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ON THE NILPOTENT GRAPH OF A FINITE GROUP

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ABSTRACT. The nilpotent graph of a finite group G, denoted by $\Gamma_{\mathfrak{N}}(G)$, is a simple graph whose vertex set is $G - \text{nil}(G)$, where $\text{nil}(G) = \{g \in G : \langle g, h \rangle \text{ is nilpotent for all } h \in G\}$, and two distinct vertices are related if they generate a nilpotent subgroup of G. In this work, lower bounds for the clique number and the number of connected components of $\Gamma_{\mathfrak{N}}(G)$ are presented in terms of the size of its Fitting subgroup and the number of its strongly self-centralizing subgroups of G, respectively. We prove that no finite non-nilpotent group has a self-complementary nilpotent graph. Furthermore, for the dihedral group D_n, it is determined that the number of connected components of its nilpotent graph is one more than n when n is odd or one more than the $2'$ part of n when n is even. In addition, a formula for the number of connected components of $\Gamma_{\mathfrak{N}}(\text{PSL}(2, q))$, where q is a prime power, is provided.

1. Introduction

Applying techniques and concepts from graph theory in the study of group theory has been a topic of growing interest in the last decades. This approach provides a visual perspective that can shed light on the group's properties in question. An outstanding example of this connection between groups and graphs is the non-commuting graph associated with a non-abelian group G, introduced by A. Abdollahi et al. in [1]. This is the graph with a set of vertices $G - Z(G)$ in which two distinct vertices are related if they do not commute. This graph captures several commutativity relations between the group elements

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and reveals interesting patterns in its inner structure. For example, if G is a finite non-abelian nilpotent group and H is a finite non-abelian group such that $|G| = |H|$ and the non-commuting graph of H is isomorphic to the non-commuting graph of G, then H is nilpotent. Later, A.K. Das et al. studied the genus of the commuting graph [6], which is precisely the complement of the non-commuting graph, i.e., the same set of vertices is considered, but two different vertices are related if they commute. That paper proved that the order of a finite non-abelian group is bounded by a function that depends on the genus of its respective commuting graph.

A few years later, A. Abdollahi et al. introduced in [2] the non-nilpotent graph associated with a group G, where the vertex set is $G - \mathrm{nil}(G)$ and in which two distinct vertices are related if they generate a non-nilpotent subgroup of G. An interesting fact about this graph is that, for finite non-nilpotent groups, the graph is always connected. A generalization of this graph involving formations of finite groups was presented by A. Lucchini et al. in [13]. Sometime later, as in the case of the non-commuting graph, research was carried out on the genus of the nilpotent graph (the complement of the non-nilpotent graph) [5].

This work studies further properties of the nilpotent graphs associated with finite non-nilpotent groups. The paper is organized as follows: Section 2 introduces several necessary concepts and results from graph and group theory that will be used later. In Section 3, a SageMath algorithm to compute the nilpotent graph of a non-nilpotent group is provided. It is also proved that the nilpotent graph of a finite group G is bipartite if and only if G is isomorphic to the symmetric group of degree three, among other results. Section 4 establishes lower bounds on the clique number and the number of connected components of the nilpotent graph of G in terms of the size of its Fitting subgroup and the number of its strongly self-centralizing subgroups of G, respectively. In Section 5, it is proved that the nilpotent graph of a finite group is connected if and only if the nilpotent graph of its direct product with any finite nilpotent group is connected. It is also proved that the nilpotent graph of any finite non-nilpotent group is never self-complementary. In Sections 6 and 7, it is demonstrated that the nilpotent graphs of the dihedral group D_n and the projective special linear groups $\mathrm{PSL}(2, q)$ where q is a prime power are always disconnected by offering formulas for the number of the connected components of their respective graphs. Finally, necessary and sufficient conditions for certain subsets of a group (that contain connected components of its nilpotent graph) to contain one of its Sylow p-subgroups are studied.

2. Preliminaries

We recall some definitions, notations, and results concerning elementary graph theory for the reader's convenience and later use. The reader is referred for undefined terms and concepts of graph theory to [9] or [10].

An undirected finite graph or simply a graph $\Gamma = (V, E)$ consists of a non-empty finite set V whose elements are called vertices and a set $E \subseteq \binom{V}{2}$ whose elements are called edges. Here $\binom{V}{2}$ is the set of two-element subsets of V. By this definition, an undirected graph cannot have self-loops. Further, all the graphs considered in this work are simple; no multiple edges connect two vertices.

In the following, Γ always denotes a graph. If x and y are vertices of Γ, then it is said that x is adjacent (or related) to y if $\{x, y\}$ is an edge. In this case, the edge's endpoints are x and y. A subgraph of Γ is a graph Γ' whose vertices and edges form subsets of the vertices and edges of Γ. The degree of a vertex w, denoted by $\deg(w)$, is the number of edges incident with w. A vertex of degree zero is called an isolated vertex or an isolated point.

Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are said to be isomorphic if there is a bijection $f : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$.

A path in Γ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \le i < k$. If $v_1, v_2, \ldots, v_k$ is a path, it is said that it goes from v_1 to v_k (or that it connects v_1 with v_k) and that the vertices v_1 and v_k are its endpoints. It is said that Γ is connected if a path exists connecting any two vertices; otherwise, Γ is disconnected. The relation defined over the set of vertices V in which two vertices are related if a path exists between them is an equivalence relation. A connected component of Γ is an equivalence class under this relation or, equivalently, a maximal connected subgraph. Some useful notations are the following: $\kappa(\Gamma)$ denotes the number of connected components of Γ, $\mathcal{C}(\Gamma) := \{C_1, \ldots, C_{\kappa(\Gamma)}\}$ is the set of all connected components of Γ, and finally, $\kappa_j(\Gamma) := |C_j|,$ for $j = 1, \ldots \kappa(\Gamma)$.

A clique in Γ is a subgraph in which any two vertices are adjacent, i.e., a complete subgraph of Γ. A clique with the largest possible size is called a maximum clique. The size $\omega(\Gamma)$ of a maximum clique in Γ is called the clique number of Γ. A bipartite graph is a graph in which the vertices can be divided into two disjoint sets such that all edges connect a vertex in one set to a vertex in another. There are no edges between vertices in the disjoint sets. The complement of Γ is a graph $\overline{\Gamma}$ on the same vertices such that two distinct vertices of $\overline{\Gamma}$ are adjacent if and only if they are not adjacent in Γ.

3. The nilpotent graph of a finite group

For details on notation and definitions about groups, see [15] or [11]. From now on, the symbol G denotes a finite group. The letter $\mathfrak{N}$ denotes the class of all finite nilpotent groups through this paper.

If $h \in G$, the nilpotentizer of h in G is the set $\mathrm{nil}_G(h) := \{g \in G : \langle g, h\rangle \text{ is nilpotent}\}$. The nilpotentizer of G is the set $\mathrm{nil}(G) := \bigcap_{h \in G} \mathrm{nil}_G(h)$. It is not known exactly when $\mathrm{nil}(G)$ is a subgroup of G, but in some cases it is. For instance, if G is a finite soluble group, then the hypercenter of G, denoted by $Z_\infty(G)$, satisfies $Z_\infty(G) = \mathrm{nil}(G)$ (see [1] or [5, Lemma 3.1]). The nilpotent graph of a finite group G, denoted by $\Gamma_{\mathfrak{N}}(G)$, is an undirected graph whose vertex set is $G - \mathrm{nil}(G)$, and two vertices g and h are adjacent if and only if $\langle g, h\rangle$ is a nilpotent subgroup of G.

Example 3.1. *$\Gamma_{\mathfrak{N}}(S_3)$ has four connected components. Note that, $\mathrm{nil}(S_3) = 1$ and if we define $\sigma :=$ (123) then the nilpotent graph of S_3 consist of the 2-path $\{\sigma, \sigma^2\}$ and three isolated points.*

Proposition 3.2. *$\Gamma_{\mathfrak{N}}(G)$ is bipartite if and only if $G \cong S_3$.*

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Proof. If $\Gamma_{\mathfrak{N}}(G)$ is bipartite, then it has no 3-cycles. Then, by [5, Proposition 3.2], $G \cong S_3$. Reciprocally, if $G \cong S_3$, note that the unique edge in $\Gamma_{\mathfrak{N}}(G)$ is incident with (123) and (132), see Example 3.1. Then, the assertion holds. □

Example 3.3. *$\Gamma_{\mathfrak{N}}(S_4)$ has five connected components. In fact, $\mathrm{nil}(S_4) = 1$, the eight 3-cycle forms four copies of K_2 and the other elements of S_4 build the fifth component.*

Contrary to what has been proven for the soluble graph of a finite group, the nilpotent graph is generally not connected (see, for example, [3], [4]).

Remark 3.4. *If $\langle x, y\rangle$ is a nilpotent subgroup of G, then this subgroup contains a non-identity element that commutes with x and y. Consequently, if $Z(G) = 1$, the connected components of the nilpotent graph of G are the same as those in the commuting graph of G. (See the proof of Proposition 7.6 in [4]). In particular, the nilpotent graph of S_n is connected if and only if its commuting graph is connected, and it follows from [12, Theorem 3.1.] that $\Gamma_{\mathfrak{N}}(S_n)$ is connected if and only if neither n nor $n-1$ is a prime number.*

The next Lemma follows similar arguments to the ones given in [3, Lemma 2.1 and Proposition 2.2].

Lemma 3.5. *The graph $\Gamma_{\mathfrak{N}}(G)$ is not a star.*

Proof. It follows immediately that for every $x \in G - \mathrm{nil}(G)$ the degree of x is

$$\deg(x) = |\mathrm{nil}_G(x)| - |\mathrm{nil}(G)| - 1.$$

Suppose now that $\Gamma_{\mathfrak{N}}(G)$ is a star. Then there exists $x \in G - \mathrm{nil}(G)$ such that $\deg(x) = |G - \mathrm{nil}(G)| - 1 = |G| - |\mathrm{nil}(G)| - 1$. Thus, $|\mathrm{nil}_G(x)| = |G|$ and so $x \in \mathrm{nil}(G)$, which is a contradiction. □

Corollary 3.6. *The nilpotent graph of a finite non-nilpotent group, G with nilpotentizer of even order, is non-Eulerian.*

Proof. Let $x \in G - \mathrm{nil}(G)$. Then, $\deg(x) = |\mathrm{nil}_G(x)| - |\mathrm{nil}(G)| - 1$. Since $|\mathrm{nil}(G)|$ divides $|\mathrm{nil}_G(x)|$ and $|\mathrm{nil}(G)|$ is even, $|\mathrm{nil}_G(x)|$ is even. Thus, $\deg(x)$ is odd, so the graph is non-Eulerian. □

4. Strongly self-centralizing subgroups and nilpotent graphs

The Fitting subgroup of G, denoted by $\mathrm{F}(G)$, is defined as the subgroup generated by all normal nilpotent subgroups of G. Then, the Fitting subgroup is a nilpotent subgroup of G. Further, it is the largest normal nilpotent subgroup of G.

From now on, G denotes a finite non-nilpotent group in the remainder of this work unless stated otherwise.

Lemma 4.1. *Let U be a nilpotent subgroup G such that $U \nsubseteq \mathrm{nil}(G)$. Then $U - (U \cap \mathrm{nil}(G))$ is the set of vertices of a clique of $\Gamma_{\mathfrak{N}}(G)$. Moreover if N is a normal nilpotent subgroup of G, $|N - (N \cap \mathrm{nil}(G))| \leq \omega(\Gamma_{\mathfrak{N}}(G))$. In particular, $|\mathrm{F}(G) - (\mathrm{F}(G) \cap \mathrm{nil}(G))| \leq \omega(\Gamma_{\mathfrak{N}}(G))$.*

Proof. For all $x, y \in U - (U \cap \mathrm{nil}(G))$ we have that $\langle x, y\rangle$ is nilpotent. Thus, the subgraph of $\Gamma_{\mathfrak{N}}(G)$ having $U - (U \cap \mathrm{nil}(G))$ as its set of vertices is a clique. Moreover, if N is a normal nilpotent subgroup of G, then

$$\begin{aligned} |N - (N \cap \mathrm{nil}(G))| &\leq \max\{|U - (U \cap \mathrm{nil}(G))| : U \trianglelefteq G \wedge U \in \mathfrak{N}\} \\ &\leq \max\{|U - (U \cap \mathrm{nil}(G))| : U \leq G \wedge U \in \mathfrak{N}\} \\ &\leq \omega(\Gamma_{\mathfrak{N}}(G)). \end{aligned}$$

The rest follows from the fact that the Fitting subgroup $\mathrm{F}(G)$ is the largest normal nilpotent subgroup of G. □

Definition 4.2. *A subgroup U of G is said to be strongly self-centralizing if $C_G(x) = U$ for all $1 \neq x \in U$.*

Theorem 4.3. *If there exists a strongly self-centralizing subgroup U of G, then:*

(1) $\mathrm{nil}_G(x) = U$ *for all* $1 \neq x \in U$.
(2) $\Gamma_{\mathfrak{N}}(G)$ *is a disconnected graph.*
(3) $|\mathrm{F}(G)| - 1 \leq \omega(\Gamma_{\mathfrak{N}}(G))$.

Proof. (1) Suppose a strongly self-centralizing subgroup U of G exists. Note that $Z(G) \leq C_G(x) = U$, for all $1 \neq x \in U$. Since U is abelian, $U \cap C_G(y) = 1$ for all $y \in G - U$; otherwise it would exist $1 \neq z \in U \cap C_G(y)$ and therefore $y \in C_G(z) = U$, which is a contradiction. This implies that $Z(G) \leq U \cap C_G(y) = 1$, and so $Z_\infty(G) = 1$.

Let $1 \neq x \in U$. If $g \in \mathrm{nil}_G(x)$, then there exists $1 \neq z \in Z(\langle x, g\rangle)$, and so $z \in C_G(x) = U$, implying that $g \in C_G(z) = U$. Thus $\mathrm{nil}_G(x) \subseteq U$ and $U = C_G(x) \subseteq \mathrm{nil}_G(x)$, i.e., $\mathrm{nil}_G(x) = U$.
(2) For every $h \in G - U$ and $1 \neq x \in U$, the subgroup $\langle x, h\rangle$ is not nilpotent, then there is no edge from h to x, so $\Gamma_{\mathfrak{N}}(G)$ is a disconnected graph.
(3) It follows immediately from Lemma 4.1. □

Theorem 4.4. *Let $\mathcal{U} \neq \emptyset$ be the set of the strongly self-centralizing subgroups of G. If $U \in \mathcal{U}$, then $U - 1$ is the set of vertices of a connected component of $\Gamma_{\mathfrak{N}}(G)$ and*

$$\kappa(\Gamma_{\mathfrak{N}}(G)) \geq \begin{cases} |\mathcal{U}| \text{ if } \bigcup_{U \in \mathcal{U}} U = G \\ |\mathcal{U}| + 1 \text{ if } \bigcup_{U \in \mathcal{U}} U \neq G. \end{cases} \tag{4.1}$$

Proof. Suppose $\mathcal{U} \neq \emptyset$. Then $\mathrm{nil}(G) = Z^*(G) = 1$, and so the set of vertices of $\Gamma_{\mathfrak{N}}(G)$ is $G - 1$. Let $U \in \mathcal{U}$, then $\mathrm{nil}_G(x) = U$ for all $1 \neq x \in U$, that is, any vertex in $G - 1$ that is adjacent to a vertex in $U - 1$ must be in $U - 1$. Note that if C is a connected component of $\Gamma_{\mathfrak{N}}(G)$ containing a vertex $x \in U - 1$ and $y \in C$, then there exists a path from x to y, so the vertex of that path that is connected to x must be in $U - 1$. Any other vertex connected to that must be in $U - 1$ as well. Applying this reasoning consecutively, one gets $y \in U - 1$. Thus, since $U - 1$ is the set of vertices of a connected subgraph of $\Gamma_{\mathfrak{N}}(G)$ (by Lemma 4.1), such a subgraph must be a connected component of $\Gamma_{\mathfrak{N}}(G)$.

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If $\bigcup_{U\in\mathcal{U}} U \subsetneq G$, then any element $h \in G - \bigcup_{U\in\mathcal{U}} U$ is a vertex $\Gamma_{\mathfrak{N}}(G)$ that is not related to an element of $\bigcup_{U\in\mathcal{U}}(U-1)$ so that h belongs to a connected component whose set of vertices has a trivial intersection with $\bigcup_{U\in\mathcal{U}}(U-1)$, implying that $|\mathcal{U}|+1 \leq \kappa(\Gamma_{\mathfrak{N}}(G))$. Therefore, (4.1) holds.

□

Example 4.5. *The set $\mathcal{U}$ of all strongly self-centralizing subgroups of S_4 is*

$$\mathcal{U} = \{\langle(132)\rangle, \langle(142)\rangle, \langle(143)\rangle, \langle(243)\rangle\}.$$

Thus, $|\mathcal{U}|+1 = 5 \leq \kappa(\Gamma_{\mathfrak{N}}(G)) = 5$ where the last equality is by Example 3.3.

5. Directed product of groups and connectedness

In this section, necessary and sufficient conditions to guarantee the connectedness of the nilpotent graph and for two nilpotent graphs to be isomorphic are provided. It is also proved that the nilpotent graph of any finite non-nilpotent group is never self-complementary.

Lemma 5.1. *Let G and H be finite groups. Then $\langle(g_1,h_1),(g_2,h_2)\rangle$ is a nilpotent subgroup of $G\times H$ if and only if $\langle g_1,g_2\rangle$ and $\langle h_1,h_2\rangle$ are nilpotent subgroups of G and H respectively.*

Proof. Suppose $U := \langle(g_1,h_1),(g_2,h_2)\rangle$ is a nilpotent subgroup of $G\times H$, and $\tilde{H} := 1\times H$. Then $U/(U\cap\tilde{H}) \cong \langle g_1,g_2\rangle$, and so $\langle g_1,g_2\rangle$ is nilpotent. Similarly $\langle h_1,h_2\rangle$ is nilpotent.

Reciprocally, suppose $\langle g_1,g_2\rangle$ and $\langle h_1,h_2\rangle$ are nilpotent groups. Then $\langle g_1,g_2\rangle\times\langle h_1,h_2\rangle$ is nilpotent. Since $\langle(g_1,h_1),(g_2,h_2)\rangle \leq \langle g_1,g_2\rangle\times\langle h_1,h_2\rangle$, then $\langle(g_1,h_1),(g_2,h_2)\rangle$ is a nilpotent group. □

Corollary 5.2. *Let G and H be groups. Then* $\mathrm{nil}(G\times H) = \mathrm{nil}(G)\times\mathrm{nil}(H)$.

Proof. It follows immediately from Lemma 5.1. □

Proposition 5.3. *Let G and H be non-nilpotent groups. Then $\Gamma_{\mathfrak{N}}(G\times H)$ is connected.*

Proof. Let (g_1,h_1), $(g_2,h_2) \in G\times H - Z^*(G\times H)$. Then $g_1 \notin Z_\infty(G)$ or $h_1 \notin Z^*(H)$ and $g_2 \notin Z_\infty(G)$ or $h_2 \notin Z^*(H)$. Consider the following cases:

Case 1. $g_1 \notin Z_\infty(G)$ and $h_2 \notin Z^*(H)$. In this case the path (g_1,h_1), $(g_1,1)$, $(1,h_2)$, (g_2,h_2) connecting the vertices (g_1,h_1), (g_2,h_2).

Case 2. $g_1 \in Z_\infty(G)$ and $g_2 \in Z_\infty(G)$. Since G is non-nilpotent, there exists $g \in G - Z_\infty(G)$. Then (g_1,h_1), $(g,1)$, (g_2,h_2) is a path connecting (g_1,h_1) and (g_2,h_2).

The remaining cases are analogous to one of the above. □

Let $\kappa_j(\Gamma_{\mathfrak{N}}(G))$ denote the number of vertices of the j-th connected component of $\Gamma_{\mathfrak{N}}(G)$. To simplify the notation, from now on, it will be used $\mathcal{C}(G)$, $\kappa(G)$, and $k_j(G)$, to denote $\mathcal{C}(\Gamma_{\mathfrak{N}}(G))$, $\kappa(\Gamma_{\mathfrak{N}}(G))$ and $\kappa_j(\Gamma_{\mathfrak{N}}(G))$, respectively.

Theorem 5.4. *For every finite nilpotent group H, it holds $\kappa(G \times H) = \kappa(G)$. In particular, $\Gamma_{\mathfrak{N}}(G)$ is connected if and only if $\Gamma_{\mathfrak{N}}(G \times H)$ is connected for every finite nilpotent group H.*

Proof. Let H be any finite nilpotent group. Then $Z^*(H) = \mathrm{nil}(H) = H$. So, by Corollary 5.2, $\mathrm{nil}(G \times H) = \mathrm{nil}(G) \times H$. Thus, by Lemma 5.1, $(g, h), (x, y)$ are adjacent in $\Gamma_{\mathfrak{N}}(G \times H)$ if and only if g and x are adjacent in $\Gamma_{\mathfrak{N}}(G)$. Then, there is no path from g to x if and only if there is no path from (g, h) to (x, y), for all $h, y \in H$. This implies that $\kappa(G \times H) = \kappa(G)$. $\square$

Proposition 5.5. *Let H be a finite nilpotent group. Then $|H|$ divides $k_i(G \times H)$ for all $i \in \{1, \ldots, \kappa(G \times H)\}$.*

Proof. Let g be a vertex in the i-th connected component of $\Gamma_{\mathfrak{N}}(G)$. Then, by Lemma 5.1, for all $h, k \in H$, (g, h) and (g, k) are in the same connected component of $\Gamma_{\mathfrak{N}}(G \times H)$. Then, for every g in i-th component of $\Gamma_{\mathfrak{N}}(G)$ there are $|H|$ elements of the form (g, h), with $h \in H$, in the i-th connected component of $\Gamma_{\mathfrak{N}}(G \times H)$. $\square$

Theorem 5.6. $\Gamma_{\mathfrak{N}}(G) \cong \Gamma_{\mathfrak{N}}(G/Z_\infty(G) \times Z_\infty(G))$. *In particular $\kappa(G) = \kappa(G/Z_\infty(G))$.*

Proof. Let $T = \{g_1, \ldots, g_m\}$ be a left transversal of $Z_\infty(G)$ in G. Let $\widetilde{G} := G/Z_\infty(G) \times Z_\infty(G)$, and define $f : \widetilde{G} \longrightarrow G$ by $f(gZ_\infty(G), z) = g_i z$, where g_i satisfies $gZ_\infty(G) = g_i Z_\infty(G)$. It is straightforward to show that f is a well-defined function, and also bijective.

Now, let $(gZ_\infty(G), z), (hZ_\infty(G), w) \in \widetilde{G}$ arbitrary. Then, by Lemma 5.1, $\langle (gZ_\infty(G), z),\ (hZ_\infty(G), w) \rangle$ is a nilpotent subgroup of $\widetilde{G}$ if and only if $\langle gZ_\infty(G),\ hZ_\infty(G) \rangle$ is a nilpotent subgroup of $G/Z_\infty(G)$. However, $\langle gZ_\infty(G), hZ_\infty(G) \rangle = \langle g_i z Z_\infty(G), g_j w Z_\infty(G) \rangle$, for some $z, w \in Z_\infty(G)$.

It follows that $\langle g_i z Z_\infty(G),\ g_j w Z_\infty(G) \rangle$ is a nilpotent subgroup of $G/Z_\infty(G)$ if and only if $\langle g_i z, g_j w \rangle = \langle f(gZ_\infty(G), z), f(hZ_\infty(G), w) \rangle$ is a nilpotent subgroup of G, so, by restricting f to $\widetilde{G} - Z^*(\widetilde{G})$, one obtains a bijective function preserving adjacency. That is an isomorphism between the corresponding graphs. Finally, since $Z_\infty(G)$ is nilpotent, it follows from Theorem 5.4 that $\kappa(G) = \kappa(G/Z_\infty(G) \times Z_\infty(G)) = \kappa(G/Z_\infty(G))$. $\square$

Lemma 5.7. $|Z_\infty(G)| = \gcd(k_1(G), \ldots, k_n(G), |G|)$, *where $n = \kappa(G)$. Furthermore, if $\gcd(k_i(G), k_j(G)) = 1$ for some i, j, then for any finite non-nilpotent group H with $\Gamma_{\mathfrak{N}}(G) \cong \Gamma_{\mathfrak{N}}(H)$, holds $|G| = |H|$.*

Proof. Using Proposition 5.5 follows that $|Z_\infty(G)|$ divides $k_i(G/Z_\infty(G) \times Z_\infty(G)) = k_i(G)$ for all $i \in \{1, \ldots, \kappa(G)\}$. Now, note that $\sum_{i=1}^{n} k_i(G) + |Z_\infty(G)| = |G|$. Thus

$$1 = \frac{|G|}{|Z_\infty(G)|} - \sum_{i=1}^{n} \frac{k_i(G)}{|Z_\infty(G)|},$$

and so

$$\gcd\left(\frac{k_1(G)}{|Z_\infty(G)|}, \ldots, \frac{k_n(G)}{|Z_\infty(G)|}, \frac{|G|}{|Z_\infty(G)|}\right) = 1,$$

which implies the first assertion.

Suppose now $\Gamma_{\mathfrak{N}}(G) \cong \Gamma_{\mathfrak{N}}(H)$, then it follows that $|G| - |Z_\infty(G)| = |H| - |Z^*(H)|$. Therefore, $|Z_\infty(G)| = 1 = |Z^*(H)|$, and so $|G| = |H|$. $\square$

Theorem 5.8. *No finite non-nilpotent group has a self-complementary nilpotent graph.*

Proof. Suppose that $\Gamma_{\mathfrak{N}}(G)$ is a self-complementary graph. Then, there exists a bijective function $f : G - Z_\infty(G) \longrightarrow G - Z_\infty(G)$ such that $\langle x, y\rangle$ is nilpotent if and only if $\langle f(x), f(y)\rangle$ is non-nilpotent. Consequently, for any $x \in G - Z_\infty(G)$ holds

$$|\mathrm{nil}_G(x)| - |Z_\infty(G)| - 1 = \deg(x) = \deg(f(x)) = |G| - |\mathrm{nil}_G(f(x))|.$$

Thus, by [14, Teorema 4.1], $|Z_\infty(G)|$ divides $|G| - |\mathrm{nil}_G(f(x))| = |\mathrm{nil}_G(x)| - |Z_\infty(G)| - 1$ and $|\mathrm{nil}_G(x)| - |Z_\infty(G)|$ for all $x \in G - Z_\infty(G)$, so $|Z_\infty(G)| = 1$.

Let C_2 be the cyclic group of order 2, $H := G \times C_2$, and consider the function $g : H - Z^*(H) \longrightarrow H - Z^*(H)$ defined by $g(x, n) = (f(x), n)$. It is easy to see that this function is well-defined and bijective. It follows from Lemma 5.1 that $\langle (x, n), (y, m)\rangle$ is nilpotent if and only if $\langle (f(x), n), (f(y), m)\rangle = \langle g(x, n), g(y, m)\rangle$ is non-nilpotent, for all $(x, n), (y, m) \in H - Z^*(H)$. Therefore, the nilpotent graph of H is self-complementary too. By a similar argument to the one used at the beginning, $|Z^*(H)| = 1$, which is a contradiction (because $|Z^*(H)| = |Z_\infty(G) \times C_2| = 2$, by Corollary 5.2). $\square$

6. Nilpotent graph of the Dihedral group

In this section, a formula for the number of connected components of the non-nilpotent dihedral group D_n is presented. It is well-known that $Z(D_n) \neq 1$ if and only if n is even; in other words, no reflection of D_n commutes with any rotation different from the identity if and only if n is odd. This fact will be used in Theorem 6.1.

Theorem 6.1. *Let $n \in \mathbb{N}$. Then*

$$\kappa(D_n) = \begin{cases} n+1 & \text{if } 2 \nmid n \\ m+1 & \text{if } 2 \mid n \wedge n \text{ is not a power of } 2 \end{cases}$$

where $n = 2^k m$, $m \geq 3$ and $2 \nmid m$.

Proof. (1) Suppose n is an odd number. Let $\langle r\rangle$ be the subgroup of rotations of D_n and s be a reflection. Since $\gcd(|r^i|, |r^j s|) = 1$ for all $i \in \{1, \dots, n-1\}$, $j \in \{0, \dots, n\}$, then $\langle r^i, r^j s\rangle$ is non-nilpotent for all $i \in \{1, \dots, n-1\}$, $j \in \{0, \dots, n-1\}$. Otherwise, some rotation different from the identity commutes with some reflection, which is not possible since n is odd.

On the other hand, if $i \neq j$, then $\langle r^i s, r^j s\rangle$ contains at least one rotation different from the identity, again implying that some reflection commutes with some rotation. Therefore $\langle r^i s, r^j s\rangle$ is nilpotent if and only if $i = j$. Then $\Gamma_{\mathfrak{N}}(D_n)$ has one component with $n-1$ vertices and n components with one element. That is, $\kappa(G) = n+1$.

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(2) From Theorem 5.6 follows that $\kappa(D_n) = \kappa(D_n/Z^*(D_n))$. It is known that $D_n/Z^*(D_n) \cong D_m$, then $\kappa(D_n/Z^*(D_n)) = \kappa(D_m)$. Since m is an odd number, by part 1, $\kappa(D_n) = \kappa(D_m) = m+1$. □

Example 6.2. *The nilpotent graphs of D_5 and D_{12} have six and four connected components respectively.*

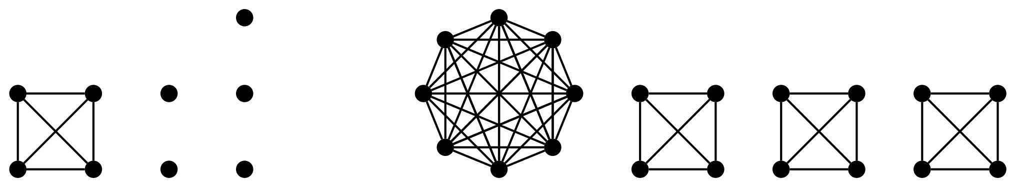

FIGURE 1. The nilpotent graph of D_5 and D_{12}

Note that, if $n = 5$ or $n = 12 = 2^2 \cdot m$, the nilpotent graph of D_n has $n+1 = 6$ or $m+1 = 3+1 = 4$ connected components, which illustrates Theorem 6.1.

7. Nilpotent graph of some projective special linear groups

If C is a connected component of the nilpotent graph of G; in this section, necessary and sufficient conditions for the union of C and the hypercenter of G to contain one of its Sylow p-subgroups are studied. If q is a power of a prime number, a formula for the number of connected components of the nilpotent graph of $\mathrm{PSL}(2, q)$ is presented.

It is easy to see that G acts by conjugation on $\mathcal{C}(G)$ as follows $g \cdot C := gCg^{-1}$. Lemma 7.1 shows that $C \subseteq N_G(C)$. Note that $N_G(C) = N_G(C \cup Z_\infty(G))$ for all $C \in \mathcal{C}(G)$.

Lemma 7.1. *Every connected component of $\Gamma_{\mathfrak{N}}(G)$ is invariant under conjugation by its element.*

Proof. Let $g \in C$, $y \in g^{-1}Cg$ be arbitrary. Then there exist $x, h_1, \ldots, h_n \in C$ such that $y = g^{-1}xg$ and $\langle g, h_1\rangle, \langle h_1, h_2\rangle, \ldots, \langle h_n, x\rangle$ are nilpotent subgroups of G, because $C \in \mathcal{C}(G)$. Then, $\langle g, g^{-1}h_1g\rangle$, $\langle g^{-1}h_1g, g^{-1}h_2g\rangle, \ldots, \langle g^{-1}h_ng, g^{-1}xg\rangle = \langle g^{-1}h_ng, y\rangle$ are nilpotent subgroups of G. That is, a path exists from g to y so that $y \in C$. Therefore $g^{-1}Cg \subseteq C$ and since $|g^{-1}Cg| = |C|$, then $g^{-1}Cg = C$. □

If $C \in \mathcal{C}(G)$, Theorem 7.2 provides necessary and sufficient conditions for the set $C \cup Z_\infty(G)$ to contain a Sylow p-subgroup of G.

Theorem 7.2. *Let $C \in \mathcal{C}(G)$, and p be a prime divisor of $|G|$.*

(1) *If P is a Sylow p-subgroup of G contained in $C \cup Z_\infty(G)$, then $p \mid |C \cup Z_\infty(G)|$.*
(2) *If $p \mid gcd(|C|+1, |N_G(C)|)$ and $Z_\infty(G) = 1$, then $C \cup 1$ contains a Sylow p-subgroup of G.*

Proof. (1) If $P \cap Z_\infty(G) \neq 1$, then $Z_\infty(G)$ contains an element of order p, i.e. $|Z_\infty(G)|$ is divisible by p. Since $|Z_\infty(G)| \mid |C|$, by Lemma 5.7, then $|C \cup Z_\infty(G)| = |C| + |Z_\infty(G)| = |Z_\infty(G)|m = pkm$, for some integer numbers k and m. Suppose now that $P \cap Z_\infty(G) = 1$. Since $P \subseteq C \cup Z_\infty(G)$, there

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exists $x \in C$ such that $\mathrm{ord}(x) = p$. Thus $x \in N_G(C)$ (by Lemma 7.1) and so $\langle x\rangle$ acts by conjugation on the elements of $C \cup Z_\infty(G)$. Note that, for $j \in \{1, \ldots, p-1\}$ and $g \in G$, $g^{x^j} = g$ if and only if $g \in C_G(x)$. In addition, if $g \in C_G(x)$, then $\langle x, g\rangle$ is nilpotent, so that $C_G(x) \subseteq C \cup Z_\infty(G)$. Thus $p = \mathrm{ord}(x) = |\mathrm{Orb}_x(g)|\ |\mathrm{Stab}_x(g)|$, for all $g \in C \cup Z_\infty(G)$. Then $|\mathrm{Orb}_x(g)| = p$ or $|\mathrm{Orb}_x(g)| = 1$ for all $g \in C \cup Z_\infty(G)$. Thus, since $|\mathrm{Orb}_x(g)| = 1$ if and only if $g \in C_G(x)$,

$$|C \cup Z_\infty(G)| = \sum_{i=1}^{n} |\mathrm{Orb}_x(g_i)| = |C_G(x)| + pn_2$$

where n and n_2 are the number of orbits and the number of orbits of size p, respectively. Hence, since $\langle x\rangle \leq C_G(x)$, $p \mid |C \cup Z_\infty(G)|$.

(2) Suppose that no Sylow p-subgroup of G is contained in $C \cup 1$. Since $p \mid |N_G(C)|$, then there exists an $x \in N_G(C)$ such that $\mathrm{ord}(x) = p$ and so $\langle x\rangle$ acts on $C \cup 1$ by conjugation. Then, $g^y \neq g$ for all $1 \neq y \in \langle x\rangle$, and $g \in C$; otherwise, $\langle y, g\rangle$ is nilpotent, and so $y \in C \cup 1$, which is a contradiction. Thus, $y \in \langle x\rangle$ stabilizes $g \in C \cup 1$ if and only if $g = 1$ or $y = 1$. Hence, by the Burnside's Lemma,

$$p\ |(C \cup 1)/\langle x\rangle| = \mathrm{ord}(x)|(C \cup 1)/\langle x\rangle| = \sum_{y \in \langle x\rangle} |\mathrm{Fix}(y)| = |C \cup 1| + (\mathrm{ord}(x) - 1) = |C| + p.$$

Thus, $p \mid |C|$, which is a contradiction. Therefore, a Sylow p-subgroup of G must be contained in $C \cup 1$. □

For the rest of this section, G denotes the projective special linear group $\mathrm{PSL}(2, q)$, where $q = p^f$ for some prime number p. In addition, $k := \gcd(q-1, 2)$, $r := \frac{q-1}{k}$, and $t := \frac{q+1}{k}$.

Theorem 7.3. [11, Chapter II, Sätze 8.2, 8.3, 8.4]

(1) *Let P be a Sylow p-subgroup of G. Then P is isomorphic to the additive group of the finite Field $\mathbb{F}_q$, an elementary abelian group of order q. Further, G has exactly $q+1$ Sylow p-subgroups.*
(2) *The subgroup U of G, which fixes the values 0 and ∞, is cyclic of order r.*
(3) *G has a cyclic subgroup S of order t.*

Theorem 7.4. [11, Chapter II, Satz 8.5] *Let P, U, S be as in Theorem 7.3, and*

$$\mathcal{P} := \{P^x, U^x, S^x : x \in G\}.$$

Then, each element of G different from 1 lies in exactly one of the groups from $\mathcal{P}$ (i.e., $\mathcal{P}$ forms a partition of G).

Thus, by Theorem 7.4, every element of G different from 1 belongs to only one of the conjugacy classes of the subgroups P, U, S, described in Theorem 7.3. Moreover, $N_G(P) \cong \mathbb{Z}_p^f \rtimes \mathbb{Z}_r$, $N_G(U) \cong D_r$ y $N_G(S) \cong D_t$.

Let C_P, C_U, C_S denote the connected component of $\Gamma_{\mathfrak{N}}(G)$ containing elements of P, U and S, respectively.

Theorem 7.5. *The following statements hold:*

(1) *If $q > 3$, then $|\mathcal{C}(G)/G| = 3$.*
(2) $|\mathrm{Orb}_G(C_P)| = q + 1$.
(3) *If $q \geq 13$, then $|\mathrm{Orb}_G(C_U)| = 1$ if and only if $q \equiv 1 \bmod 4$.*
(4) *If $q > 3$ and $q \neq 7, 9$, then $|\mathrm{Orb}_G(C_S)| = 1$ if and only if $q \equiv 3 \bmod 4$.*

Proof. (1) Let $1 \neq x \in G$ and suppose that $\langle x, y\rangle$ is nilpotent for some $y \in G - Z_\infty(G)$. If $x \in P$, then $y \in P^g$ for some $g \in G$. Otherwise, $y \in U^h$ or $y \in S^z$, for some $h, z \in G$, so that x and y commute, because their orders are relative primes. However, xy must be in some conjugate of P, U, or S, so the order of one of these groups must be divisible by p and by some other prime that divides r or t, which is false. Therefore $y \in P$. Analogously, if $x \in U$, then y must be in some conjugate of U, and if $x \in S$, then y must be in some conjugate of S. Therefore, $C_U, C_S \notin \mathrm{Orb}_G(C_P)$ and $C_U \notin \mathrm{Orb}_G(C_S)$, and since each orbit contains components with elements of one of these groups, then $|\mathcal{C}(G)/G| = 3$.
(2) Suppose $|\mathrm{Orb}_G(C_P)| \neq q + 1$. Necessarily $|\mathrm{Orb}_G(C_P)| < q + 1$, since $|\mathrm{Orb}_G(C_P)|$, by part 1, contains only components with elements in the conjugates of P. Then, there must exist $x \in P$ and y in some conjugate of P, say P', such that $\langle x, y\rangle$ is nilpotent. Consequently there exists $z \neq 1$ such that $z \in Z(\langle x, y\rangle)$, so that $z \in N_G(P) \cap N_G(P')$, since $x \in P^z \cap P$, $y \in P'^z \cap P'$ and $x \neq 1 \neq y$. Moreover, $\langle x, z\rangle$ is also nilpotent, so z must be in some conjugate of P. Since P is elementary abelian, then $|z| = p$, so, $z \in P \cap P'$, since the only subgroups of order divisible by p of $N_G(P)$ and $N_G(P')$ are P and P'. But this is impossible since $P \cap P' = 1$. Therefore, $C_P = P - 1$ so $|\mathrm{Orb}_G(C_P)| = |G : N_G(P)| = q + 1$.
(3) Suppose $|\mathrm{Orb}_G(C_U)| = 1$. Then C_U has elements of U and some conjugate of U, say U'. That is, there exist $x \in U$, $y \in U'$, $x \neq 1 \neq y$ such that $\langle x, y\rangle$ is nilpotent. Thus, there exists $z \in \langle x, y\rangle$ such that $z \neq 1$ and commutes with x and y. Therefore, $z \in N_G(U) \cap N_G(U')$, since $x \in U^z \cap U$, $y \in U'^z \cap U'$ and $x \neq 1 \neq y$.

On the other hand, if $z \in U$, then $z \notin U'$, since $U \cap U' = 1$ and $z \neq 1$, so that $\mathrm{ord}(z) = 2$, due to $z \in N_G(U') \cong D_{\frac{q-1}{k}}$. Analogously, if $z \in U'$ o $z \notin U \cup U'$, $\mathrm{ord}(z) = 2$. Since z must also be in some conjugate of U by part 1, it follows that $2 = \mathrm{ord}(z)$ divides $|U| = \frac{q-1}{k}$, so $k = 2$ and hence $q \equiv 1 \bmod 4$. Conversely, suppose that $q \equiv 1 \bmod 4$. Then $k = 2$ and $\frac{q-1}{2}$ is a even number. Note that $N_G(U) \leq N_G(C_U)$, and since $N_G(U) \cong D_{\frac{q-1}{2}}$ and $\frac{q-1}{2}$ is an even number, $Z(N_G(U)) \neq 1$, so there exists $z \notin U$ such that z commutes with some element $x \in U$. Thus $\langle x, z\rangle$ is nilpotent, and by part 1, z must belong to some conjugate of U. Then $U^g - 1 \subseteq C_U$, for some $g \in G - N_G(U)$, which implies, by Lemma 7.1, that $N_G(C_U)$ contains at least two cyclic groups of order $\frac{q-1}{2}$. Hence $N_G(U) < N_G(C_U)$. In addition, since $q \geq 13$, $N_G(U)$ is a maximal subgroup (by [8, Table 5]), so that $N_G(C_U) = G$ and $|\mathrm{Orb}_G(C_U)| = |G : N_G(C_U)| = 1$.
(4) Similarly to the previous proof, we use that $N_G(S)$ is maximal when $q \neq 7, 9$, see [8, Table 5]. □

TABLE 1. $\kappa(\mathrm{PSL}(2,q))$ for $q \leq 11$.

q	2	3	4	5	7	8	9	11
$\kappa(G)$	4	5	21	21	37	73	47	79

Theorem 7.6. *Let $q \neq 2, 3, 5$. Then*

$$\kappa(G) = \begin{cases} q^2 + q + 1 & \text{if } q \equiv 0 \bmod 4 \\ q + \frac{(q-1)q}{2} + 2 & \text{if } q \equiv 1 \bmod 4 \\ q + \frac{(q+1)q}{2} + 2 & \text{if } q \equiv 3 \bmod 4 \end{cases}$$

Proof. If $q < 13$, $q \neq 2, 3, 5$, the statement is true (by Table 1). If $q \geq 13$, there are three cases:
Case 1. If $q \equiv 0 \bmod 4$, since $N_G(U)$ and $N_G(S)$ are maximal subgroups, see [8, Table 5], follows that $N_G(U) \leq N_G(C_U)$, and $N_G(S) \leq N_G(C_S)$. By Theorem 7.5 (3) and (4), holds $|\mathrm{Orb}_G(C_U)| \neq 1$ and $|\mathrm{Orb}_G(C_S)| \neq 1$, then $N_G(U) = N_G(C_U)$, $N_G(S) = N_G(C_S)$, $|\mathrm{Orb}_G(C_U)| = |G : N_G(U)| = \frac{(q+1)q}{2}$ and $|\mathrm{Orb}_G(C_S)| = |G : N_G(S)| = \frac{(q-1)q}{2}$.

On the other hand, by Theorem 7.5 (1) and (2), $\left|\frac{\mathcal{C}(G)}{G}\right| = 3$ and $|\mathrm{Orb}_G(C_P)| = q + 1$, implying that $\kappa(G) = |\mathrm{Orb}_G(C_P)| + |\mathrm{Orb}_G(C_U)| + |\mathrm{Orb}_G(C_S)| = q^2 + q + 1$.
Case 2. If $q \equiv 1 \bmod 4$, then by Theorem 7.5 (3) and (4) follows $|\mathrm{Orb}_G(C_U)| = |G : N_G(C_U)| = 1$, and $|\mathrm{Orb}_G(C_S)| = |G : N_G(S)| = \frac{(q-1)q}{2}$. Therefore, $\kappa(G) = q + 2 + \frac{(q-1)q}{2}$.
Case 3. If $q \equiv 3 \bmod 4$, by the same argument, $\kappa(G) = q + 2 + \frac{(q+1)q}{2}$. □

Remark 7.7. *Some interesting group-theoretic consequences from the study of nilpotent graphs:*

(1) *By Theorem 4.3, the intersection of two strongly self-centralizing is always trivial.*
(2) *From Theorem 4.3 and Proposition 5.3, the direct product of two non-nilpotent groups can not contain an strongly self-centralizing subgroup.*

8. An algorithm to generate the nilpotent graph of a finite group

The following code in *SageMath* [7] is used to generate the nilpotent graph of a finite group G:

Algorithm 1: An algorithm to generate the nilpotent graph of a finite group G after removing its isolated points

Data: G = Group() # Any group supported by SageMath.

Result: The nilpotent graph of G

```
G = Group() # Any group supported by SageMath can go here.
nil_graph = Graph()
PAIRS = []
elements = G.list()
nilpotentizer = []
for element1 in elements:
    is_in_nilpotentizer = True
    temp = []
    for element2 in elements:
        if element1 != element2:
            H = G.subgroup([element1, element2])
            if H.is_nilpotent():
                temp.append((element1, element2))
            else:
                is_in_nilpotentizer = False
    if not is_in_nilpotentizer:
        for t in temp:
            PAIRS.append(t)
    elif is_in_nilpotentizer:
        nilpotentizer.append(element1)
for p in PAIRS:
    if p[0] not in nilpotentizer and p[1] not in nilpotentizer:
        nil_graph.add_edge(p)
P = nil_graph.plot(vertex_labels=False)
P.show(figsize=(8, 8))
```

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