

4-QUASINORMAL SUBGROUPS OF PRIME ORDER

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ABSTRACT. Generalizing the concept of quasinormality, a subgroup H of a group G is said to be 4-quasinormal in G if, for all cyclic subgroups K of G , $\langle H, K \rangle = HKHK$. An intermediate concept would be 3-quasinormality, but in finite p -groups - our main concern - this is equivalent to quasinormality. Quasinormal subgroups have many interesting properties and it has been shown that some of them can be extended to 4-quasinormal subgroups, particularly in finite p -groups. However, even in the smallest case, when H is a 4-quasinormal subgroup of order p in a finite p -group G , precisely how H is embedded in G is not immediately obvious. Here we consider one of these questions regarding the commutator subgroup $[H, G]$.

1. INTRODUCTION

A subgroup H of a group G is said to be 4-*quasinormal* in G if $\langle H, K \rangle = HKHK$ for every cyclic subgroup K of G and we write

$$H \text{ } qn_4 \text{ } G.$$

In [1] several properties of cyclic 4-quasinormal subgroups of finite p -groups were discovered. In particular if H is such a subgroup of G and K is also a cyclic subgroup of G , then

$$\langle H, K \rangle \text{ is a regular group provided } p \geq 5.$$

Using results from [1], the precise structure of $\langle H, K \rangle$ was described in [6] as follows:-

Lemma 1. *Let $G = \langle h, k \rangle$ be a finite p -group ($p \geq 5$) with $H = \langle h \rangle$ cyclic of order p^m and $K = \langle k \rangle$ cyclic of order p^n and suppose that $H \text{ } qn_4 \text{ } G$. Let $C = \langle [h, k] \rangle$ of order p^ℓ . Then (i) $G = HCK$; (ii) $m \geq \ell$ and $n \geq \ell$; and (iii) both H^{p^ℓ} and K^{p^ℓ} lie in $\zeta_1(G)$, the centre of G .*

Of course when $m = n = 1$, then G is either elementary abelian or E_p , the extraspecial group of order p^3 and exponent p . Also when

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$m = 1$, then

$$(1) \quad [H, K^p] = 1$$

and

$$(2) \quad [H, K, H] \text{ and } [H, K, K] \text{ lie in } K^{p^{n-1}}.$$

We shall use these facts repeatedly throughout the rest of the paper. Working with groups of exponent p , the situation is even simpler:-

Lemma 2. *Let $G = \langle H, K \rangle$ be a finite p -group with $H \text{ qn}_4 G$, $|H| = p$ and K a subgroup of exponent p . Then H^G is elementary abelian.*

Proof. Let $H = \langle h \rangle$ and $k \in K$. Then $\langle h, k \rangle$ is elementary abelian, E_p or D_8 (the dihedral group of order 8). Therefore h commutes with $[h, k]$ and hence also with h^k . Thus $H^G = H^K$ is elementary abelian. \square

When p is odd, then we even have G of exponent p . However, it is easy to see that Lemma 2 can fail when K has exponent $\geq p^2$.

Example. Let $C = \langle c \rangle$ be a cyclic group of prime order $p (\geq 3)$, $K = \langle k \rangle$ cyclic of order p^2 and form the direct product $CK = \langle c \rangle \times \langle k \rangle$. Then define $G = (CK) \rtimes \langle h \rangle$ where $H = \langle h \rangle$ is cyclic of order p and $c^h = ck^p$, $k^h = ck$. Clearly G has order p^4 and class 3 and $H \text{ qn}_4 G$. But

$$H^G = \langle h, c, k^p \rangle \cong E_p.$$

However, for an arbitrary subgroup K , the situation doesn't become more complicated provided $p \geq 5$, as we now show.

Lemma 3. *Let G be a finite p -group ($p \geq 5$) with a subgroup H of order p such that $H \text{ qn}_4 G$. Then H^G has class ≤ 2 and exponent p .*

Proof. Let $H = \langle h \rangle$. By Lemma 1, for each $g \in G$, $[h, g]$ has order at most p and g^p centralizes H . Thus G^p centralizes H and hence also H^G . Moreover by Lemma 2, H^G is elementary abelian modulo G^p . Therefore $H^G \cap G^p \leq \zeta_1(H^G)$ and so H^G has class ≤ 2 . Then H^G is regular and hence has exponent p (see [3], 10.2 Satz and 10.5 Hauptsatz). \square

In the light of the Example above and Lemma 3, a natural question is the following:-

(3) *With the hypotheses of Lemma 3, is $[H, G]$ always abelian?*

By Lemma 2, the answer is 'yes' when G has exponent p . We shall show throughout the rest of the paper that the answer remains 'yes'

in many cases. Of course when $G = \langle H, K \rangle$ with H of order p , then $[H, G] = [H, K]$.

Our notation is standard. Cyclic groups of order p are denoted by C_p and E_p denotes the extraspecial group of order p^3 and exponent p . For each integer n , $\gamma_n(G)$ is the n -th term of the lower central series of G and $\zeta_n(G)$ is the n -th term of the upper central series of G . Also the subgroup of G generated by the elements of order p is denoted by $\Omega_1(G)$.

2. CASES WHERE $[H, G]$ IS ABELIAN

We begin by considering the case $G = \langle H, K \rangle$ where $K = C_p \times C_p$.

Proposition 1. *Let $G = \langle H, K \rangle$ be a finite p -group (where $p \geq 5$), $H \cong C_p$, $K \cong C_p \times C_p$ and $H \text{ qn}_4 G$. Then G has class ≤ 2 .*

Proof. Clearly $|G| \leq p^5$ and so G has class ≤ 4 . Therefore G is regular and hence has exponent p . We may assume that $H = \langle h \rangle \not\leq G'$ and that $|G/G'| = p^3$, otherwise $G = \langle h, k \rangle$ with $|k| = p$ and then G has class at most 2.

Thus G has class at most 3. If G has class 3, then $\gamma_2(G) \cong C_p \times C_p$ and $\gamma_3(G) \cong C_p$. Then it follows that $G/\gamma_3(G) \cong E_p \times C_p$ (see [5], p. 196) and we may assume that $K = \langle k_1 \rangle \times \langle k_2 \rangle$ with

$$\gamma_2(G) = \langle [h, k_1] \rangle \times \langle [h, k_2] \rangle \text{ and } \gamma_3(G) = \langle [h, k_2] \rangle.$$

But then by the Hall-Witt identity (see [4], Section 2.1, (3))

$$[k_1, h, k_2] = 1.$$

Therefore k_2 centralizes $\gamma_2(G)$ and so G has class ≤ 2 . \square

Considering the case where $K \cong E_p$ is surely the obvious move from here.

Proposition 2. *Let $G = \langle H, K \rangle$ be a finite p -group (where $p \geq 5$), $H \cong C_p$, $K \cong E_p$ and $H \text{ qn}_4 G$. Then G has class ≤ 3 .*

Proof. Let $K = \langle k_1, k_2 \rangle$, $H = \langle h \rangle$ and $c = [k_1, k_2]$. For any $k \in K$, $\langle h, c, k \rangle$ has class ≤ 2 , by Proposition 1. Therefore $[h, c] \in \zeta_1(G)$. So $c \in \zeta_2(G)$ and thus

$$(4) \quad c \text{ commutes with } \gamma_2(G)$$

(see [4], Lemma 2.21). Also $\langle [h, c], c \rangle \triangleleft G$ and

$$(5) \quad G/\langle [h, c], c \rangle \text{ has class } \leq 2,$$

again by Proposition 1. Thus G has class at most 4. Moreover by Lemma 2

$$(6) \quad H^G \text{ is elementary abelian;}$$

and by (5)

$$(7) \quad \gamma_3(G) \text{ commutes with } K.$$

Since $\langle h \rangle \text{ qn}_4 \langle h, k_1 k_2 \rangle$, using (4), (6) and (7) we have

$$1 = [h, k_1 k_2, k_1 k_2] = [[h, k_2][h, k_1]^{k_2}, k_1 k_2] = [h, k_2, k_1 k_2][h, k_1, k_1 k_2].$$

Also $[h, k_2, k_1 k_2] = [h, k_2, k_1]$ by (7) and $[h, k_1, k_1 k_2] = [h, k_1, k_2]$. Therefore

$$(8) \quad [h, k_2, k_1] = [h, k_1, k_2]^{-1}.$$

Now by the Hall-Witt identity

$$[h, k_1, k_2][k_1^{-1}, k_2^{-1}, h][k_2, h^{-1}, k_1^{-1}] = 1$$

using (6). Thus $[h, k_1, k_2][c, h] = [h, k_2, k_1]$ (using (7)) and substituting in (8) we have

$$(9) \quad [h, k_1, k_2]^2 = [h, c].$$

Therefore G has class ≤ 3 . \square

Of course, by Lemma 2, with the hypotheses of Proposition 1 or Proposition 2, $[H, G]$ is always abelian. Also when $G = \langle H, K \rangle$ is a finite p -group ($p \geq 5$), with $|H| = p$, $H \text{ qn}_4 G$ and K is abelian, then Lemma 1 shows that $K^p \leq \zeta_1(G)$; and G/K^p has class ≤ 2 , by Proposition 1. Therefore G has class ≤ 3 and again

$$(10) \quad [H, G] \text{ is abelian.}$$

We proceed now to show that the answer to Question 3 is ‘yes’ for significantly larger classes of subgroups K . We shall need the following consequence of Proposition 2.

Lemma 4. *With the hypotheses of Proposition 2, suppose $[h, k_1] = 1$. Then G has class ≤ 2 .*

Proof. By (9), $[h, c] = 1$. Also by (7), $[h, k_2] \in \zeta_1(G)$. Since c now belongs to $\zeta_1(G)$ and $G/\langle c, [h, k_2] \rangle$ is abelian, the Lemma follows. \square

Our first main result is the following.

Theorem 1. *Let $H = \langle h \rangle \text{ qn}_4 G = \langle h, k_1, k_2 \rangle$, a finite p -group ($p \geq 5$) with $|h| = |k_1| = p$. Then $[[h, k_1], [h, k_2]] = 1$.*

Proof. Assume, for a contradiction, that the result is false and let G be a counterexample of minimal order. Then there is a unique central subgroup $\langle c \rangle$ of order p in G and

$$[[h, k_1], [h, k_2]] = c.$$

Let $|k_2| = p^n$ and $x = k_1[h, k_2]$. Since $\langle h^{k_2}, k_1 \rangle$ is either elementary abelian or E_p , it follows from Propositions 1 and 2 that $\langle h, h^{k_2}, k_1 \rangle$ has class ≤ 3 . Therefore $\langle h, h^{k_2}, k_1 \rangle$ is regular and so

$$(11) \quad x^p = 1.$$

Let $k_3 = [h, [h, k_2]]$. If $n = 1$, then $k_3 = 1$, and if $n \geq 2$, then $k_3 = k_2^{\alpha p^{n-1}}$ where $0 \leq \alpha < p$ (by (2)). Also

$$[h, x] = [h, k_1[h, k_2]] = k_3[h, k_1]c.$$

Therefore by (11)

$$\begin{aligned} 1 &= [h, x, x] = [k_3[h, k_1]c, k_1[h, k_2]] = [k_3[h, k_1], k_1[h, k_2]] \\ &= [k_3[h, k_1], [h, k_2]][k_3[h, k_1], k_1]^{[h, k_2]} = c[k_3, k_1] \end{aligned}$$

using (1). Thus

$$(12) \quad [k_1, k_3] = c$$

and so

$$(13) \quad \langle k_3 \rangle = \Omega_1 \langle k_2 \rangle \text{ with } |k_2| \geq p^2.$$

Now let $y = k_2[h, k_1]$. Then

$$[h, y] = [h, k_2[h, k_1]] = [h, k_2]^{[h, k_1]} = [h, k_2]c^{-1}.$$

Therefore

$$(14) \quad [h, y, y] = [h, k_2, y] = [h, k_2, k_2[h, k_1]] = c^{-1}[h, k_2, k_2]$$

by (1) and (2). Since $[h, h^{k_1}] = 1$, we also have, by (1) and Lemma 4,

$$\langle h, h^{k_1}, k_2 \rangle / \langle k_2^p \rangle \text{ has class } \leq 2.$$

Also $y \in \langle h, h^{k_1}, k_2 \rangle$ and so by Lemma 3

$$[h, y, y] \in \Omega_1 \langle k_2 \rangle.$$

Therefore (14) implies that $\langle c \rangle = \Omega_1 \langle k_2 \rangle = \langle k_3 \rangle$ (by (13)), contradicting (12). The Theorem follows. \square

Another case where we get a positive result is when $K = \langle k_1, k_2 \rangle$ is *powerful*, i.e. K/K^p is abelian.

Theorem 2. *Let $H = \langle h \rangle$ and $G = \langle h, k_1, k_2 \rangle$, a finite p -group ($p \geq 5$) with $|H| = p$ and suppose that $K = \langle k_1, k_2 \rangle$ is powerful. Then $[H, K]$ is abelian.*

Proof. By [2], 2.8 Corollary, $K = \langle k_1 \rangle \langle k_2 \rangle$. Then by [3], 11.5 Satz, K is metacyclic. Therefore we may assume that $\langle k_1 \rangle \triangleleft K$. We shall prove first that

$$(15) \quad [[h, k_1], [h, k_2]] = 1.$$

By Theorem 1, we may also assume that $|k_1| = m \geq 2$. Then since $\Omega_1 \langle k_1 \rangle \leq \zeta_1(G)$ (by (1)), arguing by induction on $|G|$, we may assume, for a contradiction, that

$$[[h, k_1], [h, k_2]] = z,$$

where $\langle z \rangle = \Omega_1 \langle k_1 \rangle$. Then

$$(16) \quad [h, k_1]^{k_2} = [h[h, k_2], k_1[k_1, k_2]] = [h[h, k_2], k_1] = [h, k_1]z[h, k_2, k_1],$$

using (2). Also by (2)

$$(17) \quad [h, k_1, h] = k_1^{\lambda p^{m-1}}, \quad 0 \leq \lambda < p.$$

$$\begin{aligned} \text{Therefore } [h, k_1, h] &= [h, k_1, h]^{k_2} = [[h, k_1]^{k_2}, h[h, k_2]] \\ &= [[h, k_1]z[h, k_2, k_1], h[h, k_2]] \text{ (by (16))} \\ &= [[h, k_1], h[h, k_2]] \text{ (by (1) and Proposition 1)}. \end{aligned}$$

Thus, by (1) and (17), $[h, k_1, h] = z[h, k_1, h]$, i.e. $z = 1$, a contradiction, proving (15).

Finally to prove the Theorem, let $k_3 = k_1^\alpha k_2^\beta$ and $k_4 = k_1^\gamma k_2^\delta$. Then

$$[[h, k_3], [h, k_4]] = [[h, k_1^\alpha k_2^\beta], [h, k_1^\gamma k_2^\delta]] =$$

$$[[h, k_2^\beta][h, k_1^\alpha]^{k_2^\beta}, [h, k_2^\delta][h, k_1^\gamma]^{k_2^\delta}] = [([h, k_2^\beta][h, k_1^\alpha])^{k_2^\beta}, ([h, k_2^\delta][h, k_1^\gamma])^{k_2^\delta}]$$

using (1) and (2). Therefore $[[h, k_3], [h, k_4]] =$

$$[[h, k_2^\beta][h, k_1^\alpha][[h, k_2^\beta][h, k_1^\alpha], k_2^\beta], [h, k_2^\delta][h, k_1^\gamma][[h, k_2^\delta][h, k_1^\gamma], k_2^\delta]].$$

Therefore, using (1) and Proposition 1,

$$[[h, k_3], [h, k_4]] = [[h, k_2^\beta][h, k_1^\alpha], [h, k_2^\delta][h, k_1^\gamma]] = 1,$$

since $[h, k_1^\alpha] \equiv [h, k_1]^\alpha \pmod{\langle k_1^p \rangle}$ etc. Thus the Theorem follows. \square

In order to find an answer to (3), the next case to consider should be where $G = \langle H, K \rangle$ with $K = \langle k_1, k_2 \rangle$ of class 2. We conclude by showing that the answer is still ‘yes’ in this case if, in addition, $[[k_1, k_2]] = p$.

Theorem 3. *Let $H = \langle h \rangle$ and $G = \langle h, k_1, k_2 \rangle$, a finite p -group ($p \geq 5$) with $|H| = p$, $K = \langle k_1, k_2 \rangle$ and $|K'| = p$. Then $[H, K]$ is abelian.*

Proof. By (10) it is sufficient to show that $[[h, k_1], [h, k_2]] = 1$. Let $|k_1| = p^m$ and $|k_2| = p^n$. It follows from Theorem 1 that we may assume $m \geq n \geq 2$. Suppose, for a contradiction, that the Theorem is false and let G be a counterexample with $m+n$ minimal. Since k_1^p and k_2^p are in $\zeta_1(G)$, we must have $\Omega_1\langle k_1 \rangle = \Omega_1\langle k_2 \rangle$. Then it is easy to see that we may assume

$$(18) \quad k_1^{p^{m-1}} = k_2^{p^{n-1}} \quad \text{and} \quad \langle [[h, k_1], [h, k_2]] \rangle = \Omega_1\langle k_1 \rangle.$$

Clearly K is regular and by [3], 10.8 Satz,

$$(k_1^{-p^{m-n}} k_2)^{p^{n-1}} = k_1^{-p^{m-1}} k_2^{p^{n-1}} = 1.$$

Thus by choice of G

$$(19) \quad [[h, k_1^{-p^{m-n}} k_2], [h, k_1]] = 1.$$

However, if $m > n$, then $[h, k_1^{p^{m-n}}] = 1$ and (19) contradicts (18). Therefore we must have $m = n$ and so

$$\begin{aligned} 1 &= [[h, k_1^{-1} k_2], [h, k_1]] = [[h, k_2][h, k_1^{-1}]^{k_2}, [h, k_1]] \\ &= [[h, k_2][h, k_1^{-1}][h, k_1^{-1}, k_2], [h, k_1]]. \end{aligned}$$

But G/K^p has class at most 3 (by Proposition 2) and $K^p \leq \zeta_1(G)$. Therefore G has class ≤ 4 . Also $[h, k_1^{-1}, k_2] \in \gamma_3(G)$ and so commutes with $\gamma_2(G)$ (see [4], Lemma 2.21 (ii)). Thus

$$1 = [[h, k_2][h, k_1^{-1}], [h, k_1]] = [[h, k_2], [h, k_1]]$$

by (1), contradicting (18). The Theorem follows. \square

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