THE MINIMUM SUM OF ELEMENT ORDERS OF FINITE GROUPS

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Abstract. Let \( G \) be a finite group and \( \psi(G) = \sum_{g \in G} o(g) \), where \( o(g) \) denotes the order of \( g \in G \). We show that the Conjecture 4.6.5 posed in [Group Theory and Computation, (2018) 59-90], is incorrect. In fact, we find a pair of finite groups \( G \) and \( S \) of the same order such that \( \psi(G) < \psi(S) \), with \( G \) solvable and \( S \) simple.

1. Introduction

To determine algebraic properties of a finite group from its element orders sum is an interesting problem. If \( G \) is a finite group, then \( \psi(G) = \sum_{g \in G} o(g) \) is the sum of element orders of \( G \), where \( o(g) \) denotes the order of \( g \in G \). In [1], it’s proved that the maximum value of \( \psi \) on the set of groups of order \( n \) will occur at the cyclic group \( C_n \), namely, a cyclic group of order \( n \) can be characterized by the order \( n \) and the value \( \psi \). Following this publication, many studies have been done on the function \( \psi \), for example, see [2], [3], [6], [9], [15], [13], and [16], that let to find an exact upper bound for sums of element orders in non-cyclic finite groups, in [12].

**Theorem 1.1.** [1] If \( G \) is noncyclic group and \( |G| = n \), then \( \psi(G) < \psi(C_n) \).

**Theorem 1.2.** [12] If \( G \) is noncyclic group and \( |G| = n \), then \( \psi(G) \leq \frac{7}{11} \psi(C_n) \).
It is natural to ask what one can say about the minimum of $\psi$ on groups of the same order. In [3], authors turned their attention to the minimum value of $G$, and they showed that

**Proposition 1.3.** [3] Among all nilpotent groups of order $n$, the minimum value of $\psi$ is attained by groups with all Sylow subgroups of prime exponent.

**Theorem 1.4.** [3] Let $G$ be a nilpotent group of order $n$ and there are non-nilpotent groups of order $n$. Then, there exists a non-nilpotent group $K$ of order $n$ satisfying $\psi(K) < \psi(G)$.

The minimal value of $\psi$ on finite groups of the same order was also investigated in non-abelian simple groups. For simple groups of the small orders, the $\psi$ value is the unique minimum of the values of $\psi$ for groups of the corresponding orders. Hence, the authors in [3] posed the following Conjecture:

**Conjecture 1.5.** [3] Let $S$ be a simple group. If $G$ is a non-simple group of order $|S|$, then $\psi(S) < \psi(G)$.

A counterexample has been found to Conjecture 1.5 in [15]:

**Proposition 1.6.** [15] There exists a non-simple group $G$ of order $|L_2(64)|$ such that $\psi(G) < \psi(L_2(64))$.

Since the counterexample to the Conjecture 1.5 was given by a non-solvable group, the authors in [11] stated the following new Conjecture:

**Conjecture 1.7.** [11] If $S$ is a simple group and $G$ is a solvable group satisfying $|G| = |S|$, then $\psi(S) < \psi(G)$.

Our aim in this note is to provide a counterexample to Conjecture 1.7. We prove the following theorem:

**Theorem 1.8.** There exist two finite groups $G$ and $S$ of the same order, such that $\psi(G) < \psi(S)$, with $G$ solvable and $S$ simple.

2. The groups

We describe our groups in Theorem 1.8 using the SmallGroups database in [5]. Since it does not appear to be possible to find examples in the database, our successful strategy has been to build up direct products of groups. A significant ingredient of our result is Lemma 2.1 of [3], which states that if $G$ and $H$ are finite groups, then $\psi(G \times H) \leq \psi(G) \psi(H)$, with equality if and only if $\gcd(|G|, |H|) = 1$.

We need to find two solvable groups $H$ and $K$ and a simple group $S$ with the following conditions:

i. $|H| \cdot |K| = |S|$,  
ii. $\psi(H \times K) < \psi(S)$.

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To satisfy the second condition, we must choose those groups that have the value of \( \psi(H \times K) \) as small as possible and the value of \( \psi(S) \) is sufficiently large. Notice that the first condition always holds.

It follows from Lemma 2.1 of [3] that increasing the value of \( \gcd(|H|, |K|) \) increases the probability of decreasing the value of \( \psi(H \times K) \). Therefore, we select a simple group that has in its order prime decomposition, small primes with large powers.

The projective special linear groups \( L_2(q) \) for \( q = 2^m \), is a simple group of order \((q - 1)q(q + 1) \) and its \( \psi \) value is relatively large. The simple group \( L_2(64) \) of order 63.64.65 = \( 2^6.3^2.5.7.13 \) has small primes in its order prime decomposition and \( 42|L_2(64)| < \psi(L_2(64)) \), because \( L_2(64) \) has \( \frac{1}{2}\varphi(65) = 24 \) self-centralizing elements of order 65, and \( \frac{1}{2}\varphi(63) = 18 \) self-centralizing elements of order 63, see [10, Proposition 28.4]. Note that If \( g \) is a self-centralizing element of group \( G \) then \( \langle g \rangle = C_G(g) \) and \( o(g).|g^G| = |G| \). Therefore, we select the group \( S = L_2(64) \).

By 2, we consider groups \( H \) and \( K \) so that \( |H| = 2^3.3.\alpha \), \( |K| = 2^3.3.\beta \) and \( \alpha, \beta = 5.7.13 \). For the Frobenius group \( F_{p,q} = C_p \rtimes C_q \), where \( p \) and \( q \) are two prime numbers and \( q|(p - 1) \), \( \psi(F_{p,q}) < (q - 1 + \frac{p - 1}{q})|F_{p,q}| \), and also \( \text{Aut}(C_{65}) \cong C_{12} \times C_4 \). Hence we assume that \( |H| = 2^3.3.5.13 \) and therefore \( |K| = 2^3.3.7 \).

Among all solvable groups of order 1560 and 168, we search for groups with the minimum \( \psi \) value, by Magma. Finally, we let \( H = \text{SmallGroup}(1560,150) \) and \( K = \text{SmallGroup}(168,43) \). We have that

\[
H = C_{65} \rtimes (C_{12} \times C_2)
\]

and

\[
K = (C_2^3 \rtimes C_7) \rtimes C_3
\]

are solvable groups. These groups satisfy \( \psi(S) = 12106687 \), \( \psi(H) = 19297 \) and \( \psi(K) = 855 \).

Now, we let \( G = H \times K \). We have that \( |G| = |S| \) and

\[
\psi(G) = \psi(H \times K) = 10954193,
\]

and

\[
\psi(G) < \psi(S).
\]

Although we have found some other examples, for example direct product of \( H = \text{SmallGroup}(780,16) \) or \( H = \text{SmallGroup}(780,17) \) and \( K = \text{SmallGroup}(336,218) \), they do not differ significantly from the one that we are describing above.

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3. Discussion

Although we found a solvable group such that its $\psi$ value is less than the $\psi$ value of a simple group, there is a non-solvable group such that its $\psi$ value is less than the $\psi$ value of this group,

$$5482775 = \psi(3^2 \times S_5(8)) < 10954193 = \psi(H \times K) < \psi(L_2(64)) = 12106687.$$ 

So, we propose the following new conjecture:

**Conjecture 3.1.** Let $n$ be a positive integer such that there exists a non-abelian simple group of order $n$. Then there exist a non-solvable group $G$ of order $n$ with the property that $\psi(G) < \psi(H)$ for every solvable group $H$ of order $n$.

**Algorithm 1** Search for two groups $H$ and $K$

1. Initialize $U := \{\}$ and $V := \{\}$.
2. For each $H$ in SmallGroups(168), if IsSolvable(H) then do:
   - $u := \sum_{g \in H} \text{Order}(g)$;
   - Append $u$ to $U$.
3. For each $H$ in SmallGroups(168), if IsSolvable(H) then do:
   - $u := \sum_{g \in H} \text{Order}(g)$;
   - If $u = \text{Minimum}(U)$ then do:
     - IdentifyGroup(H);
     - For each $K$ in SmallGroups(1560), if IsSolvable(K) then do:
       - $v := \sum_{g \in K} \text{Order}(g)$;
       - Append $v$ to $V$;
     - For each $K$ in SmallGroups(1560), if IsSolvable(K) then do:
       - If $v = \text{Minimum}(V)$ then do:
         - IdentifyGroup(K);
         - $G := \text{DirectProduct}(H,K)$;
         - IsSolvable(G);
         - $u := \sum_{g \in G} \text{Order}(g)$;
         - $u := \sum_{g \in \text{PSL}(2,64)} \text{Order}(g)$;
6. End if
5. End if
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