FINITE GROUPS WITH SOME SS-EMBEDDED SUBGROUPS

TAO ZHAO

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Abstract. We call $H$ an SS-embedded subgroup of $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is subnormal in $G$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ is the maximal s-permutable subgroup of $G$ contained in $H$. In this paper, we investigate the influence of some SS-embedded subgroups on the structure of a finite group $G$. Some new results were obtained.

1. Introduction

All groups considered in this paper are finite and $G$ denotes a group, $H \trianglelefteq G$ means that $H$ is a subnormal subgroup of $G$. We use conventional notions and notation, as in Huppert [6] or Gorenstein [2]. From some subgroup’s normality to investigate the structure of a finite group is a common method in the study of group theory. Recently, many new generalized normal concepts were introduced successively.

Recall that a subgroup $H$ of a group $G$ is said to be $s$-permutable [7] (or $s$-quasinormal [1]) in $G$ if $H$ is permutable with every Sylow subgroup $P$ of $G$. Wang in [11] introduced the concept of $c$-normal subgroup as follows: a subgroup $H$ is said to be $c$-normal in $G$ if $G$ has a normal subgroup $T$ such that $G = HT$ and $H \cap T \leq H_G$, where $H_G$ is the normal core of $H$ in $G$. Following Guo et al [3], a subgroup $H$ of $G$ is said to be nearly $s$-normal in $G$ if there exists $N \trianglelefteq G$ such that $HN \leq G$ and $H \cap N \leq H_{sG}$, where $H_{sG}$ is the largest $s$-permutable subgroup of $G$ contained in $H$. As a development, in [3] the concept of $S$-embedded subgroup was introduced: a subgroup $H$ is said to be $S$-embedded in $G$ if there exists a normal subgroup $N$ such that $HN$ is $s$-permutable in $G$ and $H \cap N \leq H_{sG}$. By using the $s$-permutability, $c$-normality, nearly $s$-normality or $S$-embedded properties of some subgroups,

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many interesting results have been derived (see [12],[13],[8] etc). Basing on the above concepts, in this paper we introduce that:

**Definition 1.1.** Let $H$ be a subgroup of $G$. We say that $H$ is SS-embedded in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is subnormal in $G$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ is the largest $s$-permutable subgroup of $G$ contained in $H$.

It is easy to see that all subgroups, independently of whether they are normal, $s$-permutable, $c$-normal, nearly $s$-normal or $S$-embedded in $G$ are SS-embedded subgroups of $G$. However, the converse case is not true. For example, if we let $G = [P]Q$ be a minimal non-2-nilpotent group, then it is easy to see that every maximal subgroup of $P$ is SS-embedded in $G$, but some of them are not $S$-embedded in $G$ (and hence they are not normal, $s$-permutable, $c$-normal or nearly $s$-normal in $G$). In this paper, we study the influence of some SS-embedded subgroups on the structure of a finite group $G$. Some new results are obtained.

2. Preliminaries

We list here some basic results which are useful in the sequel.

**Lemma 2.1.** ([7]) Suppose that $H$ is $s$-permutable in $G$, $H \leq G$ and $N \trianglelefteq G$. Then the following holds

1. If $H \leq K \leq G$, then $H$ is $s$-permutable in $K$.
2. $HN$ and $H \cap N$ are $s$-permutable in $G$, $HN/N$ is $s$-permutable in $G/N$.
3. $H$ is subnormal in $G$.
4. If $H$ is a $p$-group for some prime $p$, then $N_G(H) \geq O_p(G)$.

**Lemma 2.2.** Suppose that $H$ is an SS-embedded subgroup of $G$, then

1. If $H \leq K \leq G$, then $H$ is SS-embedded in $K$.
2. If $N \trianglelefteq G$ and $N \leq H$, then $H/N$ is SS-embedded in $G/N$.
3. Let $H$ be a $\pi$-subgroup and $N$ a normal $\pi'$-subgroup of $G$, then $HN/N$ is SS-embedded in $G/N$.

**Proof.** By the hypothesis, there exist a normal subgroup $T$ of $G$ and an $s$-permutable subgroup $H_{sG}$ of $G$ contained in $H$ such that $HT$ is subnormal in $G$ and $H \cap T \leq H_{sG}$.

1. It is clear that $K \cap T$ is a normal subgroup of $K$, $H(K \cap T) = K \cap HT$ is subnormal in $K$ and $H \cap (K \cap T) = H \cap T \leq H_{sG}$. By Lemma 2.1(1), $H_{sG}$ is $s$-permutable in $K$. Hence $H$ is SS-embedded in $K$.

2. Clearly, $TN/N$ is a normal subgroup of $G/N$, $(H/N)(TN/N) = HT/N$ is subnormal in $G/N$ and $(H/N) \cap (TN/N) = (H \cap TN)/N \leq H_{sG}/N$. By Lemma 2.1(2), $H_{sG}/N$ is $s$-permutable in $G/N$. Hence $H/N$ is SS-embedded in $G/N$.

3. It is easy to see that $TN/N \trianglelefteq G/N$, $(HN/N)(TN/N) = HTN/N$ is subnormal in $G/N$. Since $([H],[N]) = 1$,

\[
|H \cap TN| = \frac{|H| \cdot |TN|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HT|_\pi} = |H \cap T|.
\]
This implies that \(H \cap TN = H \cap T\), thus

\((HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N \leq HsG N/N.\)

Hence \(HN/N\) is SS-embedded in \(G/N\).

\[\square\]

**Lemma 2.3.** ([4 Lemma 2.5]) Let \(G\) be a group and \(p\) a prime such that \(p^{n+1} \nmid |G|\) for some integer \(n \geq 1\). If \((|G|, (p-1)(p^2-1) \cdots (p^n-1)) = 1\), then \(G\) is \(p\)-nilpotent.

### 3. Main results

**Theorem 3.1.** Let \(P\) be a Sylow \(p\)-subgroup of a group \(G\), where \(p\) is a prime divisor of \(|G|\) with \((|G|, (p-1)(p^2-1) \cdots (p^n-1)) = 1\). If \(N_G(P)\) is \(p\)-nilpotent and every \(n\)-maximal subgroup of \(P\) (if exists) not having a \(p\)-nilpotent supplement in \(G\) is SS-embedded in \(G\), then \(G\) is \(p\)-nilpotent.

**Proof.** Assume that the result is false and let \(G\) be a counterexample of minimal order. Then we have:

1. \(|P| \geq p^{n+1}\) and every \(n\)-maximal subgroup of \(P\) is SS-embedded in \(G\).

   By Lemma 2.3, we may assume that \(|P| \geq p^{n+1}\). If there exists an \(n\)-maximal subgroup \(P_1\) of \(P\) which has a \(p\)-nilpotent supplement \(T\) in \(G\), we prove that \(G\) is \(p\)-nilpotent. If not, let \(H\) be a non-\(p\)-nilpotent subgroup of \(G\) which contains \(P\) and is such that every proper subgroup of \(H\) is \(p\)-nilpotent. Then by [6] IV, Theorem 5.4, \(H\) is a minimal non-nilpotent group. Therefore, \(H\) has the following properties:

   (i) \(|H| = p^a q^b\), where \(p\) and \(q\) are different primes;

   (ii) \(H = [H_p]H_q\), where \(H_p = P\) is a normal Sylow \(p\)-subgroup and \(H_q\) a non-normal cyclic Sylow \(q\)-subgroup of \(H\);

   (iii) \(P/\Phi(P)\) is a chief factor of \(H\).

   Since \(G = P_1T\), \(H = H \cap P_1T = P_1(H \cap T)\). The fact \(H \cap T \leq T\) is \(p\)-nilpotent but \(H\) is not \(p\)-nilpotent implies that \(L = H \cap T\) is a proper subgroup of \(H\), and hence \(L\) is nilpotent. Let \(L = L_p \times L_q\). Obviously, \(L_q\) is also a Sylow \(q\)-subgroup of \(H\). Since \(P = P_1L_p\), \(L_p\) is not contained in \(\Phi = \Phi(P)\). Now we consider the factor group \(H/\Phi\). The fact \(L_q \leq N_H(L_p)\) implies that \(L_q\Phi/\Phi \leq N_H/\Phi(L_p\Phi/\Phi)\). On the other hand, since \(P/\Phi\) is an elementary abelian group, we have \(L_p\Phi/\Phi \leq P/\Phi\). Hence \(L_q\Phi/\Phi \leq H/\Phi\).

   Since \(L_p\Phi/\Phi \neq 1\) and \(P/\Phi\) is a chief factor of \(H\), we have \(L_p\Phi/\Phi = P/\Phi\). It follows that \(L_p = P\). Consequently, \(L = H\). This contradiction completes the proof of (1).

2. \(G\) is soluble.

   If not, then by the Feit-Thompson theorem we know \(p = 2\). First, we assume that \(O_2(G) \neq 1\). If \(|P/O_2(G)| \leq 2^{n+1}\), then by Lemma 2.3 \(G/O_2(G)\) is \(2\)-nilpotent. Hence \(G\) is soluble, a contradiction. Thus we may suppose that \(|P/O_2(G)| \geq 2^{n+1}\). Since \(N_G(P/O_2(G)) = N_G(P)/O_2(G)\) is \(2\)-nilpotent, by Lemma 2.2 we know \(G/O_2(G)\) satisfies the hypothesis of the theorem. Hence the minimal choice of \(G\) implies that \(G/O_2(G)\) is \(2\)-nilpotent. It follows that \(G\) is soluble, a contradiction. Next, we suppose that \(O_2(G) = 1\) and \(|P| \geq 2^{n+1}\). Let \(P_1\) be an \(n\)-maximal subgroup of \(P\). By the hypothesis, \(P_1\) is SS-embedded in \(G\). Hence there exists \(T \leq G\) such that \(P_1T\) is subnormal in \(G\) and \(P_1 \cap T \leq (P_1)_{sG}\). Since \((P_1)_{sG} \leq O_2(G) = 1\), \(P_1 \cap T = 1\) and so \(|T|_2 \leq 2^n\). Hence by Lemma 2.3 \(T\) is \(2\)-nilpotent and so it is soluble. Therefore, \(P_1T\) is soluble. Since \(P_1T\) is a soluble subnormal subgroup of \(G\), it is contained...
in some soluble normal subgroup $M$ of $G$. Clearly, $2^{n+1}$ does not divide $|G/M|$. So by Lemma 2.3 and the Feit-Thompson theorem, $G/M$ is soluble and so is $G$, as required.

(3) $O_p'(G) = 1$ and $O_p(G) \neq 1$.

If $O_p'(G) \neq 1$, then we know that $\overline{P} = PO_p'(G)/O_p'(G)$ is a Sylow $p$-subgroup of $\overline{G} = G/O_p'(G)$, $(|\overline{G}|, (p - 1)(p^2 - 1) \cdot \cdots (p^n - 1)) = 1$ and $N_{\overline{G}}(\overline{P}) = N_G(P)O_p'(G)/O_p'(G)$ is $p$-nilpotent. By (1), $|\overline{P}| \geq p^{n+1}$. Let $\overline{P}_1 = P_1O_p'(G)/O_p'(G)$ be an $n$-maximal subgroup of $\overline{P}$. Then we may assume that $P_1$ is an $n$-maximal subgroup of $P$. By the hypothesis and (1), $P_1$ is SS-embedded in $G$. Then by Lemma 2.2, $\overline{P}_1$ is SS-embedded in $\overline{G}$. Hence $\overline{G}$ is $p$-nilpotent by induction. It follows that $G$ is $p$-nilpotent, a contradiction. Thus we have $O_p'(G) = 1$. Since $G$ is soluble, $O_p(G) \neq 1$.

(4) $O_p(G)$ is a unique minimal normal subgroup of $G$, $\Phi(G) = 1$ and $G/O_p(G)$ is $p$-nilpotent.

Let $N$ be a minimal normal subgroup of $G$. By (2) and (3), $N$ is an elementary abelian $p$-group and $N \leq O_p(G)$. If $|P/N| \leq p^n$, then $G/N$ is $p$-nilpotent by Lemma 2.3. Now we assume that $|P/N| \geq p^{n+1}$. By Lemma 2.2, we know the hypothesis of the theorem holds for $G/N$. By the minimal choice of $G$, we have $G/N$ is $p$-nilpotent. Since the class of all $p$-nilpotent groups formed a saturated formation, $N$ is a unique minimal normal subgroup of $G$ and $\Phi(G) = 1$. Thus there is a maximal subgroup $M$ of $G$ such that $G = [N]M$. Since $O_p(G) \leq P(G) \leq C_G(N)$ and $C_G(N) \cap M \leq G$, the uniqueness of $N$ yields that $N = O_p(G) = F(G) = C_G(N)$.

(5) $|O_p(G)| \geq p^{n+1}$.

Since $G/O_p(G)$ is $p$-nilpotent, let $K/O_p(G)$ be the normal $p$-complement of $G/O_p(G)$. If $|O_p(G)| \leq p^n$, then $|K_p| \leq p^n$ and so Lemma 2.3 implies that $K$ is $p$-nilpotent. The normal $p$-complement of $K$ is also a normal $p$-complement of $G$. This contradiction shows that $|O_p(G)| \geq p^{n+1}$.

(6) The final contradiction.

Since $\Phi(G) = 1$, there exists a maximal subgroup $M$ of $G$ such that $G = [O_p(G)]M$. Let $P = O_p(G)M_p$ be a Sylow $p$-subgroup of $G$, where $M_p$ is a Sylow $p$-subgroup of $M$. Since $|O_p(G)| \geq p^{n+1}$, we can pick an $n$-maximal subgroup $P_1$ of $P$ containing $M_p$ such that $P_1 \cap O_p(G) \neq 1$. Clearly, $O_p(G) \not\leq P_1$. By the hypothesis, there exists a normal subgroup $T$ of $G$ such that $P_1T \leq G$ and $P_1 \cap T \leq (P_1)_{sG}$. Since $O_p(G)$ is an elementary abelian subgroup and $M_p \leq P_1$, we get that $P = O_p(G)P_1 \leq N_G(P_1 \cap O_p(G))$. If $T = 1$, then $P_1 = P_1T \leq G$, so $P_1 \leq O_p(G)$ which implies that $P = O_p(G)P_1 = O_p(G)$. In this case, $G = N_G(P)$ is $p$-nilpotent, a contradiction. Next, we assume that $T \neq 1$. By Lemma 2.1, $(P_1)_{sG}$ is subnormal in $G$ and so $(P_1)_{sG} \leq O_p(G)$. Therefore, $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G)$. Assume that $P_1 \cap T \neq 1$, then $(P_1)_{sG} \neq 1$ and Lemma 2.1 shows that $N_G((P_1)_{sG}) \geq O_p(G)$. Thus $1 < ((P_1)_{sG})^G = ((P_1)_{sG})^{POP(G)} = ((P_1)_{sG})^P \leq (P_1 \cap O_p(G))^P = P_1 \cap O_p(G) \leq O_p(G)$. Hence by (4) we have $((P_1)_{sG})^G = O_p(G) = P_1 \cap O_p(G)$. It follows that $O_p(G) \leq P_1$ and then $P = O_p(G)P_1 = P_1$, a contradiction. Therefore, $P_1 \cap T = 1$ and so $p^{n+1}$ does not divide $|T|$. Since $T \neq 1$ and $O_p(G)$ is the unique minimal normal subgroup of $G$, $O_p(G) \leq T$. Hence $p^{n+1}$ does not divide $|O_p(G)|$, which contradicts (5). This final contradiction completes the proof of the theorem.

Remark: Theorem 3.1 does not hold if we delete the condition that “$N_G(P)$ is $p$-nilpotent”. To see this, let $G = [P]Q$ be a minimal non-2-nilpotent group, then it is easy to see that every maximal subgroup of $P$ is SS-embedded in $G$ but $G$ is not 2-nilpotent.
From Theorem 3.1, we can easily deduce that:

**Corollary 3.2.** Let $P$ be a Sylow $p$-subgroup of $G$, where $p = \text{min} \pi (G)$. If $N_G(P)$ is $p$-nilpotent and every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $G$ is $SS$-embedded in $G$, then $G$ is $p$-nilpotent.

Next, we prove that:

**Theorem 3.3.** Let $p$ be a prime divisor of $|G|$ and $P$ a Sylow $p$-subgroup of $G$. If $N_G(P)$ is $p$-nilpotent and every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $G$ is $SS$-embedded in $G$, then $G$ is $p$-nilpotent.

**Proof.** If $p = \text{min} \pi (G)$, then by Corollary 3.2 we know that $G$ is $p$-nilpotent. Hence we only need to consider the case when $p$ is not the minimal prime divisor of $|G|$ (so it is an odd prime). Assume that the result is false and let $G$ be a counterexample of minimal order. Then we have:

(1) Every maximal subgroup of $P$ is $SS$-embedded in $G$.

See the proof of step (1) in Theorem 3.1.

(2) $O'_p(G) = 1$.

Suppose that $O'_p(G) \neq 1$. Clearly, $\overline{P} = PO'_p(G)/O'_p(G)$ is a Sylow $p$-subgroup of $\overline{G} = G/O'_p(G)$ and $N_{\overline{G}}(\overline{P}) = N_G(P)O'_p(G)/O'_p(G)$ is $p$-nilpotent. Let $T/O'_p(G)$ be a maximal subgroup of $PO'_p(G)/O'_p(G)$. Then $T = PO'_p(G)$ for some maximal subgroup $P_1$ of $P$. By (1) and Lemma 2.2 (3), we know $P_1O'_p(G)/O'_p(G)$ is $SS$-embedded in $\overline{G}$. This shows that $\overline{G}$ satisfies the hypothesis of the theorem. Thus $G/O'_p(G)$ is $p$-nilpotent by induction. It follows that $G$ is $p$-nilpotent, a contradiction.

(3) If $M$ is a proper subgroup of $G$ containing $P$, then $M$ is $p$-nilpotent.

Since $N_M(P) \leq N_G(P)$ is $p$-nilpotent. Now by (1) and Lemma 2.2 (1), we see that $M$ satisfies the hypothesis. The minimal choice of $G$ implies that $M$ is $p$-nilpotent.

(4) $G = PQ$ is soluble and $1 < O_p(G) < P$, where $Q$ is a Sylow $q$-subgroup of $G$ with $q \neq p$.

Since $G$ is not $p$-nilpotent, by Thompson’s theorem [9, Theorem 10.4.1], there is a nonidentity characteristic subgroup $H$ of $P$ such that $N_G(H)$ is not $p$-nilpotent. Since $N_G(P)$ is $p$-nilpotent, we may choose a characteristic subgroup $H$ of $G$ such that $N_G(H)$ is not $p$-nilpotent, but $N_G(K)$ is $p$-nilpotent for every characteristic subgroup $K$ of $P$ containing $H$. Obviously, $N_G(P) \leq N_G(H)$. Then by (3), $N_G(H) = G$. Therefore, $H \leq O_p(G) \neq 1$ and $O_p(G) < K$. Now by the Thompson theorem again, we see that $G/O_p(G)$ is $p$-nilpotent, and so $G$ is $p$-soluble. By [2, VI, Theorem 3.5], there exists a Sylow $q$-subgroup $Q$ of $G$ such that $PQ$ is a subgroup of $G$, where $q$ is a prime divisor of $G$ and $q \neq p$. If $PQ < G$, then $PQ$ is $p$-nilpotent by (3). This implies that $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [2 VI, Theorem 3.2], a contradiction. Thus $G = PQ$ and (4) holds.

(5) $G$ has a unique minimal normal subgroup $N$ such that $G = [N]M$, where $M$ is a maximal subgroup of $G$ and $N = O_p(G) = F(G)$.

Let $N$ be a minimal normal subgroup of $G$. Then by (2) and (4), $N$ is an elementary abelian $p$-group and $N \leq O_p(G)$. It is easy to see that $G/N$ satisfies the hypothesis. The minimal choice of $G$ implies
that $G/N$ is $p$-nilpotent. Since the class of all $p$-nilpotent groups is a saturated formation, $N$ is a unique minimal normal subgroup of $G$ and $N \not\leq \Phi(G)$. Thus, we can see that (5) holds.

(6) $|N| = p$.

Obviously, $P = NM_p$, where $M_p$ is a Sylow $p$-subgroup of $M$. Let $P_1$ be a maximal subgroup of $P$ containing $M_p$. If $P_1 = 1$, then $|N| = |P| = p$. Now suppose that $P_1 \neq 1$. By (1), there exists $K \leq G$ such that $P_1 K \leq G$ and $P_1 \cap K \leq (P_1)_{sG}$. If $K = 1$, then $P_1 = P_1 K \leq G$ which implies that $P_1 \leq O_p(G) = N$, and so $P = NP_1 = N$, a contradiction. Thus we have $K \neq 1$ and then $N \leq K$. Therefore, $P_1 \cap N \leq P_1 \cap K \leq (P_1)_{sG}$. Since $(P_1)_{sG}$ is normal in $G$ by Lemma 2.13, $(P_1)_{sG} \leq O_p(G) = N$ and so $P_1 \cap N = (P_1)_{sG}$. If $P_1 \cap N = 1$, then $|N| = p$. If $P_1 \cap N \neq 1$, then $1 \neq (P_1 \cap N)^G = ((P_1)_{sG})^G = ((P_1)_{sG})^{O_p(G)}P = (P_1 \cap N)^P = P_1 \cap N \leq N$. It follows that $(P_1 \cap N)^G = P_1 \cap N = N$, i.e., $N \leq P_1$, a contradiction. Hence (6) holds.

(7) The final contradiction.

By (5) and (6), we know $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $Aut(P)$, which is a cyclic group of order $p - 1$. Hence $M$ and in particularly $Q$, is a cyclic group. It follows from [6, IV, Theorem 2.8] that $G$ is $q$-nilpotent. Thus, $P \unlhd G$. Then by the hypothesis, $N_G(P) = G$ is $p$-nilpotent. This contradiction completes the proof of the theorem. □

Let $G$ be a group and $|G| = p_1^{r_1}p_2^{r_2} \cdots p_s^{r_s}$, where $p_1, p_2, \cdots, p_s$ are different primes. Recall that $G$ is said to be a Sylow tower group, if there exists a normal series $1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_s = G$ of $G$ such that $|G_i : G_{i-1}| = p_i^{r_i}$ for $1 \leq i \leq s$. If, in addition that, $p_1 > p_2 > \cdots > p_s$, then $G$ is called a Sylow tower group of supersoluble type.

**Theorem 3.4.** Let $G$ be a finite group. If for any non-cyclic Sylow $p$-subgroup $P$ of $G$, $N_G(P)$ is $p$-nilpotent and every maximal subgroup of $P$ is SS-embedded in $G$, then $G$ is a Sylow tower group of supersoluble type.

**Proof.** Let $p_1$ be the minimal prime divisor of $|G|$ and $P_1 \in Syl_{p_1}(G)$, we first prove that $G$ is $p_1$-nilpotent. If $P_1$ is cyclic, then by [9] Theorem 10.1.9] we know that $G$ is $p_1$-nilpotent. If $P_1$ is not cyclic, then by hypothesis $N_G(P_1)$ is $p_1$-nilpotent and every maximal subgroup of $P_1$ is SS-embedded in $G$. By Corollary 3.2, we can also conclude that $G$ is $p_1$-nilpotent. Now we let $K$ be the normal $p_1$-complement of $G$. By the hypothesis and Lemma 2.2 we know that for any non-cyclic Sylow $q$-subgroup $Q$ of $K$, $N_K(Q) \leq N_G(Q)$ is $q$-nilpotent and every maximal subgroup of $Q$ is SS-embedded in $K$. Thus, by induction we can deduce that $K$ is a Sylow tower group of supersoluble type. It follows that $G$ is a Sylow tower group of supersoluble type, as required. □

Next, by using the SS-embedded properties of some subgroups, we give some criteria for the solubility of a group $G$.

**Theorem 3.5.** Let $P$ be a Sylow $2$-subgroup of $G$. If $P$ is SS-embedded in $G$ or every maximal subgroup of $P$ is SS-embedded in $G$, then $G$ is soluble.

**Proof.** Assume that the result is false and let $G$ be a counterexample of minimal order. If $O_2(G) = P$, then $G/O_2(G)$ is a group of odd order, which is soluble by the Feit-Thompson theorem, thus $G$ is also
soluble. If $1 < O_2(G) < P$, then $P/O_2(G)$ is a Sylow 2-subgroup of $G/O_2(G)$. By the hypothesis and Lemma 2.2, $P/O_2(G)$ is SS-embedded in $G/O_2(G)$ or every maximal subgroup of $P/O_2(G)$ is SS-embedded in $G/O_2(G)$. So $G/O_2(G)$ is soluble by induction, and hence $G$ is soluble. Next, we may assume that $O_2(G) = 1$. Let $H$ denote $P$ or a maximal subgroup of $P$, respectively. By the hypothesis, there exists a normal subgroup $K$ of $G$ such that $HK \trianglelefteq G$ and $H \cap K \leq H_{sG} \leq O_2(G) = 1$. Thus $|K|_2 \leq 2$, which is 2-nilpotent and so soluble. Since $HK/K$ is a 2-group, $HK$ is soluble. Then $HK$ is contained in some soluble normal subgroup $M$ of $G$. Since $|G/M|_2 \leq 2$, it is soluble. Therefore $G$ is soluble, as required. □

**Theorem 3.6.** Let $G$ be a group. Then $G$ is soluble if and only if every maximal subgroup of $G$ is SS-embedded in $G$.

**Proof.** Suppose every maximal subgroup of $G$ is SS-embedded in $G$, we prove that $G$ is soluble. For any maximal subgroup $M$ of $G$, there exists a normal subgroup $K$ of $G$ such that $MK \trianglelefteq G$ and $M \cap K \leq M_{sG}$. If $G$ is a simple group, then $K = 1$ or $K = G$. In both case, we can conclude that $M = 1$ and $G$ is a cyclic group of prime order. Thus it is soluble, as required. Next, we let $N$ be a minimal normal subgroup of $G$. Clearly the hypothesis holds for $G/N$ by Lemma 2.2, so by induction we have $G/N$ is soluble. Hence we may assume that $N$ is non-abelian and it is the only minimal normal subgroup of $G$. By the Frattini argument [2, I, Theorem 3.7], for any prime $q$ dividing $|N|$ and for any Sylow $q$-subgroup $Q$ of $N$, there is a maximal subgroup $M$ of $G$ such that $NM = G$ and $N_G(Q) \leq M$. It is clear that $M_G = 1$ and $q$ does not divide $|G:M|$. Then by [10] Lemma 2.8, $M_{sG} = 1$. Since $M$ is SS-embedded in $G$, $G$ has a normal subgroup $T$ and an subnormal subgroup $K$ of $G$ such that $K = MT$ and $M \cap T = 1$. Since a subnormal maximal subgroup is normal in $G$, it follows that $K = G$ and $|G:M| = |T|$. Therefore, $q$ does not divide $|T|$. But $N \leq T$ as $N$ is the only minimal normal subgroup of $G$. Thus, $q$ divides $|T|$. This contradiction shows that $G$ is soluble.

Conversely, suppose that $G$ is soluble and let $M$ be a maximal subgroup of $G$. If $M$ is normal in $G$, then obviously $M$ is SS-embedded in $G$. Assume that $M$ is not normal in $G$ and let $T/K$ be a chief factor of $G$ such that $K \leq M$ and $TM = G$. Then clearly $M \cap T = K \leq M_{sG}$, so $M$ is SS-embedded in $G$, as required. □

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**References**


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Tao Zhao  
School of Science, Shandong University of Technology, 255049 Zibo, China  
Email: zht198109@163.com