NONINNER AUTOMORPHISMS OF FINITE $p$-GROUPS LEAVING THE CENTER ELEMENTWISE FIXED

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Abstract. A longstanding conjecture asserts that every finite nonabelian $p$-group admits a noninner automorphism of order $p$. Let $G$ be a finite nonabelian $p$-group. It is known that if $G$ is regular or of nilpotency class 2 or the commutator subgroup of $G$ is cyclic, or $G/Z(G)$ is powerful, then $G$ has a noninner automorphism of order $p$ leaving either the center $Z(G)$ or the Frattini subgroup $\Phi(G)$ of $G$ elementwise fixed. In this note, we prove that the latter noninner automorphism can be chosen so that it leaves $Z(G)$ elementwise fixed.

1. Introduction

One of the most widely known, although nontrivial, properties of finite $p$-groups of order greater than $p$ is that they always have a noninner automorphism $\alpha$ of $p$-power order. This fact was first proved by Gaschütz in 1966 [5]. Schmid [8] extended Gaschütz’s result by showing that if $G$ is a finite nonabelian $p$-group, then the automorphism $\alpha$ can be chosen to act trivially on the center. A longstanding conjecture that had been raised even before Gaschütz’s result is the following

Conjecture 1. Every finite nonabelian $p$-group admits a noninner automorphism of order $p$.

Indeed, in 1964 Liebeck [7] proved that if $p$ is an odd prime and $G$ is a finite $p$-group of class 2 then $G$ has a noninner automorphism of order $p$ acting trivially on the Frattini subgroup $\Phi(G)$. The corresponding result for 2-groups is false in general, as Liebeck himself produced an example of a 2-group $G$ of class 2 with the property that all automorphisms of order two leaving $\Phi(G)$ elementwise
fixed are inner. By a cohomological result of Schmid [9], it follows that finite regular nonabelian \( p \)-groups admit a noninner automorphism leaving the Frattini subgroup elementwise fixed. Deaconescu and Silberberg [4] proved that if \( C_G(Z(\Phi(G))) \neq \Phi(G) \), then the noninner automorphism can be chosen to act trivially on \( \Phi(G) \). Hence the main result of [4] reduced the verification of Conjecture 1 to finite nonabelian \( p \)-groups satisfying the condition \( C_G(Z(\Phi(G))) = \Phi(G) \). In [1, 2, 3] it is proved that if \( G \) is a finite nonabelian \( p \)-group of class at most 3 or \( G/Z(G) \) is powerful, then \( G \) has a noninner automorphism of order \( p \) leaving either \( \Phi(G) \) or \( \Omega_1(Z(G)) \) elementwise fixed. Jamali and Viseh [6] proved that every nonabelian finite 2-group with cyclic commutator subgroup has a noninner automorphism of order two leaving either \( \Phi(G) \) or \( Z(G) \) elementwise fixed. They have also observed that the results of [1, 2] can be improved, that is, if \( G \) is of nilpotency class 2 or \( G/Z(G) \) is powerful, then \( G \) has a noninner automorphism of order \( p \) leaving either the center \( Z(G) \) or Frattini subgroup elementwise fixed. Therefore the following result holds.

**Proposition 1.1.** Let \( G \) be a finite nonabelian \( p \)-group satisfying one of the following conditions:

1. \( G \) is regular;
2. \( G \) is nilpotent of class 2;
3. the commutator subgroup of \( G \) is cyclic;
4. \( G/Z(G) \) is powerful.

Then \( G \) has a noninner automorphism of order \( p \) leaving either \( Z(G) \) or \( \Phi(G) \) elementwise fixed.

The main result of our paper is the following.

**Theorem 1.2.** Let \( G \) be a finite nonabelian \( p \)-group satisfying one of the following conditions:

1. \( G \) is regular;
2. \( G \) is nilpotent of class 2;
3. the commutator subgroup of \( G \) is cyclic;
4. \( G/Z(G) \) is powerful.

Then \( G \) has a noninner automorphism of order \( p \) leaving \( Z(G) \) elementwise fixed.

2. **Proof of the main result**

We need the following result which may be well-known. We prove it for the reader’s convenience.

**Lemma 2.1.** Let \( G \) be any finite \( p \)-group. Then \( G = AH \) for some subgroups \( A \) and \( H \) such that \( A \leq Z(G) \) and \( Z(H) \leq \Phi(H) \).

**Proof.** We prove Lemma by induction on \(|G|\). If \( G \) is abelian then the assertion is clear, take \( A = G \) and \( H = 1 \). Now let \( G \) be a finite nonabelian \( p \)-group and assume that the assertion holds for all \( p \)-groups of order less than \(|G|\). Moreover we may assume that \( Z(G) \leq \Phi(G) \), otherwise one may take \( A = 1 \) and \( H = G \) to complete the proof. Thus there exist some element \( a \in Z(G) \) and a maximal subgroup \( M \) of \( G \) such that \( a \notin M \). By induction hypothesis \( M = BH \) for some subgroups \( B \) and \( H \) of \( M \) such that \( B \leq Z(M) \) and \( Z(H) \leq \Phi(H) \). Let \( A = \langle a, B \rangle \). Therefore \( A \leq Z(G) \) and \( G = AH \). This completes the proof. \( \Box \)
Remark 2.2 ([4, Remark 4.]). Let $G$ be a central product of subgroups $A$ and $B$; i.e., $G = AB$ and $[A, B] = 1$. Suppose that $\alpha \in \text{Aut}(A)$ and $\beta \in \text{Aut}(B)$ agree on $A \cap B$. Then $\alpha$ and $\beta$ admit a common extension $\gamma \in \text{Aut}(G)$. In particular, if $A$ has a noninner automorphism of order $p$ which fixes $Z(A)$ elementwise, then $G$ has a noninner automorphism of order $p$ leaving both $Z(A)$ and $B$ elementwise fixed.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $G$ be a finite nonabelian $p$-group. By Lemma 2.1, we have $G = AH$ for some subgroups $A$ and $H$ of $G$ such that $A \leq Z(G)$ and $Z(H) \leq \Phi(H)$. If $G$ is regular, or of nilpotency class 2, or with cyclic commutator subgroup, then so is $H$. Now, suppose that $G/Z(G)$ is powerful. If $p > 2$, then $H'Z(G)/Z(G) \leq G'/Z(G)/Z(G) \leq G^pZ(G)/Z(G)$. Thus $H' \leq G^pZ(G) = H^pZ(G)$, since $G^p = AH^p$. Now if $c \in H'$, then $c = ba$ for some $b \in H^p$ and $a \in Z(G)$. But $b^{-1}c = a \in Z(H)$. Therefore $H' \leq H^pZ(H)$ and this means that $H/Z(H)$ is powerful. A similar argument shows that $H/Z(H)$ is powerful for $p = 2$. Then, by Proposition 1.1, $H$ has a noninner automorphism of order $p$ fixing $Z(H)$ elementwise. Now it follows from Remark 2.2 that $G$ has a noninner automorphism of order $p$ leaving $AZ(H) = Z(G)$ elementwise fixed. This completes the proof. □

We finish the paper with the following conjecture.

Conjecture 2. Every finite nonabelian $p$-group admits a noninner automorphism of order $p$ leaving the center elementwise fixed.

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