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DIRECTED ZERO-DIVISOR GRAPH AND SKEW POWER SERIES RINGS

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Abstract. Let R be an associative ring with identity and $Z^*(R)$ be its set of non-zero zero-divisors. Zero-divisor graphs of rings are well represented in the literature of commutative and non-commutative rings. The directed zero-divisor graph of R, denoted by $\Gamma(R)$, is the directed graph whose vertices are the set of non-zero zero-divisors of R and for distinct non-zero zero-divisors x, y, $x \to y$ is an directed edge if and only if $xy = 0$. In this paper, we connect some graph-theoretic concepts with algebraic notions, and investigate the interplay between the ring-theoretical properties of a skew power series ring $R[[x;\alpha]]$ and the graph-theoretical properties of its directed zero-divisor graph $\Gamma(R[[x;\alpha]])$. In doing so, we give a characterization of the possible diameters of $\Gamma(R[[x;\alpha]])$ in terms of the diameter of $\Gamma(R)$, when the base ring R is reversible and right Noetherian with an α-condition, namely α-compatible property. We also provide many examples for showing the necessity of our assumptions.

1. **Introduction and preliminaries**

The concept of the graph of the zero-divisors of a commutative ring was first introduced by Beck [10] when discussing the coloring of a commutative ring. In his work, all elements of the ring were the graph vertices. Since then, Anderson and Livingston [5] introduced and studied the zero-divisor graph whose vertices are only non-zero zero-divisors of a ring. It has been studied extensively since it was first considered in [5] in order to illuminate interplay between graph theory and ring theory. Several fundamental results concerning $\Gamma(R)$ for a commutative ring R are given in [5]. For example, $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$, and $gr(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle [5, Theorems 2.3 and 2.4].

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Moreover, $\Gamma(R)$ is a finite graph with at least one vertex if and only if R is a finite ring which is not a field [5, Theorem 2.2].

If R is a non-commutative ring, Redmond [28] introduced various ways to define the zero-divisor graph of a non-commutative ring. He defined an undirected zero-divisor graph of a non-commutative ring R, the graph $\bar{\Gamma}(R)$, with vertices in the set $Z^*(R) = Z(R) \setminus \{0\}$ and such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$ or $ba = 0$. Then he proved that the simple (undirected) graph $\bar{\Gamma}(R)$ is connected with $\mathrm{diam}(\bar{\Gamma}(R)) \leq 3$, where $\mathrm{diam}(\bar{\Gamma}(R))$ is the diameter of $\bar{\Gamma}(R)$. The zero-divisor graphs offer a graphical representation of rings so that we may discover some new algebraic properties of rings that are hidden from the viewpoint of classical ring theorists. As an instance, using the notion of a zero-divisor graph, it has been proven in [29] that for any finite ring R, the sum $\sum_{x\in R} |r_R(x) - l_R(x)|$ is even, where $r_R(x)$ and $l_R(x)$ denote the right and left annihilators of the element x in R, respectively. He also defined a directed zero-divisor graph in a similar way. The directed zero-divisor graph of R, denoted by $\Gamma(R)$, is the directed graph whose vertices are the set of non-zero zero-divisors of R and for distinct non-zero zero-divisors x, y, $x \to y$ is an directed edge if and only if $xy = 0$. A directed graph is connected if there exists a directed path connecting any two distinct vertices. The distance and the diameter are defined in a similar way as well, having in mind that all paths in question are directed. We will be concerned with this type of directed zero-divisor graph of non-commutative rings. Several papers are devoted to studying the relationship between the zero-divisor graph and algebraic properties of rings, for instance see [1, 2, 3, 5, 10, 11, 15, 17, 16, 26, 28].

According to Cohn [12], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for each $a, b \in R$. Anderson and Camillo [5], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa and Niewieczerzal [24] took the term C_0 for it. It is obvious that every *reduced* ring (i.e., the rings with no non-zero nilpotent elements) and every commutative ring is reversible. Thus, reversible rings provide a sort of bridge between commutative and non-commutative ring theory. On the one hand, the reversible condition forces a non-commutative ring to have certain affinities with its commutative cousins (e.g., it must be Dedekind-finite, it cannot be a full matrix ring, etc.). Kim and Lee [23], studied extensions of reversible rings and showed that polynomial rings over reversible rings need not be reversible in general.

Throughout this paper, R denotes an associative ring with identity, unless otherwise stated. For $X \subseteq R$, the right ideal generated by X is denoted by $\langle X \rangle$, $\ell_R(X) = \{a \in R |\ ax = 0, \forall x \in X\}$ and $r_R(X) = \{a \in R |\ xa = 0, \forall x \in X\}$. Note that if R is a reversible ring and $X \subseteq R$, then $\ell_R(X) = r_R(X)$ is an ideal of R and we denote it by $ann(X)$. We write $Z_\ell(R)$ and $Z_r(R)$ for the set of all left zero-divisors of R and the set of all right zero-divisors of R, respectively. Clearly, if R is a reversible, then $Z_\ell(R) = Z_r(R)$.

Let R be an associative ring with identity. The set *zero-divisors* of R, denoted by $Z(R)$, is the set of elements $a \in R$ such that there exists a non-zero element $b \in R$ with $ab = 0$ or $ba = 0$ (i.e.,

$Z(R) = Z_\ell(R) \cup Z_r(R)$). Set $Z^*(R) = Z(R) \setminus \{0\}$. According to [28], the *directed graph* $\Gamma(R)$ is a graph with vertices in $Z^*(R)$, where $x \to y$ is an edge between distinct vertices x and y if and only if $xy = 0$.

We say that a directed graph Γ is *connected* if there is a path following the directed edges of Γ from any vertex of Γ to any other vertex of Γ. For two distinct vertices a and b in the graph Γ, the distance between a and b, denoted by $d(a, b)$, is the length of shortest path connecting a to b, if such a path exists; otherwise we put $d(a, b) = \infty$. Recall that the *diameter* of a graph Γ is defined as follows:

$$\mathrm{diam}(\Gamma) = \sup\{d(a, b) |\ a \text{ and } b \text{ are distinct vertices of } \Gamma\}.$$

A graph is said to be *complete* if it is connected with diameter one.

Our results in this article continues the exploration of the relationship between graph theory and algebra, which has been a hotbed of research recently. There is considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. The first such extensions that come to mind are those of polynomial and power series extensions. Axtell, Coykendall and Stickles [9] examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings. Also, Lucas [26] continued the study of the diameter of zero-divisor graphs of polynomial and power series rings over commutative rings. Moreover, Anderson and Mulay [7], studied the girth and diameter of zero-divisor graph of a commutative ring and investigated the girth and diameter of zero-divisor graphs of polynomial and power series rings over commutative rings. Since we are especially interested in non-commutative aspects of the theory, we do not wish to restrict ourselves to commutative cases.

Let R be a ring and α an endomorphism of R. The *skew power series* ring $R[[x; \alpha]]$ is the ring of power series over R in the variable x, with term-wise addition and with coefficients written on the left of x, subject to the skew-multiplication rule $xr = \alpha(r)x$ for $r \in R$. Our first main result in this paper, states that if R is a reversible right Noetherian ring and also have α-compatible property, and also $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=0}^{\infty} b_j x^j$ be non-zero elements of R such that $f(x)g(x) = 0$, then there exist elements $r, s \in R$ such that $g(x)r \neq 0 \neq sf(x)$ but $a_i b_j r = 0 = s a_i b_j$, for each i, j. Then we show that $Z(R)$ is an ideal of R if and only if $Z(R[[x; \alpha]])$ is an ideal of $R[[x; \alpha]]$. Also it is proved that (1) $\mathrm{diam}(\Gamma(R[[x; \alpha]])) = 1$ if and only if R is a non-reduced ring with $Z(R)^2 = 0$. (2) $\mathrm{diam}(\Gamma(R[[x; \alpha]])) = 2$ if and only if either (i) R is a reduced ring with exactly two minimal primes, or (ii) $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$. (3) $\mathrm{diam}(\Gamma(R[[x; \alpha]])) = 3$ if and only if R is not a reduced ring with exactly two minimal primes and $Z(R)$ is not an ideal of R. We also provide examples for showing the necessity of our assumptions.

2. Directed zero-divisor graph and skew power series rings

Let α be an endomorphism of a ring R. According to [8], a ring R is called *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Clearly, this may only happen when the endomorphism α is injective. We start by providing some examples of α-compatible rings.

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example 2.1. *Let S be any reduced ring and consider the reversible ring $R = \{(a, b) \mid a, b \in S\}$ with addition point-wise and multiplication given by $(a, b)(c, d) = (ac, ad+bc)$. Let $\alpha : R \to R$ be an automorphism defined by $\alpha((a, b)) = (a, -b)$. We will show that R is α-compatible. To see this, let $(a, b)(c, d) = 0$. Thus $ac = ad + bc = 0$. Therefore $ca = 0$, since S is reduced. Now, multiplying $ad + bc = 0$ from left by c, we get $cad + cbc = 0$ and then $cbc = 0$. Hence $(bc)^2 = bcbc = 0$ and then $bc = ad = 0$, since S is reduced. Then $(a, b)\alpha((c, d)) = (ac, bc - ad) = (0, 0)$. Similarly, one can show that $(a, b)\alpha((c, d)) = (0, 0)$ implies that $(a, b)(c, d) = (0, 0)$. Thus R is α-compatible.*

example 2.2. *Let R_1 be a ring, D a domain, $R = R_1 \oplus D[y]$ and $\alpha : D[y] \to D[y]$ be a monomorphism. Let $\overline{\alpha} : R \to R$ be an endomorphism defined by $\overline{\alpha}(a \oplus f(y)) = a \oplus \alpha(f(y))$ for each $a \in R_1$ and $f(y) \in D[y]$. Then R is $\overline{\alpha}$-compatible.*

Lemma 2.3. [19] *Let R be an α-compatible ring. Then we have the following:*

(1) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n.*
(2) *If $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$.*
(3) *If $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x; \alpha]]$ and $r \in R$, then $f(x)r = 0$ if and only if $a_i r = 0$ for each i.*
(4) *If $f(x) \in R[[x; \alpha]]$ and $r \in R$, then $rf(x) = 0$ if and only if $rxf(x) = 0$.*

Lemma 2.4. *Let R be a reversible and α-compatible ring.*

(1) *If $a_1, \ldots, a_n \in R$, then $a_1 \cdots a_n = 0$ if and only if $b_1 \cdots b_n = 0$ for each $\{b_1, \ldots, b_n\} = \{a_1, \ldots, a_n\}$.*
(2) *If $f \in R[[x; \alpha]]$ and $r \in R$, then $rf = 0$ if and only if $fr = 0$.*

Proof. It is clear. □

For the skew power series $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x; \alpha]]$, we denote by C_f the set of all coefficients of $f(x)$ in R.

Lemma 2.5. *Let R be a reversible, α-compatible and right Noetherian ring and $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero elements in $R[[x; \alpha]]$ with $f(x)g(x) = 0$. Let $C_f R = \langle a_0, a_1, \ldots, a_p\rangle$ and $C_g R = \langle b_0, b_1, \ldots, b_q\rangle$. Then there exist non-negative integers $\ell_0, \ell_1, \ldots, \ell_p, t_0, t_1, \ldots, t_q$ such that for all $0 \leq m \leq p$ and $0 \leq n \leq q$,*

$$a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m} g(x) \neq 0 = a_0^{\ell_0} a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m+1} g(x)$$

and

$$f(x) b_n^{t_n} b_{n-1}^{t_{n-1}} \cdots b_1^{t_1} b_0^{t_0} \neq 0 = f(x) b_n^{t_n+1} b_{n-1}^{t_{n-1}} \cdots b_1^{t_1} b_0^{t_0}$$

Proof. We use induction on m to show that there exist non-negative integers ℓ_i, where $0 \leq i \leq p$, such that $a_0^{\ell_0} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m} g(x) \neq 0$ but $a_0^{\ell_0} \cdots a_{m-1}^{\ell_{m-1}} a_m^{\ell_m+1} g(x) = 0$. We claim that $a_0^{j+1} b_j = 0$ for each $0 \leq j \leq q$. Taking any $i \neq 0$, since $fg = 0$ we get $\sum_{j=0}^{i} a_{i-j} \alpha^{i-j}(b_j) = 0$. In particular, the degree zero term implies that $a_0 b_0 = 0$ and since R is reversible and α-compatible, $a_0 a_1 \alpha(b_0) = 0$. Then multiplying $a_0 b_1 + a_1 \alpha(b_0) = 0$ on the left hand side by a_0, we get $a_0^2 b_1 = 0$. Now, suppose by induction that $a_0^{\ell+1} b_\ell = 0$

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for each $\ell < k$. In particular $a_0^k b_\ell = 0$ and then by α-compatible and reversibility $a_0^k a_{k-\ell}\alpha^{k-\ell}(b_\ell) = 0$ for each $\ell < k$. Multiplying $\sum_{\ell=0}^{k} a_{k-\ell}\alpha^{k-\ell}(b_\ell) = 0$ (degree k term in $f(x)g(x) = 0$) on the left hand side by a_0^k, we get $a_0^{k+1}b_k = -\sum_{\ell=0}^{k-1} a_0^k a_{k-\ell}\alpha^{k-\ell}(b_\ell) = 0$. Thus $a_0^{k+1}b_k = 0$, which finishes the proof of claim. Now, since $C_gR = \langle b_0, b_1, \ldots, b_q\rangle$ we have $a_0^{q+1}g(x) = 0$. Hence there exists non-negative integer ℓ_0 such that $a_0^{\ell_0}g(x) \neq 0$ but $a_0^{\ell_0+1}g(x) = 0$. Suppose, by induction, we have constructed $\ell_0, \ell_1, \ldots, \ell_{m-1}$ satisfying the mentioned condition and that $r = a_{m-1}^{\ell_{m-1}} \cdots a_1^{\ell_1}a_0^{\ell_0}$ and $rg(x) \neq 0$ and by reversibility of R we get $g(x)r \neq 0$. Now by taking $g'(x) := g(x)r$, we have $0 = f(x)g'(x) = (\sum_{i=m}^{\infty} a_i x^i)g'(x)$. By the base case, there exist a non-negative integer ℓ_m such that $a_m^{\ell_m}g'(x) \neq 0$ but $a_m^{\ell_m+1}g'(x) = 0$. Now by Lemma 2.4, we get $a_0^{\ell_0}a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}}a_m^{\ell_m}g(x) \neq 0$ but $a_0^{\ell_0}a_1^{\ell_1} \cdots a_{m-1}^{\ell_{m-1}}a_m^{\ell_m+1}g(x) = 0$, where $\ell_0, \ell_1, \ldots, \ell_m$ are non-negative integers. By similar argument, with small changes, one can prove the other statement. □

Theorem 2.6. *Let R be a reversible, α-compatible and right Noetherian ring and $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero power series in $R[[x;\alpha]]$. If $f(x)g(x) = 0$, then there exist non-zero elements $r, s \in R$ such that $g(x)r \neq 0 \neq sf(x)$ and $a_ib_jr = 0 = sa_ib_j$ for each i, j.*

Proof. Since R is right Noetherian, $C_fR = \langle a_0, \ldots, a_p\rangle$ and $C_gR = \langle b_0, \ldots, b_q\rangle$. There exist non-negative integers ℓ_0, ℓ_1, ..., ℓ_p satisfying the mentioned condition in Lemma 2.5 and that $r = a_0^{\ell_0}a_1^{\ell_1} \cdots a_{p-1}^{\ell_{p-1}}a_p^{\ell_p} \neq 0$ and $rg(x) \neq 0$. Since R is α-compatible and reversible, we have $g(x)r \neq 0$. Using reversibility of R and Lemma 2.5, we get $a_ig(x)a_0^{\ell_0} \cdots a_p^{\ell_p} = 0$ for any $0 \leq i \leq p$ and hence $a_tg(x)a_0^{\ell_0} \cdots a_p^{\ell_p} = 0$ for any $t > p$, since $a_t \in C_fR = \langle a_0, \ldots, a_p\rangle$. Hence $gr \neq 0$ but $a_ib_jr = 0$, for all i, j.

Similarly, by using Lemma 2.5 one can prove there is $0 \neq s \in R$ such that $sf(x) \neq 0$ but $sa_ib_j = 0$, for all i, j. □

The concept of power-serieswise McCoy ring was introduced in [30]. A ring R is said to be right *power-serieswise McCoy* if whenever non-zero power series $f(x), g(x) \in R[[x]]$ satisfy $f(x)g(x) = 0$, then there exists $0 \neq r \in R$ such that $f(x)r = 0$. Left power-serieswise McCoy rings can be defined similarly. If a ring R is both right and left power-serieswise McCoy, we say that R is power-serieswise McCoy.

Following [4], we say that a ring R with an endomorphism α is *right α-power-serieswise McCoy*, if whenever power series $f(x), g(x) \in R[[x;\alpha]] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then there exists non-zero element $c \in R$ such that $f(x)c = 0$. Left α-power-serieswise McCoy can be defined similarly. If a ring R is both right and left α-power-serieswise McCoy, we say that R is α-power-serieswise McCoy. Theorem 2.6 have the following immediate corollary.

Corollary 2.7. *If R is a reversible, α-compatible and right Noetherian ring, then R is α-power-serieswise McCoy.*

Proof. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be non-zero power series in $R[[x;\alpha]]$. If $f(x)g(x) = 0$, then by Theorem 2.6, there exist non-zero elements $r, s \in R$ such that $g(x)r \neq 0 \neq sf(x)$ and $a_ib_jr = 0 = sa_ib_j$ for each i, j. Since $g(x)r \neq 0 \neq sf(x)$, there exist i, j such that $b_jr \neq 0 \neq sa_i$. By considering $a = sa_i$ and $b = b_jr$ we have $fb = 0 = ag$, which implies that R is α-power-serieswise McCoy. □

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The following example shows that assumption "R is right Noetherian" in Corollary 2.7 is not superfluous.

example 2.8. [13, Example 3] *Let K be a commutative ring with identity and $\{Y, X_0, X_1, X_2 \dots, X_t, \dots\}$ be a set of indeterminates over K; and let*

$$R = K[Y, \{X_i\}_{i=0}^{\infty}]/\langle X_0Y, \{X_i - X_{i+1}Y\}_{i=0}^{\infty}\rangle$$

and α be the identity endomorphism on R. Clearly R is a commutative α-compatible ring. Let $f(x) = \overline{Y} - x$. Then $f(x)$ has a unit coefficient, hence certainly $rf(x) \neq 0$ for each non-zero element r of R. However, letting $g(x) = \sum_{i=0}^{\infty} \overline{X}_i x^i$, we see that $f(x)g(x) = 0$ while $g(x) \neq 0$, which shows that Corollary 2.7 fails when R is not Noetherian.

The following example shows that assumption "R is α-compatible" in Corollary 2.7 is not superfluous.

example 2.9. [4, Example 2.3] *Let S be any non-zero Noetherian reduced ring. Suppose $R = S \oplus S$ with the usual addition and multiplication. Let $\alpha : R \to R$ be an automorphism defined by $\alpha((a, b)) = (b, a)$. Then R is reduced and thus R is reversible but is not α-compatible, since $(1, 0)(0, 1) = 0$ but $(1, 0)\alpha((0, 1)) = (1, 0)$. Suppose that $f(x) = (1, 0) + (1, 0)x + (1, 0)x^2 + \cdots$ and $g(x) = (0, 1) - (1, 0)x \in R[[x; \alpha]] \setminus \{0\}$. Then $f(x)g(x) = 0$, but if there exists $(a, b) = c \in R$ such that $f(x)c = 0$, then we have $a = b = 0$ and hence $c = 0$. Therefore R is not α-power-serieswise McCoy.*

As mentioned in the introduction, Kim and Lee [23] showed that polynomials ring over reversible ring need not be reversible that so in general power series ring is not reversible. But in the following we prove that $Z_r(R[[x; \alpha]]) = Z_\ell(R[[x; \alpha]])$. Hence $\Gamma(R[[x; \alpha]])$ is connected by [28, Theorem 2.3].

Recall that an ideal $\mathcal{P}$ of R is *completely prime* if $ab \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$ for $a, b \in R$.

Lemma 2.10. *Let R be a reversible, α-compatible and right Noetherian ring. Then $Z_r(R[[x; \alpha]]) = Z_\ell(R[[x; \alpha]]) = Z(R[[x; \alpha]])$.*

Proof. Since R is reversible and right Noetherian, $Z(R) = \cup_{i=1}^{n} \mathcal{P}_i$ with each $\mathcal{P}_i$ is a completely prime ideal and $\mathcal{P}_i = ann(a_i)$, for $a_i \in Z(R)$ and each $1 \leq i \leq n$, by [16, Remark 3.2]. Then by α-compatibility assumption on R we have $\mathcal{P}_i[[x; \alpha]] \subseteq ann_{R[[x;\alpha]]}(a_i)$. Also by Lemma 2.4, we have $ann_{R[[x;\alpha]]}(a_i) \subseteq \mathcal{P}_i[[x; \alpha]]$. Therefore $\mathcal{P}_i[[x; \alpha]] = ann_{R[[x;\alpha]]}(a_i)$. Thus $\cup_{i=1}^{n} \mathcal{P}_i[[x; \alpha]] \subseteq Z_\ell(R[[x; \alpha]]) \cap Z_r(R[[x; \alpha]])$. If $f(x) \in Z_\ell(R[[x; \alpha]])$, then there exists non-zero power series $g(x) \in R[[x; \alpha]]$ such that $f(x)g(x) = 0$. By Corollary 2.7, there exists $0 \neq r \in R$ such that $f(x)r = 0$. Thus $C_f R \subseteq ann(r) \subseteq Z(R)$, hence $C_f R \subseteq \mathcal{P}_k$ for some k, by [16, Theorem 3.3]. Therefore $f(x) \in \mathcal{P}_k[[x; \alpha]]$ and $Z_\ell(R[[x; \alpha]]) \subseteq \cup_{i=1}^{n} \mathcal{P}_i[[x; \alpha]]$. Then $Z_\ell(R[[x; \alpha]]) = \cup_{i=1}^{n} \mathcal{P}_i[[x; \alpha]]$. By a similar argument one can show that $Z_r(R[[x; \alpha]]) = \cup_{i=1}^{n} \mathcal{P}_i[[x; \alpha]]$. $\square$

Remark 2.11. *Let R be a reversible, α-compatible and right Noetherian ring. Thus $Z(R) = \cup_{i=1}^{n} P_i$, by* [17, Remark 3.2]. *Also we can assume that each P_i is maximal with respect to being contained in $Z(R)$. By the proof of Lemma 2.10, $Z(R[[x; \alpha]]) = \cup_{i=1}^{n} \mathcal{P}_i[[x; \alpha]]$. We show that each $P_i[[x; \alpha]]$ is maximal*

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with respect to being contained in $Z(R[[x;\alpha]])$. Assume that Q be an ideal of $R[[x;\alpha]]$ and there exists $1 \le i \le n$ such that $P_i[[x;\alpha]] \subseteq Q \subseteq Z(R[[x;\alpha]])$. Suppose that Q_0 be an ideal of R generated by coefficients of all elements of Q. Therefore $Q \subseteq Q_0[[x;\alpha]]$ and $Q_0 \subseteq Z(R)$, since R is α-power-serieswise McCoy by Corollary 2.7. Hence $Q_0 \subseteq P_k$, for some k, by [17, Theorem 3.3]*. Thus $P_i[[x;\alpha]] \subseteq Q \subseteq Q_0[[x;\alpha]] \subseteq P_k[[x;\alpha]]$, which implies that $P_i = P_k$. Therefore $P_i[[x;\alpha]] = Q = Q_0[[x;\alpha]] = P_k[[x;\alpha]]$ and hence $P_i[[x;\alpha]]$ is maximal with respect to being contained in $Z(R[[x;\alpha]])$, for each $1 \le i \le n$.*

Now we bring the following proposition, which has a key rule in our main results.

Proposition 2.12. *Let R be a reversible, α-compatible and right Noetherian ring and $f(x), g(x) \in R[[x;\alpha]]$. The following are equivalent:*

(1) $\ell_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x)) \neq 0$.
(2) $\ell_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x)) \neq 0$.
(3) $r_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x)) \neq 0$.
(4) $r_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x)) \neq 0$.

Proof. $(1) \to (2), (3), (4)$. Let $0 \neq h(x) \in \ell_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x))$. Thus $f(x), g(x) \in r_{R[[x;\alpha]]}(h(x))$. By Lemma 2.10, we have $Z(R[[x;\alpha]]) = \cup_{i=1}^{n} \mathcal{P}_i[[x;\alpha]]$. Then $r_{R[[x;\alpha]]}(h(x)) \subseteq \cup_{i=1}^{n} \mathcal{P}_i[[x;\alpha]]$. By Remark 2.11 and minor change in the proof of [16, Theorem 3.3], one can show that $r_{R[[x;\alpha]]}(h(x)) \subseteq \mathcal{P}_k[[x;\alpha]]$ for some $1 \le k \le n$. Since $\mathcal{P}_k[[x;\alpha]] = ann_{R[[x;\alpha]]}(a_k)$, $f(x), g(x) \in ann(a_k)$ and hence $g(x)a_k = a_k g(x) = a_k f(x) = f(x)a_k = 0$.

$(4) \to (1), (2), (3)$. Let $0 \neq h(x) \in r_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x))$. Thus $f(x), g(x) \in \ell_{R[[x;\alpha]]}(h(x))$. By Lemma 2.10, we have $Z(R[[x;\alpha]]) = \cup_{i=1}^{n} \mathcal{P}_i[[x;\alpha]]$. Then $\ell_{R[[x;\alpha]]}(h(x)) \subseteq \cup_{i=1}^{n} \mathcal{P}_i[[x;\alpha]]$. By Remark 2.11 and minor change in the proof of [16, Theorem 3.3], one can show that $\ell_{R[[x;\alpha]]}(h(x)) \subseteq \mathcal{P}_k[[x;\alpha]]$ for some $1 \le k \le n$. Since $\mathcal{P}_k[[x;\alpha]] = ann_{R[[x;\alpha]]}(a_k)$, $f(x), g(x) \in ann(a_k)$ and hence $g(x)a_k = a_k g(x) = a_k f(x) = f(x)a_k = 0$.

$(2) \to (4)$. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{i=0}^{\infty} b_i x^i$ and $0 \neq h(x) = \sum_{i=0}^{\infty} c_i x^i$ be an element of $\ell_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x))$. By Lemma 2.6, there exists $0 \neq c \in R$ such that $ch(x) \neq 0$ and $cc_i a_j = 0$ for each i, j. By reversibility of R, $h(x)c \neq 0$ and $a_j c_i c = 0$ for each i, j. Hence $ch(x) \in r_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x))$.

$(3) \to (4)$. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{i=0}^{\infty} b_i x^i$ and $0 \neq h(x) = \sum_{i=0}^{\infty} c_i x^i$ be an element of $r_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x))$. Then $f(x)h(x) = 0 = h(x)g(x)$. By Lemma 2.6, there exists $0 \neq c \in R$ such that $ch(x) \neq 0$ and $cc_i b_j = 0$ for each i, j. By Lemma 2.4 and reversibility assumption on R, $h(x)c \neq 0$ and $b_j c_i c = 0$ for each i, j. Hence $h(x)c \in r_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x))$. □

We denote the set of all nilpotent elements of a ring R by $Nil(R)$.

Lemma 2.13. *Let R be a reversible, α-compatible and right Noetherian ring. If $f(x) \in Z(R[[x;\alpha]])$ and $g(x) \in Nil(R[[x;\alpha]])$, then $r_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x)) \neq \{0\}$ and $f(x) + g(x) \in Z(R[[x;\alpha]])$.*

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Proof. Since $f(x) \in Z(R[[x;\alpha]])$, there exists $0 \neq c \in R$ such that $f(x)c = 0 = cf(x)$, by Corollary 2.7. Let m be the smallest integer such that $cg(x)^m = 0$. Then $0 \neq cg(x)^{m-1} \in r_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x))$, and hence $f(x) + g(x) \in Z(R[[x;\alpha]])$, by Proposition 2.12 □

Theorem 2.14. *Let R be a reversible, α-compatible and right Noetherian ring which is not reduced. If there is a pair of zero-divisors $f(x), g(x) \in Z(R[[x;\alpha]])$ with $\ell_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x)) = \{0\}$, then* $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 3$.

Proof. By Lemma 2.10, $Z_\ell(R[[x;\alpha]]) = Z_r(R[[x;\alpha]])$ and hence $\mathrm{diam}(R[[x;\alpha]]) \leq 3$, by [28, Theorem 2.3]. If we find $\alpha, \beta \in Z(R[[x;\alpha]])$ such that $\alpha\beta \neq 0$ and $\alpha h \neq 0 \neq h\beta$ for each $0 \neq h \in Z(R[[x;\alpha]])$, then $d(\alpha, \beta) = 3$ and hence $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 3$. In the other words, we must find $\alpha, \beta \in Z(R[[x;\alpha]])$ such that $\alpha\beta \neq 0$ and α, β don't have a non-zero mutual annihilator, by Proposition 2.12. Let $f(x), g(x) \in Z(R[[x;\alpha]])$ with $\ell_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x)) = \{0\}$. Then $f(x), g(x)$ don't have a non-zero mutual annihilator, by proposition 2.12, and hence $d(f(x), g(x)) \neq 2$. By Proposition 2.12 and Lemma 2.13, neither $f(x)$ nor $g(x)$ can be nilpotent. Assume that $f(x)g(x) = 0$. We claim that $f(x)^2$ and $g(x)^2$ don't have a non-zero mutual annilator. Let $0 \neq h(x) \in \ell_{R[[x;\alpha]]}(f(x)^2) \cap r_{R[[x;\alpha]]}(g(x)^2)$. Since $f(x), g(x)$ don't have a non-zero mutual annihilator, we have $h(x)f(x) \neq 0$ or $g(x)h(x) \neq 0$. If $h(x)f(x) \neq 0$, then $0 \neq h(x)f(x) \in \ell_{R[[x;\alpha]]}(f(x)) \cap \ell_{R[[x;\alpha]]}(g(x))$, which is a contradiction. Also if $g(x)h(x) \neq 0$, then $g(x)h(x) \in r_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x))$, which is a contradiction. Thus without loss of generality we may assume there is a nilpotent $a \in R$ such that $ag(x)^2 \neq 0$. Since a is nilpotent and R is reversible and α-compatible, $ag(x)$ is nilpotent by Lemma 2.3. Thus $f(x) + ag(x) \in Z(R[[x;\alpha]])$, by Lemma 2.13. Now consider the pair $f(x) + ag(x)$ and $g(x)$. If $k(x) \in r_{R[[x;\alpha]]}(f(x) + ag(x)) \cap r_{R[[x;\alpha]]}(g(x))$, then $k(x) \in r_{R[[x;\alpha]]}(f(x)) \cap r_{R[[x;\alpha]]}(g(x))$, and hence $k(x) = 0$. Thus $(f(x) + ag(x))$ and $g(x)$ don't have non-zero mutual annihilator and $(f(x) + ag(x))g(x) = ag(x)^2 \neq 0$. By considering $\alpha = f(x) + ag(x)$ and $\beta = g(x)$ the result follows. □

Theorem 2.15. [17, Theorem 2.13] *Let R be a reversible ring.*

(1) $\mathrm{diam}(\Gamma(R)) = 0$ *if and only if* $|Z(R)| = 2$.

(2) $\mathrm{diam}(\Gamma(R)) = 1$ *if and only if $xy = 0$ for each distinct pair x, y of zero-divisors and R has at least two non-zero zero-divisors.*

(3) $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if either (i) R is a reduced with exactly two minimal primes and at least two non-zero zero-divisor, or (ii) $Z(R)$ is an ideal whose square is not $\{0\}$ and each pair of distinct zero-divisors has a non-zero annihilator.*

(4) $\mathrm{diam}(\Gamma(R)) = 3$ *if and only if there are zero-divisors $a \neq b$ such that $ann_R(\{a, b\}) = 0$ and either (i) R is a reduced ring with more than two minimal primes, or (ii) R is non-reduced.*

According to [22], a commutative ring R has *Property (A)* if every finitely generated ideal of R consisting entirely of zero-divisors has a non-zero annihilator. Property (A) was originally studied by Y. Quentel in [27] (Our Property (A) is Quentel's condition (C)). Hukaba and Keller [22] proved that if R is any ring,

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then the polynomial ring $R[x]$ satisfies Property (A). Hong, Kim, Lee and Ryu [21] extend Property (A) to non-commutative rings as following: A ring R has *right (left) Property (A)* if every finitely generated two sided ideal of R consisting entirely of left (right) zero-divisors has right (left) non-zero annihilator. A ring R is said to have Property (A) if R has right and left Property (A).

Theorem 2.16. [18, Theorem 2.6] *Let R be a reversible and right Noetherian ring. Then R has right Property (A).*

Theorem 2.17. *Let R be a reversible, α-compatible and right Noetherian ring. Then $Z(R[[x;\alpha]])$ is an ideal of $R[[x;\alpha]]$ if and only if $Z(R)$ is an ideal of R.*

Proof. $\Leftarrow$ Suppose that $Z(R)$ is an ideal of R. Since R is reversible and right Noetherian, R has right Property (A), by Theorem 2.16. Let $f(x)=\sum_{i=0}^{\infty} a_i x^i$, $g(x)=\sum_{j=0}^{\infty} b_j x^j$ be non-zero zero-divisors of $R[[x;\alpha]]$. By Corollary 2.7, there are non-zero elements $r,s\in R$ such that $rf(x)=0=g(x)s$. Since R is right Noetherian, $C_f R=\langle a_0,\ldots,a_n\rangle$ and $C_g R=\langle b_0,\ldots,b_m\rangle$. Thus $\langle a_0,\ldots,a_n,b_0,\ldots,b_m\rangle\subseteq Z(R)$, since R is reversible and $Z(R)$ is ideal. Since R has right Property (A), there exists $0\neq t\in R$ such that $\langle a_0,\ldots,a_n,b_0,\ldots,b_m\rangle t=0$. By Lemma 2.3, $\langle f(x),g(x)\rangle t=0$ and hence $f(x)+g(x)\in Z(R[[x;\alpha]])$. Now, let $f(x)\in Z(R[[x;\alpha]])$ and $h(x)\in R[[x;\alpha]]$. By Corollary 2.7, there exists $0\neq r\in R$ with $rf(x)=0$. Then $ra_i=0$ for each i, and hence $ra_i\alpha^i(b_j)=0=b_j\alpha^j(a_i)r$, by Lemma 2.3. Thus $f(x)h(x)r=0=h(x)f(x)r$, by Lemma 2.3. Therefore $Z(R[[x;\alpha]])$ is an ideal of $R[[x;\alpha]]$.

$\Rightarrow$ It is clear. $\square$

The following two examples ([9, Example 3.3] and [26, Example 5.3], respectively) show that the hypothesis "R is right Noetherian" in Theorem 2.17 is not superfluous.

example 2.18. *Let K be a field and $R=K[Y,\{X_i\}_{i=0}^{\infty}]/\langle X_0Y,\{X_i-X_{i+1}Y\}_{i=0}^{\infty}\rangle$ be the ring defined in Example 2.8. Then, R has Property (A) and $Z(R)$ is ideal but $Z(R[[x]])$ is not ideal.*

Proof. Clearly R is a commutative non Noetherian ring. In R, $\overline{X_0}$ annihilates all other variables, since $\overline{X_0Y}=0$ and $\overline{X_0X_i}=\overline{X_0X_{i+1}Y}=0$. In R, $\overline{X_1}$ annihilates all other variables but $\overline{Y}$, since $\overline{X_1X_i}=\overline{X_1X_{i+1}Y}=\overline{X_0X_{i+1}}=0$, but $\overline{X_1Y}=\overline{X_0}\neq 0$. In R, $\overline{X_2}$ annihilates all other variables but $\overline{Y}$, since $\overline{X_2X_i}=\overline{X_2X_{i+1}Y}=\overline{X_1X_{i+1}}=0$, but $\overline{X_2Y}=\overline{X_1}\neq 0$. Continuing this process, we see that $\overline{X_i}(i\geq 1)$ annihilates all other variables besides $\overline{Y}$. Observe that $\overline{X_iY}=\overline{X_{i-1}}$, $\overline{X_iY^{i+r}}=\overline{X_0Y^r}=0$, $\overline{X_iY^i}=\overline{X_0}$, and $\overline{X_{i+r}Y_i}=\overline{X_r}$. Thus, any element of R can be written in the form $r=k_0+\sum_{j=1}^{\infty} l_j\overline{Y}^j+\sum_{i=0}^{\infty} k_{i+1}\overline{X_i}$, where $k_i,l_j\in K$ and all but finitely many of the $k_i's$ and $l_j's$ are zero. Notice that if $r\in Z(R)$, then r has no constant term; hence R has Property (A) and $Z(R)$ is an ideal, since $\overline{X_0}$ annihilates all variables. Let $f(x)=\overline{Y}-x$. Then $f\in Z^*(R[[x]])$, by Example 2.8, but $f(x)-\overline{Y}\notin Z(R[[x]])$. Thus $Z(R[[x]])$ is not an ideal of $R[[x]]$. $\square$

example 2.19. *Consider domain $D=F[\mathcal{X}]_{\langle\mathcal{X}\rangle}$, where F is a field and $\mathcal{X}=\{x_n\}$ is a countably infinite set of indeterminates. Let $M(=\langle\mathcal{X}\rangle D)$ denote the maximal ideal of D. Next, we let $\mathcal{P}$ denote the primes* *of D that are generated by finite subsets of $\mathcal{X}$. The set $\mathcal{P}$ includes $P_0 = (0)$, the prime generated by the empty subset of $\mathcal{X}$. Also for $n \geq 1$, we let $P_n = \langle x_1, x_2, \ldots, x_n \rangle D$. Note that given a prime $P_\alpha \in \mathcal{P}$, there is an integer n such that $P_\alpha \subset P_k$ for each $k \geq n$. For each $P_\alpha \in \mathcal{P}$, we let $\mathcal{Q}_\alpha = M/P_\alpha$. Let $\mathcal{O} = \{P_n | n \geq 0\}$ and let $C = \sum \mathcal{Q}_n$. Let $R = D + C$ be the ring formed from the product $D \times C$ by setting $(r, b) + (s, c) = (r + s, b + c)$ and $(r, b)(s, c) = (rs, rc + sb + bc)$. Clearly R is non Noetherian ring. Lucas showed that R is a reduced ring and has Property (A) such that $Z(R) = M + C$ is an ideal of R, but $Z(R[[x]])$ is not an ideal.*

Remark 2.20. *Let R be a reduced and α-compatible ring. Then by [19], the skew power series ring $R[[x; \alpha]]$ is reduced.*

Recall that if R is a reduced ring, then each minimal prime ideal of R is completely prime, by [25, Corollary 1.4]. Also each minimal prime ideal of R is a union of annihilators, by [25, Lemma 1.5]. Thus, if P is a minimal prime ideal of a reduced α-compatible ring R, then $\alpha^{-1}(P) = P$, and hence $P[[x; \alpha]]$ is an ideal of $R[[x; \alpha]]$. Therefore $P[[x; \alpha]]$ is a completely prime ideal of $R[[x; \alpha]]$, by [14, Theorem 2.20].

Theorem 2.21. *Let R be a reversible α-compatible ring with $Z(R) \neq 0$. Then* $\mathrm{diam}(\Gamma(R[[x; \alpha]])) \geq 1$. *In particular,* $\mathrm{diam}(\Gamma(R[[x; \alpha]])) = 1$ *if and only if R is a non-reduced ring with $Z(R)^2 = 0$*

Proof. Let a be a non-zero zero-divisor of R. Then $ab = 0$, for some $0 \neq b \in R$. Then $ax \neq bx^2$ and $(ax)(bx^2) = 0$, by Lemma 2.3. Thus $d(ax, bx^2) = 1$ which implies that $diam(\Gamma(R[[x; \alpha]])) \geq 1$.

Now, assume that R is a non-reduced ring with $Z(R)^2 = 0$. We claim that $Z(R[[x; \alpha]]) \subseteq Z(R)[[x; \alpha]]$. Suppose that $f(x) = \sum_{i=0}^{\infty} a_i x^i \in Z(R[[x; \alpha]]) \setminus Z(R)[[x; \alpha]]$. Then there exists $0 \neq g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \alpha]]$ such that $f(x)g(x) = 0$. We can assume that $a_0 \neq 0 \neq b_0$. First we show that $g(x) \notin Z(R)[[x; \alpha]]$. Suppose that $g(x) \in Z(R)[[x; \alpha]]$. Since $f(x)g(x) = 0$, we have $a_0 b_0 = 0$, and hence $a_0 \in Z(R)$. Let $\mathcal{D} := \{a_i | a_i \notin Z(R)\}$. Since $f(x) \notin Z(R)[[x; \alpha]]$, $\mathcal{D} \neq \emptyset$. Therefore we can write $f(x) = f_1(x) + f_2(x)$ such that $f_1(x) \in Z(R)[[x; \alpha]]$ and all of the coefficients of $f_2(x)$ belong to $\mathcal{D}$. Then $f_1(x)g(x) = 0$, since $g(x) \in Z(R)[[x; \alpha]]$, $Z(R)^2 = 0$ and R is α-compatible. Therefore $f_2(x)g(x) = 0$. Let a be the coefficient of the smallest degree term in $f_2(x)$. Since R is α-compatible and $f_2(x)g(x) = 0$, we have $ab_0 = 0$, which is a contradiction. Therefore $g(x) \notin Z(R)[[x; \alpha]]$.

Thus we can write $g(x) = g_1(x) + g_2(x)$ where $g_1(x) \in Z(R)[[x; \alpha]]$ and non of the coefficients of $g_2(x)$ is in $Z(R)$. Since R is α-compatible and $Z(R)$ is an ideal of R, we have $f_1(x)g_1(x) = 0$ and $f_1(x)g_2(x) + f_2(x)g_1(x) \in Z(R)[[x; \alpha]]$, which imply that $f_2(x)g_2(x) \in Z(R)[[x; \alpha]]$. Let a, b be non-zero coefficient of the smallest degree term in $f_2(x)$ and $g_2(x)$, respectively. Since R is α-compatible, we have $ab = 0$, which is a contradiction. Therefore $Z(R[[x; \alpha]]) \subseteq Z(R)[[x; \alpha]]$.

Now, assume that $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0}^{\infty} b_j x^j$ be two distinct non-zero elements of $Z(R[[x; \alpha]])$. Then $a_i, b_j \in Z(R)$ for each i, j, and since R is α-compatible and $Z(R)^2 = 0$, we have $f(x)g(x) = 0$. Thus $\mathrm{diam}(\Gamma(R[[x; \alpha]])) = 1$.

Assume that $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 1$. Let $a, b \in Z(R)$. Then ax, bx^2 are distinct zero-divisors of $R[[x;\alpha]]$. Hence $(ax)(bx^2) = a\alpha(b)x^3 = 0$, since $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 1$. Thus $ab = 0$, since R is α-compatible. Therefore $Z(R)^2 = 0$ and R is non-reduced. $\square$

Corollary 2.22. *Let $R \ncong \mathbb{Z}_2\times\mathbb{Z}_2$ be a reversible α-compatible ring with $|Z(R)| \geq 3$. Then* $\mathrm{diam}(\Gamma(R)) = 1$ *if and only if* $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 1$.

Proof. Since $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathrm{diam}(\Gamma(R)) = 1$ if and only if $xy = 0$ for all $x, y \in Z(R)$ by [17, Remark 2.2]. Thus $\mathrm{diam}(\Gamma(R)) = 1$ if and only if $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 1$, by Theorem 2.21. $\square$

We proceed to characterize the diameter of $\Gamma(R[[x;\alpha]])$, when R is a reversible, α-compatible and right Noetherian ring.

Theorem 2.23. *Let R be a reversible, α-compatible and right Noetherian ring with $Z(R) \neq 0$. Then:*

(1) $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 2$ *if and only if $|Z(R)| > 3$ and either (i) R is a reduced ring with exactly two minimal primes, or (ii) $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$.*

(2) $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and $Z(R)$ is not an ideal of R.*

Proof. (1) $\Leftarrow$ Assume that R is a reduced ring with exactly two minimal primes, namely P and Q. Therefore $Z(R) = P \cup Q$ and $P \cap Q = 0$. Hence $\mathrm{diam}(\Gamma(R)) = 2$, by Theorem 2.15. By Remark 2.20, $R[[x;\alpha]]$ is a reduced ring and $P[[x;\alpha]], Q[[x;\alpha]]$ are two prime ideals of $R[[x;\alpha]]$ and $P[[x;\alpha]]\cap Q[[x;\alpha]] = 0$. Since $PQ = 0$ we have $P[[x;\alpha]] \cup Q[[x;\alpha]] \subseteq Z(R[[x;\alpha]])$. Assume that $f = \sum_{i=0}^{\infty} a_i x^i \in Z(R[[x;\alpha]])$. By Corollary 2.7, there exists $0 \neq s \in R$ such that $fs = 0$. Thus $C_f Rs = 0$. Hence $C_f R \subseteq ann(s) \subseteq Z(R) = P \cup Q$. Therefore $C_f R \subseteq P$ or $C_f R \subseteq Q$, by [17, Theorem 3.3]. Thus $Z(R[[x;\alpha]]) \subseteq P[[x;\alpha]] \cup Q[[x;\alpha]]$, which implies that $Z(R[[x;\alpha]]) = P[[x;\alpha]] \cup Q[[x;\alpha]]$. Clearly, $\alpha^{-1}(P) = P$ and $\alpha^{-1}(Q) = Q$. Assume that $f, g \in Z(R[[x;\alpha]])$. If $f, g \in P[[x;\alpha]]$, then $tf = 0 = tg$, for each $t \in Q$ and hence $ft = 0 = tg$, since R is α-compatible and reduced ring. Similarly if $f, g \in Q[[x;\alpha]]$, then one can show that $fp = 0 = pg$ for each $p \in P$. Hence $d(f, g) = 2$. If $f \in P[[x;\alpha]]$ and $g \in Q[[x;\alpha]]$, then $fg = 0$, since $P \cap Q = 0$, $\alpha^{-1}(P) = P$ and $\alpha^{-1}(Q) = Q$. Thus $d(f, g) = 1$. Therefore $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 2$.

Now, assume that $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$. Since R is right Noetherian and reversible, then R has right Property (A) by Theorem 2.16. Let $f, g \in Z(R[[x;\alpha]])$, $C_f R = \langle a_0, \ldots, a_n\rangle$ and $C_g R = \langle b_0, \ldots, b_m\rangle$. By Corollary 2.7, $Z(R[[x;\alpha]]) \subseteq Z(R)[[x;\alpha]]$ and $Z(R)$ is an ideal of R. Hence $\langle a_0, \ldots, a_n, b_0, \ldots, b_m\rangle \subseteq Z(R)$. Thus there exists $0 \neq r \in R$ such that $\langle a_0, \ldots, a_n, b_0, \ldots, b_m\rangle r = 0$, since R has right Property (A). Therefore f and g has a non-zero mutual annihilator, by Lemma 2.3. Thus $\mathrm{diam}(\Gamma(R[[x;\alpha]])) \leq 2$. Since $(Z(R))^2 \neq 0$, hence by Theorem 2.21, $\mathrm{diam}(\Gamma(R[[x;\alpha]])) \geq 2$, which implies that $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 2$.

$\Rightarrow$ Let $\mathrm{diam}(\Gamma(R[[x;\alpha]])) = 2$. Therefore $Z(R[[x;\alpha]])^2 \neq 0$ and hence $Z(R)^2 \neq 0$, since $Z(R[[x;\alpha]]) \subseteq Z(R)[[x;\alpha]]$, by Corollary 2.7. Suppose that R is not a reduced ring with exactly two minimal prime ideals. Thus R is a reduced ring with more than two minimal prime ideals, or R is not reduced. If R is a reduced ring with more than two minimal prime ideals, then by Theorem 2.15, $\operatorname{diam}(\Gamma(R)) = 3$ and hence $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 3$, which is a contradiction. Thus R is not reduced. Since $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$ and R is a reversible, α-compatible and right Noetherian ring which is not reduced, hence by Theorem 2.14 and Proposition 2.12, each pair of non-zero zero-divisor of $R[[x;\alpha]]$ has a non-zero mutual annihilator. Thus for two non-zero zero-divisors $f, g \in R[[x;\alpha]]$, $f + g \in Z(R[[x;\alpha]])$. Also, for each non-zero element $h \in R[[x;\alpha]]$, $hf, fh \in Z(R[[x;\alpha]])$, since R is α-power-serieswise McCoy by Corollary 2.7. Therefore $Z(R[[x;\alpha]])$ is an ideal of $R[[x;\alpha]]$ and hence $Z(R)$ is an ideal of R, by Theorem 2.17.

(2) $\Rightarrow$ Assume that $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 3$. If $(Z(R))^2 = 0$, then R is non-reduced and hence by Theorem 2.21, $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 1$, which is a contradiction. Thus $(Z(R))^2 \neq 0$ and hence by statement (1), R is not a reduced ring with exactly two minimal primes and $Z(R)$ is not an ideal of R.

$\Leftarrow$ Now, assume that R is not a reduced ring with exactly two minimal primes and $Z(R)$ is not an ideal of R. Then $\operatorname{diam}(\Gamma(R[[x;\alpha]])) \neq 2$, by statement (1). If $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 1$, then R is a non-reduced ring with $(Z(R))^2 = 0$, by Theorem 2.21. Thus $Z(R)$ is an ideal of R, which is a contradiction. Therefore $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 3$. $\square$

Corollary 2.24. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a reversible, α-compatible and right Noetherian ring with $|Z(R)| > 3$. Then*

(1) $\operatorname{diam}(\Gamma(R)) = 2$ *if and only if* $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$.

(2) $\operatorname{diam}(\Gamma(R)) = 3$ *if and only if* $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 3$.

Proof. (1) By Theorem 2.15 and Theorem 2.23(1), $\operatorname{diam}(\Gamma(R)) = 2$ if and only if $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$.

(2) Assume that $\operatorname{diam}(\Gamma(R)) = 3$. Since $\operatorname{diam}(\Gamma(R[[x;\alpha]])) \leq 3$ and $\Gamma(R)$ is a subgraph of $\Gamma(R[[x;\alpha]])$, we have $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 3$. Conversely, assume that $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 3$. Thus R is not a reduced ring with exactly two minimal prime ideals and $Z(R)$ is not an ideal of R, by Theorem 2.23. Thus there exist non-zero distinct zero-divisors a, b of R such that $ann(\{a, b\}) = 0$ and either R is reduced with more than two minimal prime ideals, or R is not reduced. Hence $\operatorname{diam}(\Gamma(R)) = 3$, by Theorem 2.15. $\square$

The following examples show that the assumption "R is right Noetherian" in Theorem 2.23 is not superfluous.

example 2.25. *Let D and $R = D + C$ be the rings defined in Example 2.19. Then R is a commutative reduced ring and has right Property (A) such that $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$. Thus $\operatorname{diam}(\Gamma(R)) = 2$, by Theorem 2.15. But $\operatorname{diam}(R[[x]]) = 3$, since $Z(R[[x]])$ is not an ideal of $R[[x]]$, which shows that Theorem 2.23 is not true when R is not Noetherian.*

example 2.26. [26, Example 5.6] *Let $R = \mathbb{Z}(+)\mathbb{Z}(p^\infty)$, the idealization of the integers and $\mathbb{Z}(p^\infty)$, by setting $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, ad + bc)$. Clearly R is a commutative not reduced ring. Lucas in [26] showed that R has Property (A) and $Z(R) = p\mathbb{Z}(+)\mathbb{Z}(p^\infty)$ is an ideal of R, but $\operatorname{diam}(\Gamma(R[[x]])) = 3$, which shows that Theorem 2.23 is not true when R is not Noetherian.*

We conclude the paper by giving a result, with the aim to establish a relationship between the diameters of $\Gamma(R)$ and $\Gamma(R[[x;\alpha]])$, when R is a reversible right Noetherian α-compatible ring.

Theorem 2.27. *Let R be a reversible, α-compatible and right Noetherian ring. The following cases describe all possibilities for the pair* $\operatorname{diam}(\Gamma(R))$ *and*
$\operatorname{diam}(\Gamma(R[[x;\alpha]]))$*:*

(1) $\operatorname{diam}(\Gamma(R)) = 0$ *and* $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 1$ *if and only if* $|Z(R)| = 2$.
(2) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[[x;\alpha]])) = 1$ *if and only if R is a non-reduced ring with more than one non-zero zero-divisor such that* $(Z(R))^2 = 0$.
(3) $\operatorname{diam}(\Gamma(R)) = 1$ *and* $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$ *if and only if R is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$.
(4) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$ *if and only if either (i) R is a reduced ring with exactly two minimal primes and R has more than two non-zero zero-divisors, or (ii) $Z(R)$ is an ideal with* $(Z(R))^2 \neq 0$ *and R has right Property (A).*
(5) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[[x;\alpha]])) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and there is a pair of zero-divisors a and b with no non-zero mutual annihilator.*

Proof. (1) $\Rightarrow$ It is clear.

$\Leftarrow$ Clearly $\operatorname{diam}(\Gamma(R)) = 0$. Let $Z(R) = \{0, a\}$, then ax, ax^2 are distinct zero-divisors of $R[[x;\alpha]]$ and $(ax)(ax^2) = 0$, since R is α-compatible. Thus
$\operatorname{diam}(\Gamma(R[[x;\alpha]])) \geq 1$. If $f(x) = \sum_{i=0}^{\infty} a_i \in Z(R[[x;\alpha]])$, then there exists $0 \neq c \in R$ such that $f(x)c = 0$, by Corollary 2.7. Thus $a_i \in Z(R)$ for each i, and hence $(Z(R[[x;\alpha]]))^2 = 0$, since R is α-compatible. Thus $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 1$.

(2) It follows from Theorems 2.15 (2) and 2.21.

(3) $\Rightarrow$ Since $\operatorname{diam}(\Gamma(R)) = 1$, we have $ab = 0$ for each distinct $a, b \in Z(R)$. Since $\operatorname{diam}(\Gamma(R[[x]])) = 2$, we have $(Z(R))^2 \neq 0$. Thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, by [17, Remark 2.2].

$\Leftarrow$ Clearly $\operatorname{diam}(\Gamma(R)) = 1$ and R is reduced with exactly two minimal primes. Then $\operatorname{diam}(\Gamma(R[[x;\alpha]])) = 2$, by Theorem 2.23.

(4) Note that if R has right Property (A) and $Z(R)$ is an ideal of R, then each pair of distinct zero-divisors has a non-zero annihilator. Thus the result follows from Theorems 2.15(3) and 2.23(2).

(5) It follows from Theorems 2.15(4) and 2.23(2). $\square$

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