THE PRIME GRAPH CONJECTURE FOR INTEGRAL GROUP RINGS OF SOME ALTERNATING GROUPS

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Abstract. We investigate the classical H. Zassenhaus conjecture for integral group rings of alternating groups $A_9$ and $A_{10}$ of degree 9 and 10, respectively. As a consequence of our previous results we confirm the Prime Graph Conjecture for integral group rings of $A_n$ for all $n \leq 10$.

1. Introduction and main results

Let $G$ be a finite group and let $V(ZG)$ denote the group of all normalized units of the integral group ring $ZG$ of $G$. In [30], H. Zassenhaus proposed the following conjecture

(ZC): Every torsion unit $u$ in $V(ZG)$ conjugates to some element $g$ in $G$ within the rational group algebra $QG$.

Let $\pi(H)$ denote the Gruenberg-Kegel (prime) graph of a group $H$ (not necessarily finite); i.e., the graph whose vertices are labeled by primes $p$ for which there exists an element of order $p$ in $H$ and with an edge from $p$ to a distinct prime $q$ if and only if $H$ has an element of order $pq$. In [25] (see also [23]), the following weaker version of (ZC) was proposed. We may call it the Prime Graph Conjecture:

(PGC): $\pi(V(ZG)) = \pi(G)$ for any finite group $G$.

The question about (ZC) remains open as no counterexample is known up to date. For nilpotent groups, (ZC) has been proved independently by K.W. Roggenkamp and L.L. Scott in [26] and by A. Weiss in [29]. But their method can not be applied to simple groups. However, using a new method based on the partial augmentation of a torsion unit, I.S. Luthar and I.B.S. Passi in [24] confirmed (ZC) for the alternating group $A_5$ of degree 5. Also, in [27, 28, 21] a positive answer for

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(PGC) was given for several new classes of groups, in particular for the alternating groups \(A_6, A_7\) and \(A_8\).

Recently, the (PGC) has been investigated in several papers. A positive answer has been given for solvable groups, Frobenius groups and almost for several simple groups in [23, 3] and [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19], respectively. Also, for non simple groups, see [3, 4, 16].

Here we continue our study of (ZC) for alternating groups. Our main results are given in the following two theorems.

**Theorem 1.1.** Let \(G\) denote the alternating group \(A_9\). For a torsion unit \(u\) in \(V(ZG)\) of order \(|u|\), denote the partial augmentation of \(u\) by

\[
P(u) = (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{6b},
\nu_7, \nu_9, \nu_{10a}, \nu_{10b}, \nu_{12a}, \nu_{15a}, \nu_{15b}) \in \mathbb{Z}^{17}.
\]

The following hold:

(i) There are no units of order 14, 21 and 35 in \(V(ZG)\).
(ii) If \(|u| \in \{5, 7\}\), then \(u\) is rationally conjugate to some \(g \in G\).
(iii) If \(|u| = 2\), then the tuple of the partial augmentations of \(u\) belongs to the set

\[
\{ P(u) \in \mathbb{Z}^{17} \mid (\nu_{2a}, \nu_{2b}) \in \{(0, 1), (2, -1), (1, 0), (-1, 2) \},
\nu_{kx} = 0, kx \notin \{2a, 2b\} \}
\]

(iv) If \(|u| = 3\), then the tuple of the partial augmentations of \(u\) belongs to the set

\[
\{ P(u) \in \mathbb{Z}^{17} \mid (\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{(0, -1, 2), (1, -1, 1), (1, 0, 0),
(-1, 0, 2), (0, 0, 1), (0, 2, -1),
(0, -2, 3), (0, 1, 0), (-1, 1, 1) \},
\nu_{kx} = 0, kx \notin \{3a, 3b, 3c\} \}
\]

(v) If \(|u| = 10\), then the tuple of the partial augmentations of \(u\) belongs to the set

\[
\{ P(u) \in \mathbb{Z}^{17} \mid (\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}) \in \{(0, 0, 0, 1), (1, 1, 0, -1) \},
\nu_{kx} = 0, kx \notin \{2a, 2b, 5a, 10a\} \}
\]

**Theorem 1.2.** Let \(G\) denote the alternating group \(A_{10}\). For a torsion unit \(u\) in \(V(ZG)\) of order \(|u|\), denote the partial augmentation of the element \(u\) by

\[
P(u) = (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{4a}, \nu_{4b}, \nu_{4c}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{6c}, \nu_{7}, \nu_{8a}, \nu_{9a}, \nu_{9b}, \nu_{10a}, \nu_{12a}, \nu_{12b}, \nu_{15a}, \nu_{21a}, \nu_{21b}) \in \mathbb{Z}^{23}.
\]

The following hold:

(i) There are no units of order 14 and 35 in \(V(ZG)\).
(ii) If \(|u| \in \{5, 7\}\), then \(u\) is rationally conjugate to some \(g \in G\).
(iii) If $|u| = 2$, then the tuple of the partial augmentations of $u$ belongs to the set
\[ \{ P(u) \in \mathbb{Z}^{23} \mid (\nu_{2a}, \nu_{2b}) \in \{ (0, 1), (2, -1), (1, 0), (-1, 2) \}, \nu_{kk} = 0, \ kx \notin \{2a, 2b\} \}. \]

(iv) If $|u| = 3$, then the tuple of the partial augmentations of $u$ belongs to the set
\[ \{ P(u) \in \mathbb{Z}^{23} \mid (\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{ (0, -1, 2), (0, 3, -2), (1, 0, 0), (0, 0, 1), (0, 2, -1), (-1, 2, 0), (1, 1, -1), (0, 1, 0), (-1, 1, 1) \}, \nu_{kk} = 0, \ kx \notin \{3a, 3b, 3c\} \}. \]

As an immediate consequence of the first parts of Theorems 1.1, 1.2 and [21, 27, 28] we obtain the solution of the Prime Graph Conjecture for $A_n$:

**Corollary 1.3.** For all $n \leq 10$, if $G = A_n$, then $\pi(G) = \pi(V(\mathbb{Z}G))$.

**2. Preliminaries**

Let $G$ be a finite group and let $C = \{C_1, C_{kk} \mid x \in \{a, b, \ldots\}, \ k \geq 2\}$ be the collection of all conjugacy classes of $G$, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Supposing that the torsion unit $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has order $k$, denote the partial augmentation of $u$ with respect to the conjugacy class $C_{nt}$ by $\nu_{nt} = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$. Denote the tuple of partial augmentations of the unit $u$ by
\[ P(u) = (\nu_{kk} \mid x \in \{a, b, \ldots\}, k \geq 2) \in \mathbb{Z}^l, \]
where $l + 1$ is the number of conjugacy classes of $G$.

From Higman-Berman’s Theorem [1] one knows that $\nu_1 = \alpha_1 = 0$ and
\[ \sum_{C_{nt} \in C} \nu_{nt} = 1. \]
Hence, for any character $\chi$ of $G$, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where $h_{nt}$ is a representative of a conjugacy class $C_{nt}$. Throughout the paper the $p$-Brauer character table of the group $G$ will be denoted by $\mathfrak{BC}(p)$, which can be found using the computational algebra system GAP [18]. Clearly, if $G \in \{A_9, A_{10}\}$, then the prime number $p$ has value $p \in \{2, 3, 5, 7\}$.

Through the proofs of the main results we use the following propositions from [17, 20, 22, 24].

**Proposition 2.1.** (see [21, 22]) Let either $p = 0$ or $p$ be a prime divisor of $|G|$ and let $F$ be the associated prime field. Suppose that $u \in V(\mathbb{Z}G)$ has finite order $k$ and assume $k$ and $p$ are coprime in case $p \neq 0$. If $z$ is a primitive $k$-th root of unity and $\chi$ is either a classical character or a $p$-Brauer character of $G$ then for every integer $l$ the number
\[
\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d \mid k} Tr_{F(z^d)/F} \{ \chi(u^d)z^{-dl} \}
\]
is a non-negative integer.

For \( p = 0 \) we will use the notation \( \mu_{\ell}(u, \chi, *) \) for \( \mu_{\ell}(u, \chi, 0) \).

**Proposition 2.2.** (see [17]) The order of any unit \( u \in V(\mathbb{Z}G) \) is a divisor of the exponent of \( G \).

**Proposition 2.3.** (see [21]) Let \( u \) be a torsion unit of \( V(\mathbb{Z}G) \). Let \( C \) be a conjugacy class of \( G \). If \( a \in C \) and \( p \) is a prime dividing the order of \( a \) but not the order of \( u \) then \( \varepsilon_{C}(u) = 0 \).

M. Hertweck ([20], Proposition 3.1; [22], Lemma 5.6) obtained the next results. These already yield that several partial augmentations of the torsion units are zero.

**Proposition 2.4.** Let \( G \) be a finite group and let \( u \) be a torsion unit in \( V(\mathbb{Z}G) \). If \( x \) is an element of \( G \) whose \( p \)-part, for some prime \( p \), has order strictly greater than the order of the \( p \)-part of \( u \), then \( \varepsilon_{x}(u) = 0 \).

**Proposition 2.5.** (see [21]) Let \( u \in V(\mathbb{Z}G) \) be of order \( k \). Then \( u \) is conjugate in \( \mathbb{Q}G \) to an element \( g \in G \) if and only if for each \( d \) dividing \( k \) there is precisely one conjugacy class \( C \) with partial augmentation \( \varepsilon_{C}(u^{d}) \neq 0 \).

### 3. Proof of the Theorems

**Proof of Theorem 1.1.** Let \( G = A_{9} \). It is well known that \( |G| = 181440 = 2^{6} \cdot 3^{4} \cdot 5 \cdot 7 \) and \( \exp(\mathbb{Z}G) = 1260 = 2^{2} \cdot 3^{2} \cdot 5 \cdot 7 \). The \( p \)-Brauer character tables are available for primes \( p \in \{2, 3, 5, 7\} \). The group \( G \) possesses elements of orders \( 2, 3, 4, 5, 6, 7, 9, 10, 12 \) and \( 15 \). First we investigate units of orders \( 2, 3, 5, 7 \) and \( 10 \). Secondly, according to Proposition 2.2 the order of each torsion unit divides the exponent of \( G \), so the possible orders for units are: \( 14, 18, 20, 24, 30, 35, 45 \) and \( 63 \). We prove that units of orders \( 14, 21, 25 \) and \( 35 \) do not appear in \( V(\mathbb{Z}G) \).

- Let \( u \) be an involution. Then we have \( \nu_{2a} + \nu_{2b} = 1 \) by Propositions 2.3 and 2.4. According to (2.1) we get the following system of three inequalities:

\[
\mu_{0}(u, \chi, e) = \frac{1}{2}(4\nu_{2a} + 8) \geq 0; \quad \mu_{1}(u, \chi, e) = \frac{1}{2}(-4\nu_{2a} + 8) \geq 0; \\
\mu_{0}(u, \chi, 3) = \frac{1}{2}(3\nu_{2a} - \nu_{2b} + 7) \geq 0,
\]

which has the four integral solutions listed in part (iii) of the Theorem.

- Let \( u \) be a unit of order 3. Then \( \nu_{3a} + \nu_{3b} + \nu_{3c} = 1 \) by Propositions 2.3 and 2.4. Put \( t_{1} = 5\nu_{3a} - \nu_{3b} + 2\nu_{3c} \) and \( t_{2} = 4\nu_{3a} + \nu_{3b} - 2\nu_{3c} \). Then by (2.1) we have that

\[
\mu_{0}(u, \chi, e) = \frac{1}{3}(2t_{1} + 8) \geq 0; \quad \mu_{1}(u, \chi, e) = \frac{1}{3}(-t_{1} + 8) \geq 0; \\
\mu_{0}(u, \chi, 5) = \frac{1}{3}(18\nu_{3a} + 27) \geq 0; \quad \mu_{1}(u, \chi, 5) = \frac{1}{3}(-9\nu_{3a} + 27) \geq 0; \\
\mu_{0}(u, \chi, 3) = \frac{1}{3}(-2t_{2} + 8) \geq 0; \quad \mu_{1}(u, \chi, 3) = \frac{1}{3}(t_{2} + 8) \geq 0.
\]

From the first two inequalities we get that \( t_{1} \in \{-4, -1, 2, 5, 8\} \) and from the next two we get that \( \nu_{3a} \in \{-1, 0, 1, 2, 3\} \). Considering the last two inequalities, we obtain the 6 non-trivial and 3 trivial integral solutions listed in part (iv) of the Theorem.
• Let \( u \) be a unit of order either 5 or 7. Then by Propositions \( 2.3 \) and \( 2.4 \), the only nonzero partial augmentation is \( \nu_{5a} = 1 \) or \( \nu_{7a} = 1 \), respectively. According to Proposition \( 2.5 \), such unit \( u \) is rationally conjugate to an element \( g \) in \( G \).

• Let \( u \) be a unit of order 10. Then \( \nu_{2a} + \nu_{2b} + \nu_{5a} + \nu_{10a} = 1 \) by Propositions \( 2.3 \) and \( 2.4 \). Put \( t_1 = 4\nu_{2a} + 3\nu_{5a} - \nu_{10a} \), \( t_2 = \nu_{2a} - 3\nu_{2b} + \nu_{5a} + \nu_{10a} \) and \( t_3 = 7\nu_{2a} + 3\nu_{2b} + 2\nu_{5a} + 2\nu_{10a} \). Since \( u^5 \) is an involution, we need to consider the following four cases:

\[
\chi(u^5) \in \{\chi(2a), \; \chi(2b), \; 2\chi(2a) - \chi(2b), \; -\chi(2a) + 2\chi(2b)\}.
\]

Consider each case separately:

Case 1. Let \( \chi(u^5) = \chi(2a) \) and \( \chi(u^2) = \chi(5a) \). By \( (2.1) \) we obtain the following system of inequalities:

\[
\begin{align*}
\mu_1(u, \chi_2, *) &= \frac{1}{10}(t_1 + 1) \geq 0; & \mu_5(u, \chi_2, *) &= \frac{1}{10}(-4t_1 + 16) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{10}(4t_2 + 26) \geq 0; & \mu_5(u, \chi_3, *) &= \frac{1}{10}(-4t_2 + 24) \geq 0; \\
\mu_1(u, \chi_3, *) &= \frac{1}{10}(t_2 + 19) \geq 0; & \mu_1(u, \chi_5, *) &= \frac{1}{10}(t_3 + 18) \geq 0; \\
\mu_0(u, \chi_5, *) &= \frac{1}{10}(4t_5 + 42) \geq 0; & \mu_5(u, \chi_5, *) &= \frac{1}{10}(-4t_3 + 28) \geq 0.
\end{align*}
\]

It is easy to check that \( t_1 = -1, \; t_2 = 1 \) and \( t_3 \in \{-8, 2\} \), which has the following integral solution: \( (0, 0, 0, 1) \).

Case 2. Let \( \chi(u^5) = \chi(2b) \) and \( \chi(u^2) = \chi(5a) \). By \( (2.1) \) we have

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{10}(4t_1 + 20) \geq 0; & \mu_2(u, \chi_2, *) &= \frac{1}{10}(-4t_1 + 5) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{10}(4t_2 + 22) \geq 0; & \mu_5(u, \chi_3, *) &= \frac{1}{10}(-4t_2 + 28) \geq 0; \\
\mu_1(u, \chi_3, *) &= \frac{1}{10}(t_2 + 23) \geq 0; & \mu_1(u, \chi_5, *) &= \frac{1}{10}(t_3 + 22) \geq 0; \\
\mu_0(u, \chi_5, *) &= \frac{1}{10}(4t_5 + 38) \geq 0; & \mu_5(u, \chi_5, *) &= \frac{1}{10}(-4t_3 + 32) \geq 0; \\
\mu_0(u, \chi_2, 3) &= \frac{1}{10}(12\nu_{2a} - 4\nu_{2b} + 8\nu_{5a} - 8\nu_{10a} + 14) \geq 0.
\end{align*}
\]

It follows that \( t_1 \in \{-5, 5\}, \; t_2 \in \{-3, 7\} \) and \( t_3 \in \{-2, 8\} \), which has the following integral solution: \( (1, 1, 0, -1) \).

Case 3. Let \( \chi(u^5) = 2\chi(2a) - \chi(2b) \) and \( \chi(u^2) = \chi(5a) \). By \( (2.1) \) we have

\[
\begin{align*}
\mu_1(u, \chi_2, *) &= \frac{1}{10}(t_1 - 3) \geq 0; & \mu_5(u, \chi_2, *) &= \frac{1}{10}(-4t_1 + 12) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{10}(4t_2 + 30) \geq 0; & \mu_5(u, \chi_3, *) &= \frac{1}{10}(-4t_2 + 20) \geq 0; \\
\mu_1(u, \chi_3, *) &= \frac{1}{10}(t_2 + 15) \geq 0; & \mu_1(u, \chi_5, *) &= \frac{1}{10}(t_3 + 14) \geq 0; \\
\mu_0(u, \chi_5, *) &= \frac{1}{10}(4t_5 + 46) \geq 0; & \mu_5(u, \chi_5, *) &= \frac{1}{10}(-4t_3 + 24) \geq 0; \\
\mu_0(u, \chi_7, *) &= \frac{1}{10}(-20\nu_{2a} + 12\nu_{2b} + 22) \geq 0.
\end{align*}
\]

It follows that \( t_1 = 3, \; t_2 \in \{-5, 5\} \) and \( t_3 \in \{-4, 6\} \), which has no integral solution.
Case 4. Let $\chi(u^7) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Using (2.1) we obtain the following system of inequalities:

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{14}(4t_1 + 16) \geq 0; & \mu_2(u, \chi_2, *) &= \frac{1}{14}(-t_1 + 1) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{14}(4t_2 + 18) \geq 0; & \mu_5(u, \chi_3, *) &= \frac{1}{14}(-4t_2 + 32) \geq 0; \\
\mu_1(u, \chi_3, *) &= \frac{1}{14}(t_2 + 27) \geq 0; & \mu_1(u, \chi_5, *) &= \frac{1}{10}(t_3 + 26) \geq 0; \\
\mu_0(u, \chi_5, *) &= \frac{1}{14}(4t_3 + 34) \geq 0; & \mu_5(u, \chi_5, *) &= \frac{1}{14}(-4t_3 + 36) \geq 0.
\end{align*}
\]

It follows that $t_1 = 1$, $t_2 = 3$ and $t_3 \in \{-6, 4\}$, which has no integral solution.

- Let $u$ be a unit of order 14. Then we have $\nu_{2a} + \nu_{2b} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $\nu_1 = 4\nu_{2a} + \nu_{7a}$ and $t_2 = \nu_{2a} - 3\nu_{2b}$. Since $\chi(u^7)$ has order 2, according to the previous cases we need to consider the following four cases:

\[\chi(u^7) \in \{\chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b)\}.\]

Consider each case separately:

Case 1. Let $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{14}(6t_1 + 18) \geq 0; & \mu_7(u, \chi_2, *) &= \frac{1}{14}(-6t_1 + 10) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{14}(6t_2 + 22) \geq 0; & \mu_7(u, \chi_3, *) &= \frac{1}{14}(-6t_2 + 20) \geq 0; \\
\mu_1(u, \chi_3, *) &= \frac{1}{13}(t_2 + 20) \geq 0,
\end{align*}
\]

which has no integral solutions.

Case 2. Let $\chi(u^7) = \chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{14}(6t_1 + 14) \geq 0; & \mu_7(u, \chi_2, *) &= \frac{1}{14}(-6t_1 + 14) \geq 0; \\
\mu_1(u, \chi_2, *) &= \frac{1}{13}(t_1 + 7) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{14}(6t_2 + 26) \geq 0; & \mu_7(u, \chi_3, *) &= \frac{1}{14}(-6t_2 + 16) \geq 0,
\end{align*}
\]

which has no integral solutions.

Case 3. Let $\chi(u^7) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we obtain the following system of inequalities

\[
\begin{align*}
\mu_1(u, \chi_2, *) &= \frac{1}{14}(t_1 - 1) \geq 0; & \mu_7(u, \chi_2, *) &= \frac{1}{14}(-6t_1 + 6) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{14}(6t_2 + 26) \geq 0; & \mu_7(u, \chi_3, *) &= \frac{1}{14}(-6t_2 + 16) \geq 0.
\end{align*}
\]

It follows that $t_1 = 1$ and $t_2 = -2$ which has no integral solutions.

Case 4. Let $\chi(u^7) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{14}(6t_1 + 10) \geq 0; & \mu_2(u, \chi_2, *) &= \frac{1}{14}(-t_1 + 3) \geq 0; \\
\mu_0(u, \chi_3, *) &= \frac{1}{14}(6t_2 + 14) \geq 0; & \mu_7(u, \chi_3, *) &= \frac{1}{14}(-6t_2 + 28) \geq 0.
\end{align*}
\]

It follows that $t_1 = 3$ and $t_2 = 0$ which has no integral solutions.
Let $u$ be a unit of order 21. Then $\nu_{3a} + \nu_{3b} + \nu_{3c} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 5\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{7a}$ and $t_2 = \nu_{3a} - \nu_{3b}$. Since $\chi(u^7)$ has order 3, according to previous cases we need to consider the following nine cases:

\[
\chi(u^7) \in \{\chi(3a), \chi(3b), \chi(3c), -2\chi(3b) + 3\chi(3c), \\
\chi(3a) - \chi(3b) + \chi(3c), -\chi(3a) + \chi(3b) + \chi(3c) \\
-\chi(3a) + 2\chi(3c), 2\chi(3b) - \chi(3c), -\chi(3b) + 2\chi(3c)\}.
\]

Consider each case separately:

**Case 1.** Let $\chi(u^7) = \chi(3a)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have that:

\[
\mu_0(u, \chi_2, *) = \frac{1}{21} (12t_1 + 24) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{21} (-6t_1 + 9) \geq 0;
\]

\[
\mu_7(u, \chi_3, *) = \frac{1}{21} (18t_2 + 24) \geq 0; \quad \mu_0(u, \chi_3, *) = \frac{1}{21} (-36t_2 + 15) \geq 0,
\]

which has no integral solutions.

**Case 2.** Let $\chi(u^7) = \chi(3b)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the following system of two inequalities:

\[
\mu_0(u, \chi_2, *) = \frac{1}{21} (12t_1 + 12) \geq 0; \quad \mu_3(u, \chi_2, *) = \frac{1}{21} (-2t_1 + 5) \geq 0.
\]

Clearly, it has no integral solution.

**Case 3.** Let $\chi(u^7) = \chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have that:

\[
\mu_0(u, \chi_2, *) = \frac{1}{21} (12t_1 + 18) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{21} (-6t_1 + 12) \geq 0;
\]

\[
\mu_1(u, \chi_2, *) = \frac{1}{21} (t_1 + 5) \geq 0.
\]

It has no integral solution.

**Case 4.** Let $\chi(u^7) = -2\chi(3b) + 3\chi(3c)$ and $\chi(u^3) = \chi(7a)$. According to (2.1) we are able to construct the following system of inequalities:

\[
\mu_1(u, \chi_2, *) = \frac{1}{21} (t_1 - 1) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{21} (-6t_1 + 6) \geq 0;
\]

\[
\mu_3(u, \chi_3, *) = \frac{1}{21} (6t_2 + 9) \geq 0; \quad \mu_0(u, \chi_3, *) = \frac{1}{21} (-36t_2 + 9) \geq 0,
\]

which has no integral solutions.

**Case 5.** Let $\chi(u^7) = \chi(3a) - \chi(3b) + \chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the system:

\[
\mu_1(u, \chi_2, *) = \frac{1}{21} (t_1 - 1) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{21} (-6t_1 + 6) \geq 0;
\]

\[
\mu_3(u, \chi_3, *) = \frac{1}{21} (6t_2 + 9) \geq 0; \quad \mu_0(u, \chi_3, *) = \frac{1}{21} (-36t_2 + 9) \geq 0,
\]

which has no integral solutions.

**Case 6.** Let $\chi(u^7) = -\chi(3b) + 2\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the system of four inequalities:

\[
\mu_0(u, \chi_2, *) = \frac{1}{21} (12t_1 + 24) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{21} (-6t_1 + 9) \geq 0;
\]

\[
\mu_7(u, \chi_3, *) = \frac{1}{21} (18t_2 + 24) \geq 0; \quad \mu_0(u, \chi_3, *) = \frac{1}{21} (-36t_2 + 15) \geq 0,
\]

which has no integral solutions.
Case 7. Let $\chi(u^7) = -\chi(3a) + 2\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we get the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{27}(5t_1 + 12) \geq 0; \quad \mu_3(u, \chi_2, *) = \frac{1}{27}(-12t_1 + 5) \geq 0,$$

which has no integral solutions.

Case 8. Let $\chi(u^7) = -\chi(3a) + \chi(3b) + \chi(3c)$ and $\chi(u^3) = \chi(7a)$. According to (2.1) we are able to construct the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{27}(12t_1 + 6) \geq 0; \quad \mu_3(u, \chi_2, *) = \frac{1}{27}(-2t_1 - 1) \geq 0,$$

which has no integral solutions.

Case 9. Let $\chi(u^7) = 2\chi(3b) - \chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we obtain the following unsolvable system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{27}(t_1 + 6) \geq 0; \quad \mu_3(u, \chi_2, *) = \frac{1}{27}(-t_1 - 1) \geq 0.$$

Let $u$ be a unit of order 35. Then $\nu_a + \nu_7 = 1$ by Propositions 2.3 and 2.4. Clearly $\chi(u^7) = \chi(5a)$ and $\chi(u^3) = \chi(7a)$. Put $t_1 = 3\nu_a + \nu_7$. Then by (2.1) we have the system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24t_1 + 26) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{35}(-6t_1 + 11) \geq 0,$$

which has no integral solution.

Proof of Theorem 1.2 Let $G = A_{10}$. It is well known that $|G| = 1814400 = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ and $\exp(G) = 2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. The group $G$ possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15 and 21. First we investigate units of orders 2, 3, 5 and 7. Secondly, according to Proposition 2.2 the order of each torsion unit divides the exponent of $G$, so the possible orders for units are: 14, 18, 20, 24, 30, 35, 45 and 63. We prove that units of orders 14 and 35 do not appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

• Let $u$ be a unit of order 2. Then $\nu_{2a} + \nu_{2b} = 1$ by Propositions 2.3 and 2.4. Put $t = 5\nu_{2a} + \nu_{2b}$. By (2.1) we obtain that

$$\mu_0(u, \chi_2, *) = \frac{1}{2}(t + 9) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{2}(-t + 9) \geq 0,$$

which has the 5 integral solutions listed in part (iii) of the Theorem.

• Let $u$ be a unit of order 3. Then we have $\nu_{3a} + \nu_{3b} + \nu_{3c} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 2\nu_{3a} + \nu_{3b}, t_2 = 14\nu_{3a} + 2\nu_{3b} - \nu_{3c}$ and $t_3 = 8\nu_{3a} - 4\nu_{3b} + 2\nu_{3c}$. According to (2.1) we obtain the system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{3}(6t_1 + 9) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{3}(-3t_1 + 9) \geq 0;$$

$$\mu_0(u, \chi_3, *) = \frac{1}{3}(2t_2 + 35) \geq 0; \quad \mu_1(u, \chi_3, *) = \frac{1}{3}(-t_2 + 35) \geq 0;$$

$$\mu_0(u, \chi_3, 2) = \frac{1}{3}(-2t_3 + 16) \geq 0; \quad \mu_1(u, \chi_3, 2) = \frac{1}{3}(t_3 + 16) \geq 0.$$

It is easy to see that this system has 6 non-trivial and 3 trivial integral solutions, which are listed in Theorem 1.2 (iv).
• Let \( u \) be a unit of order 5. Then \( \nu_{5a} + \nu_{5b} = 1 \) by Propositions 2.3 and 2.4. The system of two inequalities constructed by (2.1)
\[
\mu_0(u, \chi_2, *) = \frac{1}{5}(16\nu_{5a} - 4\nu_{5b} + 9) \geq 0; \\
\mu_0(u, \chi_3, 2) = \frac{1}{5}(-16\nu_{5a} + 4\nu_{5b} + 16) \geq 0,
\]
has only two trivial integral solutions: \((\nu_{5a}, \nu_{5b}) \in \{(0, 1), (1, 0)\}\).

• Let \( u \) be a unit of order 7. Then by Propositions 2.3 and 2.4 \( \nu_{7a} = 1 \) and \( \nu_{7a} = 0 \) for \( kx \neq 7a \). According to Propositions 2.5 the unit \( u \) is rationally conjugate to an element \( g \in G \).

• Let \( u \) be a unit of order 14. Then \( \nu_{2a} + \nu_{2b} + \nu_{7a} = 1 \) by Propositions 2.3 and 2.4. Put \( t_1 = 5\nu_{2a} + \nu_{2b} + 2\nu_{7a} \) and \( t_2 = 11\nu_{2a} + 3\nu_{2b} \). Since
\[
\chi(u^7) \in \{\chi(2a), \ \chi(2b), \ 2\chi(2a) - \chi(2b), \\
-\chi(2a) + 2\chi(2b), \ -2\chi(2a) + 3\chi(2b)\}
\]
we need to consider the following five cases:

Case 1. Let \( \chi(u^7) = \chi(2a) \) and \( \chi(u^2) = \chi(7a) \). By (2.1) we obtain that
\[
\mu_1(u, \chi_2, *) = \frac{1}{14}(t_1 + 2) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{14}(-6t_1 + 16) \geq 0; \\
\mu_0(u, \chi_3, *) = \frac{1}{14}(6t_2 + 46) \geq 0; \quad \mu_1(u, \chi_3, *) = \frac{1}{14}(t_2 + 24) \geq 0; \\
\mu_7(u, \chi_3, *) = \frac{1}{14}(-6t_2 + 24) \geq 0.
\]
It follows that \( t_1 = 5 \) and \( t_2 = 4 \), which has no integral solution.

Case 2. Let \( \chi(u^7) = \chi(2b) \) and \( \chi(u^2) = \chi(7a) \). According to (2.1)
\[
\mu_0(u, \chi_2, *) = \frac{1}{14}(6t_1 + 22) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{14}(-6t_1 + 20) \geq 0; \\
\mu_1(u, \chi_2, *) = \frac{1}{14}(t_1 + 6) \geq 0.
\]
From the first two equations we get \( t_1 = 1 \), which contradicts the third one. So this system has no integral solutions.

Case 3. Let \( \chi(u^7) = 2\chi(2a) - \chi(2b) \) and \( \chi(u^2) = \chi(7a) \). By (2.1) we have
\[
\mu_1(u, \chi_2, *) = \frac{1}{14}(t_1 - 2) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{14}(-6t_1 + 12) \geq 0; \\
\mu_0(u, \chi_3, *) = \frac{1}{14}(6t_2 + 54) \geq 0; \quad \mu_1(u, \chi_3, *) = \frac{1}{14}(t_2 + 16) \geq 0; \\
\mu_7(u, \chi_3, *) = \frac{1}{14}(-6t_2 + 16) \geq 0.
\]
It follows that \( t_1 = 2 \) and \( t_2 = 2 \), which has no integral solution.

Case 4. Let \( \chi(u^7) = -\chi(2a) + 2\chi(2b) \) and \( \chi(u^2) = \chi(7a) \). By (2.1)
\[
\mu_0(u, \chi_2, *) = \frac{1}{14}(6t_1 + 18) \geq 0; \quad \mu_2(u, \chi_2, *) = \frac{1}{14}(-t_1 + 4) \geq 0; \\
\mu_0(u, \chi_3, *) = \frac{1}{14}(6t_2 + 30) \geq 0; \quad \mu_1(u, \chi_3, *) = \frac{1}{14}(t_2 + 40) \geq 0; \\
\mu_7(u, \chi_3, *) = \frac{1}{14}(-6t_2 + 40) \geq 0.
\]
Clearly, \( t_1 = 4 \) and \( t_2 = 2 \). It is easy to check that such system of equations has no integral solution.
Case 5. Let \( \chi(u^7) = -2\chi(2a) + 3\chi(2b) \) and \( \chi(u^2) = \chi(7a) \). By (2.1) we have the following system of inequalities:
\[
\mu_0(u, \chi_2, *) = \frac{1}{14} (6t_1 + 14) \geq 0; \quad \mu_2(u, \chi_2, *) = \frac{1}{14} (-t_1) \geq 0;
\]
\[
\mu_0(u, \chi_3, *) = \frac{1}{14} (6t_2 + 22) \geq 0; \quad \mu_1(u, \chi_3, *) = \frac{1}{14} (t_2 + 48) \geq 0;
\]
\[
\mu_7(u, \chi_3, *) = \frac{1}{14} (-6t_2 + 48) \geq 0.
\]
It follows that \( t_1 = 0 \) and \( t_2 = 8 \), which has no integral solutions.

\[\bullet\] Let \( u \) be of order 35. Then by Propositions 2.3 and 2.4 we get that
\[
\nu_{5a} + \nu_{5b} + \nu_{7a} = 1.
\]
Put \( t_1 = 4\nu_{5a} - \nu_{5b} + 2\nu_{7a} \). We consider the following two cases:

Case 1. Let \( \chi(u^7) = \chi(5a) \) and \( \chi(u^5) = \chi(7a) \). Using (2.1) we obtain that
\[
\mu_0(u, \chi_2, *) = \frac{1}{35} (24t_1 + 37) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{35} (-6t_1 + 17) \geq 0,
\]
which has no integral solutions.

Case 2. Let \( \chi(u^7) = \chi(5b) \) and \( \chi(u^5) = \chi(7a) \). By (2.1) we construct the system of two inequalities:
\[
\mu_0(u, \chi_2, *) = \frac{1}{35} (24t_1 + 17) \geq 0; \quad \mu_5(u, \chi_2, *) = \frac{1}{35} (-4t_1 + 3) \geq 0,
\]
which has no integral solutions. \( \square \)

Acknowledgments

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References


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