ON THE COMPLEXITY OF THE COLORFUL DIRECTED PATHS IN VERTEX COLORING OF DIGRAPHS

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Abstract. The colorful paths and rainbow paths have been considered by several authors. A colorful directed path in a digraph $G$ is a directed path with $\chi(G)$ vertices whose colors are different. A $v$-colorful directed path is such a directed path, starting from $v$. We prove that for a given 3-regular triangle-free digraph $G$ determining whether there is a proper $\chi(G)$-coloring of $G$ such that for every $v \in V(G)$, there exists a $v$-colorful directed path is $\mathbf{NP}$-complete.

1. Introduction

Graph coloring is a well-studied area of graph theory. For a graph $G$, a proper $k$-coloring of $G$ is a function $c : V(G) \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every two adjacent vertices $u, v \in V(G)$. The chromatic number of $G$ denoted by $\chi(G)$, is the smallest $k$ for which $G$ has a proper $k$-coloring. For a given coloring of a graph $G$, we say path $P$ of $G$ is a rainbow path if all vertices of $P$ have different colors. A $v$-rainbow path is a rainbow path starting from the vertex $v$. A $v$-colorful path is a rainbow path starting from the vertex $v$ with $\chi(G)$ vertices. Let $G$ be a graph. We recall that a path in $G$ is said to represent all $\chi(G)$ colors if all the colors $1, \ldots, \chi(G)$ appear on this path. A colorful directed path in a digraph $G$ is a directed path with $\chi(G)$ vertices whose colors are different. A $v$-colorful directed path is such a directed path, starting from $v$. The colorful paths and rainbow paths have been considered by several authors, for instance see [1, 2, 3, 4, 6, 7]. In 2007, Lin posed the following problem [4].

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Problem 1.1. [7] Let $G$ be a connected graph. Does there always exist a proper vertex coloring of $G$ with $\chi(G)$ colors such that every vertex of $G$ is on a path with $\chi(G)$ vertices which represents all $\chi(G)$ colors?

Afterwards, Akbari et al. proposed the following stronger conjecture [1].

Conjecture 1.2. [1] Let $G \neq C_7$ be a connected graph. Then there exists a proper $\chi(G)$-coloring of $G$ such that for every $v \in V(G)$, there exists a $v$-colorful path.

In [2] this was proved with $\lfloor \frac{\chi(G)}{2} \rfloor$ vertices instead of $\chi(G)$ vertices. Afterwards, Alishahi et al. strengthen this to $\chi(G) - 1$ vertices [3]. Also in [2] it was proved that, there exists a proper $(\Delta(G) + 1)$-coloring of $G$ with a $v$-colorful path for every $v \in V(G)$. Furthermore, in [2] it was proved that this result is true if one replaces $(\Delta(G) + 1)$ colors with $2\chi(G)$ colors.

A proper vertex coloring of a digraph $D$ is defined, simply a vertex coloring of its underlying graph $G$, and its chromatic number $\chi(D)$ is defined to be the chromatic number $\chi(G)$ of $G$. The chromatic number of a digraph provides interesting information about its subdigraphs. The following well-known result, due to Gallai, gives a relationship between the length of the longest path and the chromatic number (for example see [9]).

Theorem 1.3. [Gallai Theorem] Every digraph $G$ has a directed path with at least $\chi(G)$ vertices.

In 2001, Li generalized the Gallai Theorem by specifying the starting vertex of the directed path [6].

Theorem 1.4. [6] If $G$ is a digraph in which $v$ is a vertex that can reach all other vertices, then $G$ has a directed path starting at $v$ with at least $\chi(G)$ vertices.

Li gave the following conjecture for the digraph [6].

Conjecture 1.5. [6] For any proper $\chi(G)$-coloring of a digraph $G$ and any vertex $v \in V(G)$ that can reach all other vertices, there is a directed path starting at $v$ whose vertices use all $\chi(G)$ colors.

Chang et al. gave a counterexample to the above conjecture [4]. In this note, we are interested in the following problem.

Problem: Colorful Directed Paths

INPUT: A connected digraph $G$.

QUESTION: Is there a proper $\chi(G)$-coloring of $G$ such that for every $v \in V(G)$, there is a $v$-colorful directed path?

Our main result is that Colorful Directed Paths is $\text{NP}$-complete for 3-regular triangle-free digraphs. In contrast, we show that Colorful Directed Paths can be solved in polynomial time for 2-regular digraphs.

In [8] it was proved that, it is $\text{NP}$-complete to decide whether $G$ is colorable with $\chi(G)$ colors in such a way that for a given vertex $v \in V(G)$ there is a path starting at $v$ representing all $\chi(G)$ colors.
Next, by a similar argument, we prove that the following problem is \textbf{NP}-complete for disconnected graphs.

**Problem:** Colorful Paths

\textbf{Input:} A graph \( G \)

\textbf{Question:} Is there a proper \( \chi(G) \)-coloring of \( G \) such that for every \( v \in V(G) \), there exists a \( v \)-colorful path?

We follow [5, 9] for terminology and notation not defined here, and we consider finite simple graphs and digraphs. We denote the vertex set and the edge set of \( G \) by \( V(G) \) and \( E(G) \), respectively. We denote the maximum degree and the minimum degree of \( G \) by \( \Delta(G) \) and \( \delta(G) \), respectively. The union of simple graphs \( G \) and \( H \) is the graph \( G \cup H \) with vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \). If \( G \) and \( H \) are disjoint, we refer to their union as a disjoint union, and generally denote it by \( G + H \).

By starting with a disjoint union of two graphs \( G \) and \( H \) and adding edges joining every vertex of \( G \) to every vertex of \( H \), one obtains the join of \( G \) and \( H \), denoted \( G \lor H \). Also, for every \( v \in V(G) \), \( d(v) \) denotes the degree of \( v \). For a natural number \( r \), a graph \( G \) is called an \( r \)-regular graph if \( d(v) = r \), for each \( v \in V(G) \).

2. \textbf{NP-completeness}

**Theorem 2.1.** Colorful Directed Paths is \textbf{NP}-complete for 3-regular triangle-free digraphs and it can be solved in polynomial time for 2-regular digraphs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The auxiliary digraphs \( H(x, y, z) \), \( L(x) \) and \( M(z) \).}
\end{figure}

\textit{Proof.} First, we show that Colorful Directed Paths can be solved in polynomial time for 2-regular digraphs. Let \( G \) be a connected 2-regular digraph. We have the following straightforward characterization. If \( G \) is a connected 2-regular digraph, then there exists a proper \( \chi(G) \)-coloring of \( G \) such that for every \( v \in V(G) \), there exists a \( v \)-colorful directed path, if and only if, for every vertex \( v \in V(G) \),
$d^+(v) = 1$ and $|V(G)| = 2k$ or $3k$. Next, we prove that Colorful Directed Paths is $\text{NP}$-complete for 3-regular triangle-free digraphs. Clearly, the problem is in $\text{NP}$. We reduce 3-Sat to our problem. Let $\Phi$ be a 3-Sat formula with clauses $C = \{c_1, \ldots, c_k\}$ and variables $X = \{x_1, \ldots, x_n\}$. Also let $d = 10(k + n)$. We use the auxiliary digraphs $T$, $M(z)$, $H(x,y,z)$, $A(x_j)$, $B(c_j)$ and $L(x)$, which are shown in Figure 1 and Figure 2. We construct a digraph $G(\Phi)$ as the digraph arising from the following construction:

**Algorithm 1 : Construction of $G(\Phi)$.**

1. We start $H(x,y,z)$ as the digraph $G(\Phi)$.
2. For each variable $x_j$, put a copy of $A(x_j)$ also put two directed edges $x_jv^1_{2j-1}$, $-x_jv^1_{2j}$ from $x_j$ and $-x_j$ to $v^1_{2j-1}$ and $v^1_{2j}$, respectively.
3. For each clause $c_j$, put a copy of $B(c_j)$ also put five directed edges $v^2_{1j}c_j^1$, $v^1_{1j+2n}c_j^2$, $c_j^1v^1_{4j-1+2n}$, $c_j^2v^1_{4j-2+2n}$ and $c_j^3v^1_{4j-3+2n}$.
4. For each clause $c_j = l_1 \lor l_2 \lor l_3$, for every $i$, $1 \leq i \leq 3$, add the directed edge $c_j^i a^i_1$ from $a^i_1$ to $a^i_2$.
5. For each vertex $v$, if $d(v) = 1$, put two auxiliary graphs $L(v_x)$, $L(v_{x'})$ and also put two directed edges $vv_x$ and $vv_{x'}$, from $v$ to $v_x$ and $v_{x'}$.
6. For each vertex $v$, if $d(v) = 2$, put the auxiliary graphs $L(v_x)$ and the directed edges $vv_x$ from $v$ to $v_x$.

Next, we discuss basic properties of the digraph $G(\Phi)$. The digraph $G(\Phi)$ is 3-regular and triangle-free. Let $f$ be a proper 3-coloring, such that for every vertex $v$, there exists a $v$-colorful directed path. We have:

$$\{f(\neg x_j), f(x_j)\} = \{\text{Red, Blue}\}, \quad f(x_j) = f(a^1_{x_j}) = \cdots = f(a^n_{x_j}), \quad f(\neg x_j) = f(a^1_{\neg x_j}) = \cdots = f(a^n_{\neg x_j}), \quad f(s^1_j) = f(s^2_j) = \text{Blue},$$

![Figure 2. The auxiliary digraphs $A(x_j)$, $B(c_j)$ and $T$.](image-url)
Moreover, for every copy of $T$ we have $f(w^1_j) = f(w^1_j)$ and for every copy of $M(z)$ we have $f(z) = f(z_1) = \cdots = f(z_d)$. Also for every copy of $H(x, y, z)$ we have $f(z) = f(z_1) = \cdots = f(z_d)$, $f(x) = f(x_1) = \cdots = f(x_d)$, $f(y) = f(y_1) = \cdots = f(y_d)$, $f(x) = f(v^2_1) = \cdots = f(v^2_{d/2})$ and $f(z) = f(v^1_1) = \cdots = f(v^1_{d/2})$.  First, suppose that $\Phi$ is satisfiable with the satisfying assignment $\Gamma$. Now we present the proper 3-coloring $f$ for $G(\Phi)$, such that for every $v \in V(G(\Phi))$, there exists a $v$-colorful directed path. Let $f(x) = Red$, $f(y) = Blue$ and $f(z) = Black$. Now, for every vertex $v$, $v \in V(H(x, y, z))$, the color of $v$, is determined uniquely. For each variable $x_i$, if $x_i = True$, then let $f(x_i) = Red$ and $f(\neg x_i) = Blue$. Otherwise let $f(x_i) = Blue$ and $f(\neg x_i) = Red$. For every $c_j = l_1 \lor l_2 \lor l_3$, color the vertices of $B(c_j)$ according to the Figure 3. Now, for every vertex $v$, $v \in V(A(x_j))$, the color of $v$, is determined uniquely. Finally, color the vertices of every copy of $L(x)$. It is easy to see that for every $v \in V(G(\Phi))$, there exists a $v$-colorful directed path.

Next, suppose that $G(\Phi)$ has the proper 3-coloring $f$, such that for every $v \in V(G(\Phi))$, there exists a $v$-colorful directed path. With no loss of generality suppose that $f(x) = Red$, $f(y) = Blue$ and $f(z) = Black$. For each variable $x_i$, let $x_i = True$ in $\Gamma$ if and only if $f(x_i) = Red$. Let $c_j = l_1 \lor l_2 \lor l_3$ be an arbitrary clause. We have $f(s^1_j) = f(s^2_j) = Blue$, therefore $Blue \in \{f(c^1_j), f(s^3_j)\}$, so $Blue \in \{f(c^1_j), f(c^2_j), f(c^3_j)\}$. Consequently $Red \in \{f(s^1_{l_1}), f(s^2_{l_1}), f(s^3_{l_1})\}$, so $Red \in \{f(l_1), f(l_2), f(l_3)\}$. Therefore; $\Gamma$ is a satisfying assignment for $\Phi$.

$\square$

**Theorem 2.2.** For every $r \geq 3$, the following problem is NP-complete: ”given an $r$-regular triangle-free digraph $G$ with $\chi(G) = 3$, does there exist a proper 3-coloring of $G$ such that for every $v \in V(G)$, there exists a $v$-colorful directed path?”

**Proof.** The proof is similar to the proof of Theorem 2.1. Consider two disjoint copies of $G(\Phi)$ ($G(\Phi)$ is introduced in the proof of the previous theorem), then for every vertex $v$, put a directed edge from $v$ to the corresponding vertex in the second copy of $G(\Phi)$. By repeating this procedure, we find an $r$-regular triangle-free digraph $G'$ with $2^{r-3}|V(G(\Phi))|$ vertices. Clearly, there exists a proper 3-coloring
of \( G' \) such that for every \( v \in V(G') \), there exists a \( v \)-colorful directed path, if and only if, there exists a proper 3-coloring of \( G(\Phi) \) such that for every \( v \in V(G(\Phi)) \), there exists a \( v \)-colorful directed path.

\( \square \)

**Theorem 2.3.** Colorful Paths is \( \text{NP-complete} \).

**Proof.** Clearly, Colorful Paths is in \( \text{NP} \). We reduce Hamilton Path to this problem (for a given graph \( G \), does \( G \) have a Hamilton path? [5]). Consider a graph \( G \) with \( |V(G)| = n \), as an instance of Hamilton Path. We construct a new graph \( G' \) with the property that, there exists a proper \( \chi(G') \)-coloring of \( G' \) such that for every \( v \in V(G') \), there exists a \( v \)-colorful path, if and only if \( G \) has a Hamilton path. Let \( G' = (G \lor K_1) + K_{n+1} \). If \( G \) has a Hamilton path then \( G \lor K_1 \) has a Hamilton cycle, so there exists a proper \( \chi(G') \)-coloring of \( G' \) such that for every \( v \in V(G') \), there exists a \( v \)-colorful path. Next, suppose that there exists a proper \( \chi(G') \)-coloring \( f \) of \( G' \) such that for every \( v \in V(G') \), there exists a \( v \)-colorful path. Now consider a \( u \)-colorful path \( uv_1v_2 \ldots v_n \) for \( G' \), clearly \( v_1v_2 \ldots v_n \) is a Hamilton path for \( G \). So Colorful Paths is \( \text{NP-complete} \).

\( \square \)

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