CH_{p,q}) \leq \begin{cases} \frac{(p+1)(q+3)}{4} - 2 & \text{if } q \equiv 1 \pmod{4}; \\ \frac{(p+1)(q+2)}{4} - 2 & \text{if } q \equiv 2 \pmod{4}; \\ \frac{(p+1)(q+1)}{4} & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

4.3. **Total and Paired Domination Numbers of Some Web Graphs.** We present the total and the paired domination numbers of a web graph $W_{p,q}$, where the values of p and q are divided as follows: $p \in \{2, 3\}$ and $q \geq 3$; $p \geq 4$ and $q \in \{3, 4\}$; $p \in \{4, 5, 6\}$ and $q \geq 5$.

Theorem 4.25. *For any integer $q \geq 3$,*

$$\gamma_t(W_{2,q}) = q + 1 \text{ and } \gamma_{pr}(W_{2,q}) = \begin{cases} q + 1 & \text{if } q \text{ is odd}; \\ q + 2 & \text{if } q \text{ is even}. \end{cases}$$

Proof. Note that each $\gamma_t(W_{2,q})$-set must contain the q support vertices and one vertex of C^1 to dominate c, so $\gamma_t(W_{2,q}) \geq q+1$. Since $\gamma_{pr}(W_{2,q})$ is even, $\gamma_{pr}(W_{2,q}) \geq q+1$ if q is odd, and $\gamma_{pr}(W_{2,q}) \geq q+2$ if q is even. Note that $\{u_{1,1}\} \cup C^2$ is a total dominating set with cardinality $q+1$, and it is also a paired dominating set if q is odd. If q is even, then $\{c, u_{1,1}\} \cup C^2$ is a paired dominating set with cardinality $q + 2$. □

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Theorem 4.26. *For any integer $q \geq 3$,*

$$\gamma_t(W_{3,q}) = q+2 \text{ and } \gamma_{pr}(W_{3,q}) = \begin{cases} q+2 & \text{if } q \text{ is even;} \\ q+3 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Note that all q support vertices of C^3 are in every $\gamma_t(W_{3,q})$-set. Then the induced subgraph $W_{3,q}[\{v : v \notin N(C^3)\}] \cong W_q$. Since $\gamma_t(W_q) = 2$, $\gamma_t(W_{3,q}) \geq q+2$. We also get that $\gamma_{pr}(W_{3,q}) \geq q+2$ if q is even, and $\gamma_{pr}(W_{3,q}) \geq q+3$ if q is odd. We note that $D = \{c, u_{1,1}\} \cup C^3$ is total dominating set. If q is even, then D is also a paired dominating set; otherwise, $D \cup \{u_{2,1}\}$ is a paired dominating set. □

Before we prove the other results, we need the following lemma which shows that the total (paired) domination numbers of $W_{p,q}$ can be calculated from the total (paired) domination numbers of $CH_{p-2,q}$.

Lemma 4.27. *For any integers $p \geq 4$ and $q \geq 3$,*

$$\gamma_t(W_{p,q}) = \gamma_t(CH_{p-2,q}) + q$$

and

$$\gamma_{pr}(W_{p,q}) = \begin{cases} \gamma_{pr}(CH_{p-2,q}) + q & \text{if } q \text{ is even;} \\ \gamma_{pr}(CH_{p-2,q}) + q + 1 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Note that the union of a $\gamma_t(CH_{p-2,q})$-set and C^p is a total dominating set of $W_{p,q}$. Futhermore, if $|C^p| = q$ is even, then the union of a $\gamma_{pr}(CH_{p-2,q})$-set and C^p is a paired dominating set of $W_{p,q}$; otherwise, the union of a $\gamma_{pr}(CH_{p-2,q})$-set and $C^p \cup \{v_1\}$ is a paired dominating set of $W_{p,q}$. Then we get the upper bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$.

We know that all q support vertices of C^p are in every $\gamma_t(W_{p,q})$-set and every $\gamma_{pr}(W_{p,q})$-set. Then the induced subgraph $W_{p,q}[\{v : v \notin N(C^p)\}]$ is $CH_{p-2,q}$. Therefore, $\gamma_t(W_{p,q}) \geq \gamma_t(CH_{p-2,q}) + q$. Similarly, $\gamma_{pr}(W_{p,q}) \geq \gamma_{pr}(CH_{p-2,q}) + q$ if q is even, and $\gamma_{pr}(W_{p,q}) \geq \gamma_{pr}(CH_{p-2,q}) + q + 1$ if q is odd. □

By Lemma 4.27 together with Theorems 4.6 - 4.10, we obtain Theorems 4.28 - 4.32, respectively.

Theorem 4.28. *For any integer $p \geq 4$,*

$$\gamma_t(W_{p,3}) = \begin{cases} \lceil \frac{4(p-2)}{5} \rceil + 3 & \text{if } p \equiv 0,1,4 \pmod 5; \\ \lceil \frac{4(p-2)}{5} \rceil + 4 & \text{if } p \equiv 2,3 \pmod 5; \end{cases}$$

and

$$\gamma_{pr}(W_{p,3}) = \begin{cases} \lceil \frac{4(p-2)}{5} \rceil + 5 & \text{if } p \equiv 0,3 \pmod 5; \\ \lceil \frac{4(p-2)}{5} \rceil + 4 & \text{if } p \equiv 1,4 \pmod 5; \\ \lceil \frac{4(p-2)}{5} \rceil + 6 & \text{if } p \equiv 2 \pmod 5. \end{cases}$$

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Theorem 4.29. *For any integer $p \geq 4$,*

$$\gamma_t(W_{p,4}) = \gamma_{pr}(W_{p,4}) = \begin{cases} p+3 & \text{if } p \text{ is odd;} \\ p+4 & \text{if } p \text{ is even.} \end{cases}$$

Theorem 4.30. *For any integer $q \geq 5$,*

$$\gamma_t(W_{4,q}) = \begin{cases} 10 & \text{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil + q & \text{if } q \equiv 0,1 \pmod{4}; \\ 2\lceil \frac{q+3}{4} \rceil + q - 1 & \text{if } q \equiv 2,3 \pmod{4} \text{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(W_{4,q}) = \begin{cases} 10 & \text{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil + q & \text{if } q \text{ is even and } q \neq 6; \\ 2\lceil \frac{q+3}{4} \rceil + q + 1 & \text{if } q \text{ is odd.} \end{cases}$$

Theorem 4.31. *For any integer $q \geq 5$,*

$$\gamma_t(W_{5,q}) = \begin{cases} 2\lceil \frac{q+2}{3} \rceil + q & \text{if } q \equiv 0,2,3 \pmod{6}; \\ 2\lceil \frac{q+3}{3} \rceil + q - 1 & \text{otherwise;} \end{cases}$$

and

$$\gamma_{pr}(W_{5,q}) = \begin{cases} 2\lceil \frac{q+3}{3} \rceil + q & \text{if } q \text{ is even;} \\ 2\lceil \frac{q+3}{3} \rceil + q + 1 & \text{if } q \text{ is odd.} \end{cases}$$

Theorem 4.32. *For any integer $q \geq 5$,*

$$\gamma_t(W_{6,q}) = \begin{cases} 11 & \text{if } q = 5; \\ 2q+2 & \text{if } q \geq 6; \end{cases}$$

and

$$\gamma_{pr}(W_{6,q}) = \begin{cases} 12 & \text{if } q = 5; \\ 2q+2 & \text{if } q \text{ is even;} \\ 2q+4 & \text{if } q \text{ is odd and } q \neq 5. \end{cases}$$

4.4. **Upper Bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$.** Previously, we showed the exact values of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ for some small values of p and q. We now present the upper bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ for $p \geq 7$ and $q \geq 5$. By using Lemma 4.27 together with Theorems 4.18 - 4.24, we easily get Theorems 4.33 - 4.39, respectively.

Theorem 4.33. *For any integer $p \geq 7$,*

$$\gamma_t(W_{p,5}) \leq \begin{cases} \lceil \frac{9(p-3)}{7} \rceil + 7 & \textit{if } p \equiv 0 \pmod{7}; \\ \lceil \frac{9(p-3)}{7} \rceil + 8 & \textit{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(W_{p,5}) \leq \begin{cases} \lceil \frac{4(p-3)}{3} \rceil + 10 & \textit{if } p \equiv 0 \pmod{3}; \\ \lceil \frac{4(p-3)}{3} \rceil + 8 & \textit{if } p \equiv 1 \pmod{3}; \\ \lceil \frac{4(p-3)}{3} \rceil + 9 & \textit{if } p \equiv 2 \pmod{3}. \end{cases}$$

Theorem 4.34. *For any integer $q \geq 6$,*

$$\gamma_t(W_{7,q}) \leq \begin{cases} \lfloor \frac{11q+8}{5} \rfloor + 1 & \textit{if } q \equiv 0 \pmod{5}; \\ \lfloor \frac{11q+8}{5} \rfloor + 2 & \textit{if } q \equiv 1,2,4 \pmod{5}; \\ \lfloor \frac{11q+8}{5} \rfloor + 3 & \textit{if } q \equiv 3 \pmod{5}; \end{cases}$$

and

$$\gamma_{pr}(W_{7,q}) \leq \begin{cases} \lfloor \frac{11q+8}{5} \rfloor + 1 & \textit{if } q \equiv 0 \pmod{10}; \\ \lfloor \frac{11q+8}{5} \rfloor + 3 & \textit{if } q \equiv 1,7,8,9 \pmod{10}; \\ \lfloor \frac{11q+8}{5} \rfloor + 2 & \textit{if } q \equiv 2,4,5,6 \pmod{10}; \\ \lfloor \frac{11q+8}{5} \rfloor + 4 & \textit{if } q \equiv 3 \pmod{10}. \end{cases}$$

Theorem 4.35. *For any integer $q \geq 6$,*

$$\gamma_t(W_{8,q}) \leq \begin{cases} 18 & \textit{if } q = 6; \\ \lfloor \frac{5q}{2} \rfloor + 2 & \textit{if } q \equiv 0 \pmod{4}; \\ \lfloor \frac{5q}{2} \rfloor + 4 & \textit{if } q \equiv 1,2,3 \pmod{4} \textit{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(W_{8,q}) \leq \begin{cases} 18 & \textit{if } q = 6; \\ \lfloor \frac{5q}{2} \rfloor + 2 & \textit{if } q \equiv 0 \pmod{4}; \\ \lfloor \frac{5q}{2} \rfloor + 6 & \textit{if } q \equiv 1 \pmod{4}; \\ \lfloor \frac{5q}{2} \rfloor + 5 & \textit{if } q \equiv 2,3 \pmod{4} \textit{ and } q \neq 6. \end{cases}$$

Theorem 4.36. *For any integer $q \geq 6$,*

$$\gamma_t(W_{9,q}) \leq \begin{cases} \lfloor \frac{19q}{7} \rfloor + 2 & \text{if } q \equiv 0 \pmod{7}; \\ \lfloor \frac{19q}{7} \rfloor + 5 & \text{if } q \equiv 1, 2 \pmod{7}; \\ \lfloor \frac{19q}{7} \rfloor + 3 & \text{if } q \equiv 3 \pmod{7}; \\ \lfloor \frac{19q}{7} \rfloor + 4 & \text{if } q \equiv 4, 6 \pmod{7}; \\ \lfloor \frac{19q}{7} \rfloor + 6 & \text{if } q \equiv 5 \pmod{7}; \end{cases}$$

and

$$\gamma_{pr}(W_{9,q}) \leq \begin{cases} \lfloor \frac{19q}{7} \rfloor + 2 & \text{if } q \equiv 0 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 6 & \text{if } q \equiv 1, 9, 12 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 5 & \text{if } q \equiv 2, 8, 11, 13 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 4 & \text{if } q \equiv 3, 4, 6 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 7 & \text{if } q \equiv 5 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 3 & \text{if } q \equiv 7, 10 \pmod{14}. \end{cases}$$

Theorem 4.37. *For any integers $p \geq 10$ and $q \geq 6$,*

$$\gamma_t(W_{p,q}) \leq 2\lceil \frac{p-2}{2} \rceil \lceil \frac{q}{4} \rceil + q + 2$$

and

$$\gamma_{pr}(W_{p,q}) \leq \begin{cases} 2\lceil \frac{p-2}{2} \rceil \lceil \frac{q}{4} \rceil + q + 2 & \text{if } q \text{ is even}; \\ 2\lceil \frac{p-2}{2} \rceil \lceil \frac{q}{4} \rceil + q + 3 & \text{if } q \text{ is odd}. \end{cases}$$

The next two theorems provide some better bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ than those in Theorem 4.37 for some special values of p and q.

Theorem 4.38. *If $10 \leq p \equiv 0 \pmod{2}$ and $9 \leq q \equiv 1 \pmod{4}$, then $\gamma_t(W_{p,q}) \leq \frac{(p+2)(q+1)}{4} + 1$.*

Theorem 4.39. *If $11 \leq p \equiv 1 \pmod{2}$ and $q \geq 6$, then*

$$\gamma_t(W_{p,q}) \leq \begin{cases} \frac{(p+3)(q+1)}{4} - 1 & \text{if } q \equiv 1, 3 \pmod{4}; \\ \frac{(p+3)(q+2)}{4} - 4 & \text{if } q \equiv 2 \pmod{4}; \end{cases}$$

and

$$\gamma_{pr}(W_{p,q}) \leq \begin{cases} \frac{(p+3)(q+3)}{4} - 4 & \text{if } q \equiv 1 \pmod{4}; \\ \frac{(p+3)(q+2)}{4} - 4 & \text{if } q \equiv 2 \pmod{4}; \\ \frac{(p+3)(q+1)}{4} & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

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5. **Open Problems**

We have obtained upper bounds of $\gamma_t(CH_{p,q})$ and $\gamma_{pr}(CH_{p,q})$ for $p, q \geq 5$ in Subsection 4.2 and the upper bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ for $p \geq 7$ and $q \geq 5$ in Subsection 4.4. To approach the exact values for these parameters, we provide the following problems.

- Determine lower bounds of $\gamma_t(CH_{p,q})$ and $\gamma_{pr}(CH_{p,q})$ for $p, q \geq 5$.
- Determine lower bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ for $p \geq 7$ and $q \geq 5$.

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